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FORMULAS FOR THE SUMS OF SOME CONDITIONALLY CONVERGENT SERIES

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Abstract. *In this paper, we find formulas for the sums of two families of conditionally convergent series containing $p(\geq 1)$ arbitrary real parameters, in terms of elementary functions. These families, when the parameters are chosen appropriately, coincide with the formulas for $\ln 2$ and $\pi/4$ when $p = 1$, and with the formulas known to Newton when $p = 2$ and to Euler when $p = 2, 4$.*

Keywords: conditionally convergent series, Newton-Euler summation formulas for some conditionally convergent series

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1. Introduction

Consider two divergent series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \quad \text{and} \quad 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots .$$

Let p be a natural number. Without rearranging the terms of these series, we change their signs so that p positive terms are followed by the same number of negative terms, then p positive terms again, and so on. This results in conditionally convergent series

$$S_{1,p} := 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p} - \frac{1}{p+1} - \frac{1}{p+2} - \ldots - \frac{1}{2p} + \frac{1}{2p+1} + \ldots + \frac{1}{3p} - \ldots ,$$

$$S_{2,p} := 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2p-1} - \frac{1}{2p+1} - \frac{1}{2p+3} - \ldots - \frac{1}{4p-1} + \frac{1}{4p+1} + \ldots + \frac{1}{6p-1} - \ldots .$$

These series are special cases of a conditionally convergent series

$$S_{b,\,p}\left(a_0, a_1, \ldots, a_{p-1}\right) := \sum_{n=0}^{+\infty} (-1)^n \left(\frac{a_0}{bpn+1} + \ldots + \frac{a_{p-1}}{bpn + b(p-1) + 1} \right), \qquad (1)$$

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where $a_0, a_1, \ldots, a_{p-1}$ are given real numbers, and the parameter b is equal to 1 or 2 when $a_0 = \ldots = a_{p-1} = 1$.

The classical formulas $S_{1,1} = \ln 2$ and $S_{2,1} = \pi/4$ are well known. They are obtained from the Taylor series expansions of the functions $\ln(1+x)$ and $\operatorname{arctg} x$, respectively, and the limit transition in the resulting series as $x \to 1$. The very interesting history of the discovery of these formulas is described in [5].

In this paper, we present a method (see Section 2) that allows us to find formulas for the sum $S_{b,p}(a_0, a_1, \ldots, a_{p-1})$ in terms of elementary functions. In particular, in Section 3, we prove the following theorem.

Theorem. *Let $p \in \mathbb{N}$ and $p \geq 2$. Then the following equalities hold*

$$S_{1,p} = \frac{\ln 2}{p} + \frac{\pi}{p} \sum_{j=0}^{[(p-2)/2]} \left(1 - \frac{1+2j}{p}\right) \operatorname{ctg}\left(\frac{\pi + 2\pi j}{2p}\right) \tag{2}$$

and

$$S_{2,p} = \frac{\pi}{2p} \left(\frac{\varepsilon_p}{2} + \sum_{j=0}^{[(p-2)/2]} \frac{1}{\sin \dfrac{\pi + 2\pi j}{2p}} \right), \tag{3}$$

where $[x]$ is the integer part of x, and $\varepsilon_p = 0$ if p is an even number, and $\varepsilon_p = 1$ if p is an odd number.

Letting $p = 2, 3, 4, 5$ and 6 in the formulas (2) and (3), we obtain the following identities

$$S_{1,2} = \frac{\ln 2}{2} + \frac{\pi}{4}, \qquad S_{1,3} = \frac{\ln 2}{3} + \frac{2\pi}{3\sqrt{3}}, \qquad S_{1,4} = \frac{\ln 2}{4} + \frac{(2\sqrt{2}+1)\pi}{8},$$

$$S_{1,5} = \frac{\ln 2}{5} + \frac{2\sqrt{25 + 10\sqrt{5}} \cdot \pi}{25}, \qquad S_{1,6} = \frac{\ln 2}{6} + \frac{(15 + 4\sqrt{3})\pi}{36},$$

$$S_{2,2} = \frac{\pi}{2\sqrt{2}}, \quad S_{2,3} = \frac{5\pi}{12}, \quad S_{2,4} = \frac{\sqrt{2+\sqrt{2}} \cdot \pi}{4},$$

$$S_{2,5} = \frac{(4\sqrt{5}+1)\pi}{20}, \quad S_{2,6} = \frac{\sqrt{2}(1 + 2\sqrt{3})\pi}{12}$$

accordingly.

In conclusion of this paragraph, we note that some of these formulas are well known, for example, the formulas for $S_{1,2}$ and $S_{2,3}$ are given in [4, clause 5.1.2, formula 5, and clause 5.1.4, formula 5], the formula for $S_{2,2}$ belongs to Newton (see, for example, [1, Chapter 1, problem No 52(b)] and [3]), and the formula for $S_{2,4}$ belongs to Euler (see, for example, [2, p. 180]).

2. The Method of Calculating the Amount $S_{b,\,p}\left(a_0, a_1, \ldots, a_{p-1}\right)$

2. 1.

Let the numbers $b, p, a_0, a_1, \ldots a_{p-1}$ be the same as in Section 1. This section provides a method for finding formulas for the sums of conditionally convergent series (1).

Consider the power series

$$f_{b,\,p}(x)(\equiv f_{b,\,p}(a_0, a_1, \ldots, a_{p-1}; x)) := \sum_{n=0}^{+\infty}(-1)^n\left(\frac{a_0 x^{bpn+1}}{bpn+1} + \ldots + \frac{a_{p-1}x^{bpn+b(p-1)+1}}{bpn+b(p-1)+1}\right).$$

It is easy to see that this series converges for $|x| < 1$ and allows for pointwise differentiation. Therefore

$$f'_{b,\,p}(x) = \sum_{n=0}^{+\infty}(-x^{bp})^n\left(a_0 + \ldots + a_{p-1}x^{b(p-1)}\right) = \frac{P_{b,\,p}(x)}{1+x^{bp}},$$

where $P_{b,\,p}(x) := a_0 + a_1x^b + \ldots + a_{p-1}x^{b(p-1)}$. Given that $f_{b,\,p}(0) = 0$ and integrating (2) over the interval $[0; x]$, we find that

$$f_{b,\,p}(x) = \int_0^x \frac{P_{b,\,p}(t)}{1+t^{bp}}\,dt. \tag{4}$$

Next, let $x_0, x_1, \ldots, x_{bp-1}$ denote the roots of the equation

$$1 + x^{bp} = 0 \tag{5}$$

and by expanding the integrand into simple fractions, we find that

$$\frac{P_{b,\,p}(t)}{1+t^{bp}} = \frac{A_0}{t-x_0} + \frac{A_1}{t-x_1} + \ldots \frac{A_{bp-1}}{t-x_{bp-1}}, \tag{6}$$

where the unknown coefficients $A_0, \ldots, A_{bp-1}$ are given by the formula

$$A_j = \lim_{t\to x_j}\frac{P_{b,\,p}(t)(t-x_j)}{1+t^{bp}} = -\frac{x_j}{bp}P_{b,\,p}(x_j), \quad j = 0, 1, \ldots, bp-1. \tag{7}$$

2. 2.

First, consider the case $b = 1$ and $p = 2k$, $k = 1, 2, \ldots$. In this case, the roots of the equation (5) are found by the formulas $x_j = e^{i\pi(1+2j)/(2k)}$, $j = 0, 1, \ldots, 2k-1$. From these equalities and the realness of the numbers $a_0, a_1, \ldots, a_{p-1}$, it follows that for $j = 0, \ldots, k-1$, the equalities $x_{2k-1-j} = \overline{x_j}$ and $A_{2k-1-j} = \overline{A_j}$. Therefore, by grouping the corresponding terms in (6), taking into account the obtained result in (4) and integrating, we arrive at the validity of the relation

$$f_{1,\,2k}(x)=\sum_{j=0}^{k-1}\mathrm{Re}\,A_j\cdot\ln\left(x^2-2\mathrm{Re}\,x_j\cdot x+1\right)-$$

$$-2\sum_{j=0}^{k-1}\mathrm{Im}\,A_j\cdot\left(\mathrm{arctg}\left(\frac{x-\mathrm{Re}\,x_j}{\mathrm{Im}\,x_j}\right)+\frac{\pi}{2}-\frac{\pi+2\pi j}{2k}\right). \tag{8}$$

Taking the limit of both sides of this equality as $x\to 1$ and using Abel's second theorem, we find that

$$S_{1,\,2k}(a_0,a_1,\ldots,a_{2k-1})=f_{1,2k}(a_0,a_1,\ldots,a_{2k-1};1). \tag{9}$$

Now consider the case when $b=1$ and $p=2k+1$, $k=1,2,\ldots$. In this case, the roots of equation (5) are given by the formula

$$x_j=e^{\frac{i\pi(1+2j)}{2k+1}},\quad j=0,1,\ldots,2k. \tag{10}$$

Using arguments similar to those above, we can prove that the following equalities are true

$$f_{1,\,2k+1}(x)=\frac{P_{1,\,2k+1}(-1)}{2k+1}\ln(x+1)+\sum_{j=0}^{k-1}\mathrm{Re}\,A_j\ln\left(x^2-2\mathrm{Re}\,x_j\cdot x+1\right)-$$

$$-2\sum_{j=0}^{k-1}\mathrm{Im}\,A_j\left(\mathrm{arctg}\left(\frac{x-\mathrm{Re}\,x_j}{\mathrm{Im}\,x_j}\right)+\frac{\pi}{2}-\frac{\pi+2\pi j}{2k+1}\right), \tag{11}$$

$$S_{1,\,2k+1}(a_0,a_1,\ldots,a_{2k})=f_{1,2k+1}(a_0,a_1,\ldots,a_{2k};1), \tag{12}$$

where A_j is calculated using formula (7), and x_j is calculated using formula (10).

2. 3.

If $b=2$, then repeating the reasoning above, we conclude that

$$f_{2,\,p}(x)=\sum_{j=0}^{p-1}\mathrm{Re}\,A_j\ln\left(x^2-2\mathrm{Re}\,x_j\cdot x+1\right)-$$

$$-2\sum_{j=0}^{p-1}\mathrm{Im}\,A_j\left(\mathrm{arctg}\left(\frac{x-\mathrm{Re}\,x_j}{\mathrm{Im}\,x_j}\right)+\frac{\pi}{2}-\frac{\pi+2\pi j}{2p}\right), \tag{13}$$

where x_j are defined as the roots of equation (5), and A_j are calculated using formula (7) with $b=2$, and

$$S_{2,\,p}(a_0,a_1,\ldots,a_{p-1})=f_{2,p}(a_0,a_1,\ldots,a_{p-1};1). \tag{14}$$

Formulas (8), (11), (13) allow us to find the sums of power series $f_{b,\,p}(x)$, and formulas (9), (12), (14) - sums of conditionally convergent numerical series $S_{b,\,p}\,(a_0,a_1,\ldots,a_{p-1})$ in terms of elementary functions.

3. Proof of Theorem

3. 1.

Let's move on to the proof of the formula (2). When $b = 1$ and $a_0 = a_1 = \ldots = a_{p-1} = 1$, it follows from (7) that

$$A_j = \frac{2x_j}{p(x_j - 1)}, \quad j = 0, 1, \ldots, p - 1.$$

From this equality, it can be deduced that

$$\operatorname{Re} A_j = \frac{1}{p}, \qquad \operatorname{Im} A_j = -\frac{1}{p} ctg\left(\frac{\pi + 2\pi j}{2p}\right), \quad j = 0, \ldots, p - 1.$$

Thus, in the case $p = 2k$, $k = 1, 2, \ldots$, the equality (9) can be written as

$$S_{1,\,2k} = \frac{\ln 2}{2} + \frac{1}{2k}\sum_{j=0}^{k-1} \ln\left(1 - \cos\frac{\pi + 2\pi j}{2k}\right) + \frac{1}{2k}\sum_{j=0}^{k-1} \operatorname{ctg}\left(\frac{\pi + 2\pi j}{4k}\right) \cdot \left(\pi - \frac{\pi + 2\pi j}{2k}\right).$$

Using the formula

$$\prod_{j=0}^{\left[\frac{n-1}{2}\right]} \sin\frac{\pi j}{n} = \sqrt{2^{1-n} n}$$

(see, for example, [1, p. 6.1.2, formula 3]), we transform the first sum

$$\sum_{j=0}^{k-1} \ln\left(1 - \cos\frac{\pi + 2\pi j}{2k}\right) = k\ln 2 + 2\left(\ln\left(\prod_{j=0}^{\left[\frac{4k-1}{2}\right]} \sin\frac{\pi j}{4k}\right) - \ln\left(\prod_{j=0}^{\left[\frac{2k-1}{2}\right]} \sin\frac{\pi j}{2k}\right)\right) = (1-k)\ln 2.$$

Thus, we finally get that

$$S_{1,\,2k} = \frac{\ln 2}{2k} + \frac{\pi}{2k}\sum_{j=0}^{k-1}\left(1 - \frac{1+2j}{2k}\right)\operatorname{ctg}\left(\frac{\pi + 2\pi j}{4k}\right).$$

If $p = 2k + 1, k = 1, 2, \ldots$, then

$$S_{1,\,2k+1} = \frac{k+1}{2k+1}\ln 2 + \frac{1}{2k+1}\sum_{j=0}^{k-1}\left[\ln\left(1 - \cos\frac{\pi + 2\pi j}{2k+1}\right) + \left(\pi - \frac{\pi + 2\pi j}{2k+1}\right)\operatorname{ctg}\frac{\pi + 2\pi j}{2(2k+1)}\right],$$

where, as calculations show,

$$\sum_{j=0}^{k-1} \ln\left(1 - \cos\frac{\pi + 2\pi j}{2k+1}\right) = -k\ln 2.$$

Therefore, formula (2) is also valid for $p = 2k + 1$.

3. 2.

In the case $b = 2$ and $a_0 = a_1 = \ldots = a_{p-1} = 1$, it follows from formula (7) that

$$A_j = \frac{x_j}{p(x_j^2 - 1)}, \qquad j = 0, 1, \ldots, 2p - 1.$$

From this equality, it can be deduced that

$$\operatorname{Re} A_j = 0\,, \qquad \operatorname{Im} A_j = -\frac{1}{2p} \cdot \frac{1}{\sin \dfrac{\pi + 2\pi j}{2p}}.$$

Considering these equalities in (14), we arrive at the identity

$$S_{2,p} = \frac{\pi}{4p^2} \sum_{j=0}^{p-1} \frac{2(p-j) - 1}{\sin \dfrac{\pi + 2\pi j}{2p}},$$

from which, in turn, the equality (3) follows.

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