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# INVERSE PROBLEM OF DETERMINING INITIAL CONDITIONS IN A MIXED PROBLEM FOR A TWO-DIMENSIONAL HYPERBOLIC EQUATION

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**Abstract.** *The article deals with the determination of initial conditions in a mixed problem for a two-dimensional hyperbolic equation. First, the uniqueness of the solution to the corresponding initial–boundary value problem is established. Then, by imposing certain restrictions on the given data, the existence of a solution to this problem is demonstrated. Furthermore, existence and uniqueness theorems are proved for the associated inverse problem of determining the initial conditions.*

**Keywords**: inverse problem, hyperbolic equation, existence, uniqueness, classical solution

**Mathematics Subject Classification (2020):** 35R30, 35M10, 35L10, 35A01, 35A02

## 1. Introduction

Inverse boundary value problems constitute one of the most important classes of problems in mathematics and applied sciences. The primary objective of such problems is to determine unknown coefficients and/or the right-hand side of partial differential equations using additional measurement data. These problems typically arise when the characteristics of an object of interest cannot be observed directly. For example, they include the reconstruction of field source properties from their measured values at certain points, or the recovery and interpretation of an original signal based on a known output signal. Moreover, inverse problems occur in a wide range of fields, including medical imaging, geophysics, non-destructive testing, acoustics, oil and gas exploration, and electromagnetic or $X$-ray tomography, etc.

In addition, the inverse boundary value problem may also involve reconstructing the initial conditions from boundary observations over time or reconstructing the original terms in the equation. In this article we will consider the inverse problem of determining the initial conditions in a mixed problem for a two-dimensional hyperbolic equation. It should be noted that inverse problems for hyperbolic equations are of crucial importance in various fields, providing insight into systems in which wave propagation plays a key role. The fundamentals of the theory and practice of studying inverse problems were established and developed in the fundamental works of A.N. Tikhonov [25], M.M. Lavrentiev et al. [15], V.K. Ivanov et al. [12], A.I. Prilepko et al. [20], and others. However, sufficiently complete bibliographies

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of recent works related to the study of inverse problems for partial differential equations are reflected in many monographs and articles (e.g., [2]-[4], [6], [10], [11], [13], [14], [16]-[19], [21], [22], [24], [26], [27]).

We provide a brief overview of related works on inverse boundary value problems for time-fractional parabolic equations. In the monograph by Yuldashev [27], the unique generalized solvability of multidimensional mixed problems for higher-order nonlinear partial differential equations was investigated. In this work, a new method was developed for analyzing the unique solvability of mixed problems for differential equations that contain elementary operators of higher-order mathematical physics on the left-hand side and a nonlinear function on the right-hand side. In the paper by Eskin [6], inverse problems for second-order hyperbolic equations of general form with time-dependent coefficients were investigated, and the time-dependent Lorentzian metric was determined from boundary measurements. The article [19] investigates an inverse boundary value problem for a two-dimensional hyperbolic equation with overdetermination conditions and establishes existence and uniqueness theorems for the classical solution by applying the contraction mapping principle. In the work of Sabitov and Zaynullov [23], inverse problems aimed at determining the initial conditions for the string and telegraph equations were studied. The authors established criteria for uniqueness and provided estimates ensuring the separation of small denominators from zero with the corresponding asymptotic behavior, which made it possible to justify the convergence within the class of regular solutions of these equations. Denisov [5] proposed an iterative method for solving the inverse coefficient problem for a hyperbolic equation by reducing it to a nonlinear operator equation with respect to the unknown coefficient and proved the uniform convergence of the iterations to the solution of the inverse problem.

The numerical aspects of inverse problems for hyperbolic equations under various boundary conditions have been extensively investigated in [1], [7]-[9] and the references therein.

The paper is organized as follows. Section 1 establishes the relevance of the study, formulates its main objectives, and provides a comprehensive review of the related literature with detailed comparisons to previous works. In Section 2, the mathematical formulation of the problem under consideration is presented, and the definition of a classical solution is introduced. Section 3 states and proves a theorem on the uniqueness of the solution to the initial–boundary value problem. The existence of a classical solution is analyzed in Section 4. Section 5 is devoted to determining the initial conditions for the inverse boundary value problem, and Section 6 summarizes the key findings of the study.

## 2. Mathematical Formulation of the Problem

Let $D_T = \bar{Q}_{xy} \times \{0 \le t \le T\}$ be the closed bounded region in space, where $Q_{xy}$ defined by the inequalities $0 < x < 1,\ 0 < y < 1$, and let $f(x,y,t), \varphi(x,y), \psi(x,y)$ be sufficiently smooth functions of $x, y \in [0,1]$ and $t \in [0,T]$. We first consider an initial boundary value problem to find a function $u(x,y,t)$ satisfying the following equation

$$u_{tt}(x,y,t) - \Delta u(x,y,t) = f(x,y,t) \quad (x,y,t) \in D_T, \tag{1}$$

with the initial conditions

$$u(x,y,0) = \varphi(x,y),\ u_t(x,y,0) = \psi(x,y) \quad (0 \le x \le 1,\ 0 \le y \le 1), \tag{2}$$

and boundary conditions

$$u_x(0,y,t) = u(1,y,t) = 0 \quad (0 \le y \le 1,\ 0 \le t \le T), \tag{3}$$

$$u(x,0,t) = u_y(x,1,t) = 0 \quad (0 \le x \le 1,\ 0 \le t \le T), \tag{4}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

**Definition 1.** *By a classical solution to problem* (1)–(4) *we mean a function* $u(x,y,t) \in C^2(D_T)$ *that satisfies* (1) *in* $D_T$, *condition* (2) *in* $\bar{Q}_{xy}$, *and conditions* (3), (4) *on the sets in* $Q_{yt} = \{(y,t) : 0 \le y \le 1, 0 \le t \le T\}$ *and* $Q_{xt} = \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$, *respectively, in the usual sense.*

## 3. Uniqueness of the Solution to the Initial Boundary Value Problem

**Theorem 1.** *Problem* (1)–(4) *cannot have more than one solution, i.e. if problem* (1)–(4) *has a solution, then it is unique.*

*Proof.* Assume that the functions $u_1(x,y,t)$ and $u_2(x,y,t)$ are two distinct solutions to the considered problem and their difference is $v(x,y,t) = u_1(x,y,t) - u_2(x,y,t)$.

Obviously, the function $v(x,y,t)$ satisfies the homogeneous equation

$$v_{tt}(x,y,t) - \Delta v(x,y,t) = 0, \tag{5}$$

with the conditions

$$v(x,y,0) = v_t(x,y,0) = 0 \quad (0 \le x \le 1, 0 \le y \le 1), \tag{6}$$

$$v_x(0,y,t) = v(1,y,t) = 0 \quad (0 \le y \le 1, 0 \le t \le T), \tag{7}$$

$$v(x,0,t) = v_y(x,1,t) = 0 \quad (0 \le x \le 1, 0 \le t \le T). \tag{8}$$

Let us prove that the function $v(x,y,t)$ is identically equal to zero. Multiplying both sides of the Equation (5) by the special function $2v_t(x,y,t)$ and integrating the resulting equality with respect to $x$ and $y$ over the interval $[0,1]$, we get

$$2\int_0^1\int_0^1 v_{tt}(x,y,t)v_t(x,y,t)dxdy - 2\int_0^1\int_0^1 \Delta v(x,y,t)v_t(x,y,t)dxdy = 0. \tag{9}$$

Using boundary conditions (7), (8), we have

$$2\int_0^1\int_0^1 v_{tt}(x,y,t)v_t(x,y,t)dxdy = \frac{d}{dt}\int_0^1\int_0^1 v_t^2(x,y,t)dxdy,$$

$$2\int_0^1 v_{xx}(x,y,t)v_t(x,y,t)dx = 2v_x(1,y,t)v_t(1,y,t) - 2v_x(0,y,t)v_t(0,y,t)$$

$$-2\int_0^1 v_x(x,y,t)v_{tx}(x,y,t)dx = -\frac{d}{dt}\int_0^1 v_x^2(x,y,t)dx,$$

$$2\int_0^1 v_{yy}(x,y,t)v_t(x,y,t)dy = 2v_y(x,1,t)v_t(x,1,t) - 2v_y(x,0,t)v_t(x,0,t)$$

$$-2\int_0^1 v_y(x,y,t)v_{ty}(x,y,t)dy = -\frac{d}{dt}\int_0^1 v_y^2(x,y,t)dy.$$

Then, from (9), we obtain

$$\frac{d}{dt}\int_0^1\int_0^1 (v_t^2(x,y,t)+v_x^2(x,y,t)+v_y^2(x,y,t))dxdy=0 \quad (0\le t\le T). \tag{10}$$

If we use the notation

$$H(t)=\int_0^1\int_0^1 (v_t^2(x,y,t)+v_x^2(x,y,t)+v_y^2(x,y,t))dxdy \quad (0\le t\le T),$$

then from equality (10) it follows that the derivative of function $H(t)$ is equal to zero, i.e.

$$H'(t)=0 \quad (0\le t\le T).$$

Hence

$$H(t)=\int_0^1\int_0^1 (v_t^2(x,y,t)+v_x^2(x,y,t)+v_y^2(x,y,t))dxdy=C, \tag{11}$$

where $C$ is an arbitrary constant.

So, from the initial condition (6) it follows that

$$H(0)=\int_0^1\int_0^1 (v_t^2(x,y,0)+v_x^2(x,y,0)+v_y^2(x,y,0))dxdy=0.$$

From (11) it is obvious that

$$H(0)=C=0.$$

Setting $C=0$ in (11), the procedure yields

$$\int_0^1\int_0^1 (v_t^2(x,y,t)+v_x^2(x,y,t)+v_y^2(x,y,t))dxdy=0 \quad (0\le t\le T).$$

Since the integrand is non-negative, the following equalities hold:

$$v_t(x,y,t)=0,\ v_x(x,y,t)=0,\ v_y(x,y,t)=0.$$

Thus, we get

$$v(x,y,t)=const=C_0.$$

Using the condition (7), we see that

$$v(x,y,0)=C_0=0.$$

Hence, we conclude that $C_0=0$. So, it proves that

$$v(x,y,t)\equiv 0,$$

or

$$u_1(x,y,t)\equiv u_2(x,y,t).$$

It follows that if a solution to problem (1)–(4) exists, then it is unique. The theorem is proved. ◀

# 4. Existence of a Solution to the Problem

We seek a solution $u(x,y,t)$ of problem (1)–(4) in the form

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}u_{k,n}(t)\cos\lambda_k x\sin\gamma_n y, \tag{12}$$

where

$$\lambda_k=\frac{\pi}{2}(2k-1)\ (k=1,2,...),\quad \gamma_n=\frac{\pi}{2}(2n-1)\ (n=1,2,...),$$

$$u_{k,n}(t)=4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\ (k,n=1,2,...).$$

Applying the method of separation of variables to determine the desired coefficients $u_{k,n}(t)$ $(k,n=1,2,...)$ for the function $u(x,y,t)$ from (1), (2), we obtain

$$u''_{k,n}(t)+\mu_{k,n}^2u_{k,n}(t)=f_{k,n}(t)\quad (k,n=1,2,...;0\le t\le T), \tag{13}$$

$$u_{k,n}(0)=\varphi_k,u'_{k,n}(0)=\psi_{k,n}\quad (k,n=1,2,...), \tag{14}$$

where

$$\mu_{k,n}^2=\lambda_k^2+\gamma_n^2\quad (k,n=1,2,...),$$

$$f_{k,n}(t)=4\int_0^1\int_0^1 f(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\quad (k,n=1,2,...),$$

$$\varphi_{k,n}=4\int_0^1\int_0^1 \varphi(x,y)\cos\lambda_k x\sin\gamma_n y dxdy\quad (k,n=1,2,...),$$

$$\psi_{k,n}=4\int_0^1\int_0^1 \psi(x,y)\cos\lambda_k x\sin\gamma_n y dxdy\quad (k,n=1,2,...).$$

Solving the problem (13), (14), gives

$$u_{k,n}(t)=\varphi_{k,n}\cos\mu_{k,n}t+\frac{1}{\mu_{k,n}}\psi_{k,n}sin\mu_{k,n}t$$

$$+\frac{1}{\mu_{k,n}}\int_0^t f_{k,n}(\tau)\sin\mu_{k,n}(t-\tau)d\tau\quad (k,n=1,2,...,0\le t\le T). \tag{15}$$

Substituting the expressions of $u_{k,n}(t)$ $(k,n=1,2,\ldots)$ into (12) yields

$$u(x,y,t)=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left\{\varphi_{k,n}\cos\mu_{k,n}t+\frac{1}{\mu_{k,n}}\psi_{k,n}\sin\mu_{k,n}t\right.$$

$$\left.+\frac{1}{\mu_{k,n}}\int_0^t F_k(\tau;u,a,b)\sin\mu_{k,n}(t-\tau)d\tau\right\}\cos\lambda_k x\sin\gamma_n y. \tag{16}$$

The following theorem is valid.

**Theorem 2.** *Let the data of problem* (1)–(4) *satisfy the following conditions:*

*$D_1$)* $\varphi(x,y), \varphi_x(x,y), \varphi_{xx}(x,y), \varphi_y(x,y), \varphi_{xy}(x,y), \varphi_{yy}(x,y) \in C(\bar{Q}_{xy})$,
$\varphi_{xxy}(x,y), \varphi_{xyy}(x,y), \varphi_{xxx}(x,y), \varphi_{yyy}(x,y) \in L_2(Q_{xy})$,
$\varphi_x(0,y) = \varphi(1,y) = \varphi_{xx}(1,y) = 0 \quad (0 \le y \le 1)$,
$\varphi(x,0) = \varphi_y(x,1) = \varphi_{yy}(x,0) = 0 \quad (0 \le x \le 1)$;

*$D_2$)* $\psi(x,y), \psi_x(x,y), \psi_y(x,y) \in C(\bar{Q}_{xy})$,
$\psi_{xx}(x,y), \psi_{yy}(x,y) \in L_2(Q_{xy})$,
$\psi_x(0,y) = \psi(1,y) = 0 \quad (0 \le y \le 1)$,
$\psi(x,0) = \psi_y(x,1) = 0 \quad (0 \le x \le 1)$;

*$D_3$)* $f(x,y,t) \in C(D_T), f_x(x,y,t), f_y(x,y,t) \in L_2(D_T)$,
$f_x(0,y,t) = f(1,y,t) = 0 \quad (0 \le y \le 1, 0 \le t \le T)$,
$f(x,0,t) = f_y(x,1,t) = 0 \quad (0 \le x \le 1, 0 \le t \le T)$,

*then the function*

$$u(x,y,t) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left\{\varphi_{k,n}\cos\mu_{k,n}t + \frac{1}{\mu_{k,n}}\psi_{k,n}\sin\mu_{k,n}t\right.$$

$$\left. + \frac{1}{\mu_{k,n}}\int_0^t F_k(\tau;u,a,b)\sin\mu_{k,n}(t-\tau)d\tau\right\}\cos\lambda_k x\sin\gamma_n y,$$

*is a classical solution to problem* (1)–(4).

*Proof.* It is easy to see that

$$\mu_{k,n}^3 \le (\lambda_k^2 + \gamma_k^2)(\lambda_k + \gamma_n) = \lambda_k^3 + \lambda_k^2\gamma_n + \gamma_k^2\lambda_k + \gamma_k^3.$$

By considering the foregoing relations, we have

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^3\|u_{k,n}(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}} \le 2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^3|\varphi_{k,n}|)^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\gamma_n|\varphi_{k,n}|)^2\right)^{\frac{1}{2}} + 2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\gamma_n^2|\varphi_{k,n}|)^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^3|\varphi_{k,n}|)^2\right)^{\frac{1}{2}} + 2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2|\psi_{k,n}|)^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2|\psi_{k,n}|)^2\right)^{\frac{1}{2}} + 2\sqrt{2T}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2|f_{k,n}(\tau)|)^2d\tau\right)^{\frac{1}{2}}\right.$$

$$\left.+2\sqrt{2T}\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2|f_{k,n}(\tau)|)^2d\tau\right)^{\frac{1}{2}}\right).$$

From this inequality, we get

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^3\|u_{k,n}(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}} \le 2\sqrt{2}\|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})}$$

$$+2\sqrt{2}\|\varphi_{xyy}(x,y)\|_{L_2(Q_{xy})} + 2\sqrt{12}\|\varphi_{xxy}(x,y)\|_{L_2(Q_{xy})}$$

$$+2\sqrt{2}\left\|\varphi_{yyy}(x,y)\right\|_{L_2(Q_{xy})}+2\sqrt{2}\left\|\psi_{yy}(x,y)\right\|_{L_2(Q_{xy})}+2\sqrt{2}\left\|\varphi_{yy}(x,y)\right\|_{L_2(Q_{xy})}$$

$$+2\sqrt{2T}(\|f_{xx}(x,y,t)\|_{L_2(D_T)}+\|f_{yy}(x,y,t)\|_{L_2(D_T)}). \quad (17)$$

Taking (17) into account, it is obvious that

$$|u_{xx}(x,y,t)| \le \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\lambda_k^2\,|u_{k,n}(t)|$$

$$\le \left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_{k,n}^{-2}\right\}^{\frac{1}{2}}\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^3\,\|u_{k,n}(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}} < \infty, \quad (18)$$

$$|u_{yy}(x,y,t)| \le \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\gamma_n^2\,|u_{k,n}(t)|$$

$$\le \left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_{k,n}^{-2}\right\}^{\frac{1}{2}}\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^3\,\|u_{k,n}(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}} < \infty. \quad (19)$$

From (18) and (19) it follows that the functions $u(x,t), u_{xx}(x,t)$, and $u_{yy}(x,t)$ are continuous in $D_T$.

Now, from (13) it is not hard to see that

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}\left\|u''_{k,n}(t)\right\|_{C[0,T]})^2\right\}^{\frac{1}{2}} \le \sqrt{2}\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^3\,\|u_{k,n}(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}}$$

$$+\sqrt{2}\left\|\|f_x(x,y,t)+f_y(x,y,t)\|_{C[0,T]}\right\|_{L_2(Q_{xy})}.$$

It follows that the function $u_{tt}(x,y,t)$ is continuous in $D_T$.

By direct verification one can see that function $u(x,y,t)$ satisfies equation (1) and conditions (2)–(4), in the usual sense. Thus, Theorem 2 is proved completely. ◄

## 5. Inverse Problem of Finding Initial Conditions

Based on the direct problem (1)–(4), we consider the following inverse problems to find the initial functions $\varphi(x,y)$ and $\psi(x,y)$. We are primarily interested in the functions $u(x,y,t), \varphi(x,y)$ and $\psi(x,y)$ satisfying conditions (1)–(4), and the conditions

$$u(x,y,T)=h(x,y),\ u_t(x,y,T)=g(x,y) \quad (0\le x\le 1, 0\le y\le 1), \quad (20)$$

where $h(x,y)$ and $g(x,y)$ are sufficiently smooth function.

From (15) it is obvious that

$$u'_{k,n}(t)=-\mu_{k,n}\varphi_{k,n}\sin\mu_{k,n}t+\psi_{k,n}\cos\mu_{k,n}t$$

$$+\int_0^t f_{k,n}(\tau)\cos\mu_{k,n}(t-\tau)d\tau \quad (k,n=1,2,...;0\le t\le T).$$

Now, from (20), taking into account (16), we obtain

$$\varphi_{k,n}\cos\mu_{k,n}T+\frac{1}{\mu_{k,n}}\psi_{k,n}sin\mu_{k,n}T$$

$$+\frac{1}{\mu_{k,n}}\int\limits_0^T f_{k,n}(\tau)\sin\mu_{k,n}(T-\tau)d\tau = h_{k,n} \quad (k,n=1,2,...), \tag{21}$$

$$-\mu_{k,n}\varphi_{k,n}\sin\mu_{k,n}T+\psi_{k,n}\cos\mu_{k,n}T$$

$$+\int\limits_0^T f_{k,n}(\tau)\cos\mu_{k,n}(T-\tau)d\tau = g_{k,n} \quad (k,n=1,2,...), \tag{22}$$

where

$$h_{k,n}=4\int\limits_0^1\int\limits_0^1 h(x,y)\cos\lambda_k x\sin\gamma_n y dxdy \quad (k,n=1,2,...),$$

$$g_{k,n}=4\int\limits_0^1\int\limits_0^1 g(x,y)\cos\lambda_k x\sin\gamma_n y dxdy \quad (k,n=1,2,...).$$

Multiplying both sides of the equations (21) and (22) by $\mu_{k,n}\cos\mu_{k,n}T$ and $\sin\mu_{k,n}T$, gives, in turn,

$$\mu_{k,n}\varphi_{k,n}\cos^2\mu_{k,n}T+\psi_{k,n}sin\mu_{k,n}T\cos\mu_{k,n}T$$

$$+\cos\mu_{k,n}T\int\limits_0^T f_{k,n}(\tau)\sin\mu_{k,n}(T-\tau)d\tau = h_{k,n}\mu_{k,n}\cos\mu_{k,n}T \quad (k,n=1,2,...)$$

and

$$-\mu_{k,n}\varphi_{k,n}\sin^2\mu_{k,n}T+\psi_{k,n}\cos\mu_{k,n}T\sin\mu_{k,n}T$$

$$+\sin\mu_{k,n}T\int\limits_0^T f_{k,n}(\tau)\cos\mu_{k,n}(T-\tau)d\tau = g_{k,n}\sin\mu_{k,n}T \quad (k,n=1,2,...).$$

Subtracting the second result from the first yields

$$\varphi_{k,n}=h_{k,n}\cos\mu_{k,n}T-g_{k,n}\frac{1}{\mu_{k,n}}\sin\mu_{k,n}T$$

$$+\frac{1}{\mu_{k,n}}\int\limits_0^T f_{k,n}(\tau)\sin\mu_{k,n}\tau d\tau \quad (k,n=1,2,...). \tag{23}$$

Multiplying both sides of the equations (21) and (22) by $\mu_{k,n}\sin\mu_{k,n}T$ and $\cos\mu_{k,n}T$, respectively, we will have

$$\mu_{k,n}\varphi_{k,n}\sin\mu_{k,n}T\cos\mu_{k,n}T+\psi_{k,n}sin^2\mu_{k,n}T$$

$$+\sin\mu_{k,n}T\int\limits_0^T f_{k,n}(\tau)\sin\mu_{k,n}(T-\tau)d\tau = \mu_{k,n}h_{k,n}\sin\mu_{k,n}T \quad (k,n=1,2,...),$$

$$-\mu_{k,n}\varphi_{k,n}\cos\mu_{k,n}T\sin\mu_{k,n}T+\psi_{k,n}\cos^2\mu_{k,n}T$$

$$+\cos\mu_{k,n}T\int\limits_0^T f_{k,n}(\tau)\cos\mu_{k,n}(T-\tau)d\tau = g_{k,n}\cos\mu_{k,n}T \quad (k,n=1,2,...).$$

Adding the last two relations term by term, it follows that

$$\psi_{k,n} = \mu_{k,n} h_{k,n} \sin \mu_{k,n} T + g_{k,n} \cos \mu_{k,n} T - \int_0^T f_{k,n}(\tau) \cos \mu_{k,n} \tau d\tau \quad (k,n = 1,2,...). \quad (24)$$

After substituting the expressions of $\varphi_{k,n}, \psi_{k,n}$ $(k,n = 1,2,...)$ into (15), we obtain

$$u_{k,n}(t) = h_{k,n} \cos \mu_{k,n}(T-t) - \frac{1}{\mu_{k,n}} g_{k,n} \sin \mu_{k,n}(T-t)$$

$$-\frac{1}{\mu_{k,n}} \int_0^T f_k(\tau) \sin \mu_{k,n}(t-\tau) d\tau + \frac{1}{\mu_{k,n}} \int_0^t f_k(\tau) \sin \mu_{k,n}(t-\tau) d\tau. \quad (25)$$

**Theorem 3.** *We impose the following restrictions on the data of problem* (1)–(4), (20)*:*

$I_1$) $h(x,y), h_x(x,y), h_{xx}(x,y), h_y(x,y), h_{xy}(x,y), h_{yy}(x,y) \in C(\bar{Q}_{xy})$,
$h_{xxy}(x,y), h_{xyy}(x,y), h_{xxx}(x,y), h_{yyy}(x,y) \in L_2(Q_{xy})$,
$h_x(0,y) = h(1,y) = h_{xx}(1,y) = 0 \quad (0 \le y \le 1)$,
$h(x,0) = h_y(x,1) = h_{yy}(x,0) = 0 \quad (0 \le x \le 1)$;

$I_2$) $g(x,y), g_x(x,y), g_y(x,y) \in C(\bar{Q}_{xy})$,
$g_{xx}(x,y), g_{yy}(x,y) \in L_2(Q_{xy})$,
$g_x(0,y) = g(1,y) = 0 \quad (0 \le y \le 1)$,
$g(x,0) = g_y(x,1) = 0 \quad (0 \le x \le 1)$;

$I_3$) $f(x,y,t) \in C(D_T), f_x(x,y,t), f_y(x,y,t) \in L_2(D_T)$,
$f_x(0,y,t) = f(1,y,t) = 0 \quad (0 \le y \le 1, 0 \le t \le T)$,
$f(x,0,t) = f_y(x,1,t) = 0 \quad (0 \le x \le 1, 0 \le t \le T)$.

*Then problem* (1)–(4), (20) *has a unique classical solution, and it is determined by the series,*

$$u(x,y,t) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left\{ h_{k,n} \cos \mu_{k,n}(T-t) - \frac{1}{\mu_{k,n}} g_{k,n} \sin \mu_{k,n}(T-t) \right.$$

$$\left. -\frac{1}{\mu_{k,n}} \int_0^T f_k(\tau) \sin \mu_{k,n}(t-\tau) d\tau + \frac{1}{\mu_{k,n}} \int_0^t f_k(\tau) \sin \mu_{k,n}(t-\tau) d\tau \right\} \cos \lambda_k x \sin \gamma_n y,$$

$$\varphi(x,y) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left\{ h_{k,n} \cos \mu_{k,n}(T-t) - \frac{1}{\mu_{k,n}} g_{k,n} \sin \mu_{k,n}(T-t) \right.$$

$$\left. + \frac{1}{\mu_{k,n}} \int_0^T f_{k,n}(\tau) \sin \mu_{k,n} \tau d\tau \right\} \cos \lambda_k x \sin \gamma_n y,$$

$$\psi(x,y) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \mu_{k,n} h_{k,n} \sin \mu_{k,n} T + g_{k,n} \cos \mu_{k,n} T \right.$$

$$\left. + \int_0^T f_{k,n}(\tau) \cos \mu_{k,n} \tau d\tau \right\} \cos \lambda_k x \sin \gamma_n y.$$

*Proof.* **Proof of uniqueness.** Let $\{u_1(x,y,t),\varphi_1(x,y),\psi_1(x,y)\}$ and $\{u_2(x,y,t),\varphi_2(x,y),\psi_2(x,y)\}$ be two solutions of the problem (1)-(4), (20). We denote by $v(x,y,t) = u_2(x,y,t) - u_1(x,y,t)$, $\bar\varphi(x,y) = \varphi_2(x,y) - \varphi_1(x,y)$, and $\bar\psi(x,y) = \psi_2(x,y) - \psi_1(x,y)$ the differences of these solutions. Then functions $v(x,y,t),\bar\varphi(x,y)$ and $\bar\psi(x,y)$ satisfy the equation

$$v_{tt}(x,y,t) - \Delta v(x,y,t) = 0,$$

with the conditions

$$v(x,y,0) = \bar\varphi(x,y), v_t(x,y,0) = \bar\psi(x,y) \quad (0 \le x \le 1, 0 \le y \le 1),$$

$$v_x(0,y,t) = v(1,y,t) = 0 \quad (0 \le y \le 1, 0 \le t \le T),$$

$$v(x,0,t) = v_y(x,1,t) = 0 \quad (0 \le x \le 1, 0 \le t \le T),$$

$$u(x,y,T) = 0, \quad u_t(x,y,T) = 0 \quad (0 \le x \le 1, 0 \le y \le 1).$$

Then, taking into account (20), (23), (24), we find

$$v_{k,n}(t) = 4\int_0^1\int_0^1 v(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy = 0 \quad (k,n = 1,2,...),$$

$$\bar\varphi_{k,n} = 4\int_0^1\int_0^1 \bar\varphi(x,y)\cos\lambda_k x\sin\gamma_n y dxdy = 0 \quad (k,n = 1,2,...),$$

$$\bar\psi_{k,n} = 4\int_0^1\int_0^1 \bar\psi(x,y)\cos\lambda_k x\sin\gamma_n y dxdy = 0 \quad (k,n = 1,2,...).$$

As a result, we found that for any fixed $t \in [0,T]$ the functions $v(x,y,t),\bar\varphi(x,y)$, and $\bar\psi(x,y)$ are orthogonal to the system of functions $\{\cos\lambda_k x\sin\gamma_n y\}$ $(k,n = 1,2,...)$, which is complete in $L_2(Q_{xy})$. This proves that $v(x,y,t) = 0$, $\bar\varphi(x,y) = 0$, and $\bar\psi(x,y) = 0$. Thus, if there are two solutions $u_1(x,y,t),\varphi_1(x,y),\psi_1(x,y)$ and $u_2(x,y,t)$, $\varphi_2(x,y),\psi_2(x,y)$ to problem (1)–(4), (20), then $u_1(x,y,t) \equiv u_2(x,y,t)$, $\varphi_1(x,y) \equiv \varphi_2(x,y)$, and $\psi_1(x,y) \equiv \psi_2(x,y)$. It follows that if a solution to problem (1)–(4), (20) exists, then it is unique. So, the uniqueness of the solution to problem (1)–(4), (20) is proved.

**Proof of existence.** From (25) it is easy to see that

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\left\|u_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac12} \le 2\sqrt2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^3\left|h_{k,n}\right|\right)^2\right)^{\frac12}$$

$$+2\sqrt2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\gamma_n\left|h_{k,n}\right|\right)^2\right)^{\frac12} + 2\sqrt2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\gamma_n^2\left|h_{k,n}\right|\right)^2\right)^{\frac12}$$

$$+2\sqrt2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^3\left|h_{k,n}\right|\right)^2\right)^{\frac12} + 2\sqrt2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\left|g_{k,n}\right|\right)^2\right)^{\frac12}$$

$$+2\sqrt2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^2\left|g_{k,n}\right|\right)^2\right)^{\frac12} + 2\sqrt{2T}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\left|f_{k,n}(\tau)\right|\right)^2 d\tau\right)^{\frac12}\right.$$

$$+2\sqrt{2T}\left(\int\limits_0^T \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2\left|f_{k,n}(\tau)\right|)^2 d\tau\right)^{\frac{1}{2}}\Bigg).$$

From the last inequality we get

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^3\left\|u_{k,n}(t)\right\|_{C[0,T]})^2\right\}^{\frac{1}{2}}$$

$$\leq 2\sqrt{2}\left\|h_{xxx}(x,y)\right\|_{L_2(Q_{xy})}+2\sqrt{2}\left\|h_{xyy}(x,y)\right\|_{L_2(Q_{xy})}+2\sqrt{12}\left\|h_{xxy}(x,y)\right\|_{L_2(Q_{xy})}$$

$$+2\sqrt{2}\left\|h_{yyy}(x,y)\right\|_{L_2(Q_{xy})}+2\sqrt{2}\left\|g_{yy}(x,y)\right\|_{L_2(Q_{xy})}+2\sqrt{2}\left\|g_{yy}(x,y)\right\|_{L_2(Q_{xy})}$$

$$+2\sqrt{2T}(\left\|f_{xx}(x,y,t)\right\|_{L_2(D_T)}+\left\|f_{yy}(x,y,t)\right\|_{L_2(D_T)}). \tag{26}$$

From (18) and (19), by virtue of (26), we conclude that functions $u(x,t), u_{xx}(x,t)$ and $u_{yy}(x,t)$ are continuous in $D_T$.

Now from (23) and (24), respectively, we have

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^3\left|\varphi_{k,n}\right|)^2\right\}^{\frac{1}{2}}\leq 2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^3\left|h_{k,n}\right|)^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\gamma_n\left|h_{k,n}\right|)^2\right)^{\frac{1}{2}}+2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\gamma_n^2\left|h_{k,n}\right|)^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^3\left|h_{k,n}\right|)^2\right)^{\frac{1}{2}}+2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\left|g_{k,n}\right|)^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2\left|g_{k,n}\right|)^2\right)^{\frac{1}{2}}+2\sqrt{2T}\Bigg(\left(\int\limits_0^T \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\left|f_{k,n}(\tau)\right|)^2 d\tau\right)^{\frac{1}{2}}$$

$$+2\sqrt{2T}\left(\int\limits_0^T \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2\left|f_{k,n}(\tau)\right|)^2 d\tau\right)^{\frac{1}{2}}\Bigg),$$

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^2\left|\psi_{k,n}\right|)^2\right\}^{\frac{1}{2}}\leq 2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^3\left|h_{k,n}\right|)^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\gamma_n\left|h_{k,n}\right|)^2\right)^{\frac{1}{2}}+2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\gamma_n^2\left|h_{k,n}\right|)^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^3\left|h_{k,n}\right|)^2\right)^{\frac{1}{2}}+2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\left|g_{k,n}\right|)^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2\left|g_{k,n}\right|)^2\right)^{\frac{1}{2}}+2\sqrt{2T}\Bigg(\left(\int\limits_0^T \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\left|f_{k,n}(\tau)\right|)^2 d\tau\right)^{\frac{1}{2}}$$

$$+2\sqrt{2T}\left(\left(\int\limits_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right),$$

or

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^3\left|\varphi_{k,n}\right|)^2\right\}^{\frac{1}{2}}\leq 2\sqrt{2}\left\|h_{xxx}(x,y)\right\|_{L_2(Q_{xy})}$$
$$+2\sqrt{2}\left\|h_{xyy}(x,y)\right\|_{L_2(Q_{xy})}+2\sqrt{12}\left\|h_{xxy}(x,y)\right\|_{L_2(Q_{xy})}$$
$$+2\sqrt{2}\left\|h_{yyy}(x,y)\right\|_{L_2(Q_{xy})}+2\sqrt{2}\left\|g_{yy}(x,y)\right\|_{L_2(Q_{xy})}+2\sqrt{2}\left\|g_{yy}(x,y)\right\|_{L_2(Q_{xy})}$$
$$+2\sqrt{2T}(\left\|f_{xx}(x,y,t)\right\|_{L_2(D_T)}+\left\|f_{yy}(x,y,t)\right\|_{L_2(D_T)}), \tag{27}$$
$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^2\left|\psi_{k,n}\right|)^2\right\}^{\frac{1}{2}}\leq 2\sqrt{2}\left\|h_{xxx}(x,y)\right\|_{L_2(Q_{xy})}$$
$$+2\sqrt{2}\left\|h_{xyy}(x,y)\right\|_{L_2(Q_{xy})}+2\sqrt{12}\left\|h_{xxy}(x,y)\right\|_{L_2(Q_{xy})}$$
$$+2\sqrt{2}\left\|h_{yyy}(x,y)\right\|_{L_2(Q_{xy})}+2\sqrt{2}\left\|g_{yy}(x,y)\right\|_{L_2(Q_{xy})}+2\sqrt{2}\left\|g_{yy}(x,y)\right\|_{L_2(Q_{xy})}$$
$$+2\sqrt{2T}(\left\|f_{xx}(x,y,t)\right\|_{L_2(D_T)}+\left\|f_{yy}(x,y,t)\right\|_{L_2(D_T)}).$$

It's obvious that

$$\varphi(x,y)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\varphi_{k,n}\cos\lambda_k x\sin\gamma_n y,$$

$$\varphi_{xxx}(x,y)=-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\lambda_k^3\varphi_{k,n}\sin\lambda_k x\sin\gamma_n y,$$

$$\varphi_{yyy}(x,y)=-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\gamma_k^3\varphi_{k,n}\cos\lambda_k x\cos\gamma_n y,$$

$$\varphi_{xxy}(x,y)=-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\lambda_k^2\gamma_k\varphi_{k,n}\cos\lambda_k x\sin\gamma_n y,$$

$$\varphi_{xyy}(x,y)=-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\lambda_k\gamma_k^2\varphi_{k,n}\cos\lambda_k x\sin\gamma_n y.$$

It is not hard to see that

$$\left\|\varphi_{xxx}(x,y)\right\|_{L_2(Q_{xy})}\leq\frac{1}{4}\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_k^3\left|\varphi_{k,n}\right|)^2\right\}^{\frac{1}{2}},$$

$$\left\|\varphi_{yyy}(x,y)\right\|_{L_2(Q_{xy})}\leq\frac{1}{4}\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_k^3\left|\varphi_{k,n}\right|)^2\right\}^{\frac{1}{2}},$$

$$\left\|\varphi_{xxy}(x,y)\right\|_{L_2(Q_{xy})}\leq\frac{1}{4}\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_k^3\left|\varphi_{k,n}\right|)^2\right\}^{\frac{1}{2}},$$

$$\left\|\varphi_{xyy}(x,y)\right\|_{L_2(Q_{xy})}\leq\frac{1}{4}\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_k^3\left|\varphi_{k,n}\right|)^2\right\}^{\frac{1}{2}}.$$

Thus, taking into account (27), we obtain

$$\varphi(x,y), \varphi_x(x,y), \varphi_{xx}(x,y), \varphi_y(x,y), \varphi_{xy}(x,y), \varphi_{yy}(x,y) \in C(\bar{Q}_{xy}),$$

$$\varphi_{xxy}(x,y), \varphi_{xyy}(x,y), \varphi_{xxx}(x,y), \varphi_{yyy}(x,y) \in L_2(Q_{xy}),$$

$$\varphi_x(0,y) = \varphi(1,y) = \varphi_{xx}(1,y) = 0 \quad (0 \le y \le 1),$$

$$\varphi(x,0) = \varphi_y(x,1) = \varphi_{yy}(x,0) = 0 \quad (0 \le x \le 1).$$

Similarly, it can be shown that

$$\psi(x,y), \psi_x(x,y), \psi_y(x,y) \in C(\bar{Q}_{xy}),$$

$$\psi_{xx}(x,y), \psi_{yy}(x,y) \in L_2(Q_{xy}),$$

$$\psi_x(0,y) = \psi(1,y) = 0 \quad (0 \le y \le 1),$$

$$\psi(x,0) = \psi_y(x,1) = 0 \quad (0 \le x \le 1),$$

$$\psi_{xx}(x,y), \psi_{yy}(x,y) \in L_2(Q_{xy}).$$

It follows that the functions $\varphi(x,y)$ and $\psi(x,y)$ satisfy the conditions of Theorem 2.

It is easy to check that equation (1) and conditions (2)–(4), (20) are satisfied in the usual sense. The theorem is proved. ◄

## 6. Concluding Remark

This paper investigates an initial-boundary value problem for the two-dimensional wave equation in a rectangular domain subject to mixed Dirichlet and Neumann boundary conditions. The objective is to recover the initial data, including the wave displacement and velocity, from the final position and final velocity. By introducing the change of variables $k = x$, $\delta = y$, and $\tau = T - t$, the inverse problem can be reformulated as a direct problem, where the known final position and final velocity of the original system serve as the initial data for the transformed problem. Consequently, the inverse problem considered in this work plays a primarily formal role. Nevertheless, this limitation can be overcome by allowing the observation point to vary within the spatial domain, a direction that will be addressed in future work.

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