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ASYMPTOTIC BEHAVIOUR OF THE SKEWNESS COEFFICIENT AND EXCESS KURTOSIS OF THE RENEWAL-REWARD PROCESS WITH DEPENDENT COMPONENTS

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Abstract. *This paper investigates the skewness and excess kurtosis of renewal-reward process with dependent components, expanding upon classical renewal theory by incorporating dependency structures among the components. We goal to derive expressions for the skewness coefficient and excess kurtosis and explore how dependencies influence the behavior of the renewal-reward process.*

Keywords: renewal-reward process, high-order central moments, standard deviation, skewness coefficients, excess kurtosis

Mathematics Subject Classification (2020): 60H30, 60G50, 60K05

1. Introduction

Let us random vectors (ξ_n, η_n), $n \geq 1$ be independent and identically distributed. In the general case, the random variable η_n is assumed to depend on the random variable ξ_n. Let us random variables ξ_n takes only positive values and denote the distribution function of ξ_n by F: $F(x) = P\{\xi_n \leq x\}$.

Let us introduce the following sum:

$$S_{\nu(t)} = \sum_{n=1}^{\nu(t)} \eta_n, \tag{1}$$

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where $\nu(t) = \max\{n : T_n \le t\}$, $t > 0$ is the renewal process and $T_n = \sum\limits_{i=1}^{n} \xi_i$, $n = 1, 2, \ldots$. The process $S_{\nu(t)}$, $t \ge 0$ is called as renewal-reward process and represents the sum of the rewards obtained up to time t ([3]-[5]). It is easy to see that, the renewal-reward process is a generalization of the renewal process. So, in the special case if $\eta_n \equiv 1$, $n \ge 1$ we obtain that $S_{\nu(t)} \equiv \nu(t)$.

Define

$$H(t) = E(v(t)),$$

$$R(t) = H(t) - \frac{1}{\mu_1} t - \frac{\mu_2}{2\mu_1^2} + 1,$$

$$D_n(t) = E\left(S_{\nu(t)}\right)^n = E\left(\sum_{k=1}^{\nu(t)} \eta_k\right)^n, \quad n \ge 1,$$

$$M_n(t) = E\left(\sum_{k=1}^{\nu(t)} \eta_k^n\right), \quad n \ge 1,$$

$$L_n(t) = M_n(t) - \frac{\lambda_n}{\mu_1} t - \frac{\lambda_n \mu_2}{2\mu_1^2} - \frac{n_{1,n}}{\mu_1}, \quad n \ge 1,$$

where

$$\mu_k = E\xi_1^k, \quad \lambda_k = E\eta_1^k = \int_0^\infty E\left(\eta_1^k | \xi_1 = t\right) dF(t), \quad k \ge 1,$$

$$n_{k,s} = E\left(\xi_1^k \eta_1^s\right) = \int_0^\infty x^k E\left(\eta_1^s | \xi_1 = x\right) dF(x), \quad k \ge 1, \quad s \ge 1.$$

Let

$$r_k = \int_0^\infty t^k R(t) dt, \ k \ge 0,$$

where $\int_0^\infty t^k |R(t)| \, dt < \infty$ and

$$l_{k,s} = \int_0^\infty t^s L_k(t) dt, \ k \ge 1, \ s \ge 0,$$

where $\int_0^\infty t^s |L_k(t)| \, dt < \infty$.

Definition. *([1]) A distribution function F is said to belong to the class ϑ if some convolution of F has an absolutely continuous component.*

In [4] there were obtained expressions for r_0 and $l_{1,0}$:
- if $F \in \vartheta$ and $\mu_3 < \infty$, then

$$r_0 = \frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2};$$

- if $F \in \vartheta$, μ_3, λ_1 and $n_{2,1}$ exist, then

$$l_{1,0} = \frac{\lambda_1 \mu_1^2}{4\mu_1^3} - \frac{\lambda_1 \mu_3}{6\mu_1^2} + \frac{n_{2,1}}{2\mu_1} - \frac{\mu_2 n_{1,1}}{2\mu_1^2}.$$

In [2] there were obtained expressions for r_1, r_2, $l_{1,1}$ and $l_{1,2}$.

Lemma 1. *([2]) If $F \in \vartheta$, μ_5 exist and finite, then*

$$r_1 = -\frac{\mu_2^3}{8\mu_1^4} + \frac{\mu_2\mu_3}{6\mu_1^3} - \frac{\mu_4}{24\mu_1^2},$$

$$r_2 = \frac{\mu_2^4}{8\mu_1^5} - \frac{\mu_3\mu_2^2}{4\mu_1^4} + \frac{\mu_3^2}{18\mu_1^3} + \frac{\mu_2\mu_4}{12\mu_1^3} - \frac{\mu_5}{60\mu_1^2}.$$

Lemma 2. *([2]) If $F \in \vartheta$, μ_5, λ_2 and $n_{4,2}$ are exist, then*

$$l_{1,1} = \lambda_1 r_1 + n_{1,1} r_0 + \frac{n_{3,1}}{6\mu_1} - \frac{\mu_2 n_{2,1}}{4\mu_1^2},$$

$$l_{1,2} = \lambda_1 r_2 + 2n_{1,1} r_1 + n_{2,1} r_0 + \frac{n_{4,1}}{12\mu_1} - \frac{\mu_2 n_{3,1}}{6\mu_1^2},$$

$$l_{2,2} = \lambda_2 r_2 + 2n_{1,2} r_1 + n_{2,2} r_0 + \frac{n_{4,2}}{12\mu_1} - \frac{\mu_2 n_{3,2}}{6\mu_1^2}.$$

2. Main Results

Our main purpose is to obtain an asymptotic expansions for the skewness and excess kurtosis of the process (1):

$$\gamma_3(t) = \frac{\alpha_3(t)}{\sigma_S^3(t)}, \quad \gamma_4(t) = \frac{\alpha_4(t)}{\sigma_S^4(t)},$$

where

$$\alpha_n(t) = E\left(S_{\nu(t)} - E(S_{\nu(t)})\right)^n, \quad n = 3, 4,$$

$$\sigma_s^n(t) = \left(E\left(S_{\nu(t)} - E(S_{\nu(t)})\right)^2\right)^{\frac{n}{2}}, \quad n = 3, 4.$$

Theorem 1. *Let us random vectors (ξ_n, η_n), $n \geq 1$ be independent and identically distributed. Assume that, also $F \in \vartheta$ and μ_5, λ_3, $n_{4,3}$ are exist and finite. Then as $t \to \infty$:*

$$\alpha_3(t) = A_{3,2} t^2 + A_{3,1} t + o(t), \tag{2}$$

$$\alpha_4(t) = A_{4,3} t^3 + A_{4,2} t^2 + o\left(t^2\right), \tag{3}$$

where coefficients $A_{i,j}$ are expressed by the moments of λ_s, μ_k, $n_{k,s}$ as follows:

$$A_{3,1} = 6a_1 b_2 + 6a_2 b_1 + 6a_1 b_1^2 - 6a_1 b_1 + 6a_1^2 l_{1,0} + a_3,$$

$$A_{3,2} = 3a_1a_2 - 3a_1^2,$$

$$A_{4,2} = 12a_1^2b_1^2 + 72a_1^2b_1 - 24a_1^2b_2 - 36a_1a_2b_1 + 3a_2^2 - 24a_1^3l_{1,0} + 24a_1l_{1,0}\ ,$$

$$A_{4,3} = 12a_1^3 - 12a_1^2a_2,$$

$$a_k = \frac{\lambda_k}{\mu_1},\quad b_k = \frac{\lambda_k\mu_2}{2\mu_1^2} - \frac{n_{1,k}}{\mu_1}.$$

Proof. Using by definitions of third central moment, we have:

$$\alpha_3(t) = E\left(S_{\nu(t)} - E\left(S_{\nu(t)}\right)\right)^3 = D_3(t) - 3D_2(t)M_1(t) + 2\left(M_1(t)\right)^3. \tag{4}$$

By the similar way:

$$\alpha_4(t) = E\left(S_{\nu(t)} - E\left(S_{\nu(t)}\right)\right)^4 = D_4(t) - 4D_3(t)M_1(t) + 6D_2(t)M_1^2(t) - 3M_1^4(t). \tag{5}$$

Our goal is to obtain asymptotic expansions for $\alpha_3(t)$ and $\alpha_4(t)$ as $t\to\infty$.
For this, we will use following asymptotic expansions as $t\to\infty$ [3]:

$$M_k(t) = a_kt + b_k + o\left(t^{-3}\right) \tag{6}$$

and

$$D_2(t) = a_1^2t^2 + \left(a_1 + 4a_1b_1\right)t + 2b_1^2 + b_1 + 4a_1l_{1,0} + o\left(t^{-2}\right), \tag{7}$$

where

$$a_k = \frac{\lambda_k}{\mu_1},\quad b_k = \frac{\lambda_k\mu_2}{2\mu_1^2} - \frac{n_{1,k}}{\mu_1}.$$

We will also use asymptotic expansion as $t\to\infty$ for the 3$^{\text{rd}}$ moment of renewal-reward process [1]:

$$D_3(t) = D_{3,3}t^3 + D_{3,2}t^2 + D_{3,1}t + D_{3,0} + o(1), \tag{8}$$

where

$$D_{3,3} = 6a_1^{*(3)},$$

$$D_{3,2} = 6b_1^{*(3)} + 6a_{1*2},$$

$$D_{3,1} = 6c_1^{*(3)} + 6b_{1*2} + a_3,\ D_{3,0} = 6d_1^{*(3)} + 6c_{1*2} + b_3,$$

$$a_{i*j} = \frac{1}{2}a_ia_j,\ b_{i*j} = a_jb_i + a_ib_j,\ c_{i*j} = b_ib_j + a_jl_{i,0} + a_il_{j,0},$$

$$a_k^{*(2)} = \frac{1}{2}a_k^2,\ b_k^{*(2)} = 2a_kb_k,\ c_k^{*(2)} = b_k^2 + 2a_kl_{k,0},$$

$$a_1^{*(3)} = \frac{1}{3}a_1a_1^{*(2)},\ b_1^{*(3)} = \frac{1}{2}a_1b_1^{*(2)} + a_1^{*(2)}b_1,\ c_1^{*(3)} = a_1c_1^{*(2)} + b_1b_1^{*(2)} + 2a_1^{*(2)}l_{1,0},$$

$$d_1^{*(3)} = a_1l_1^{*(2)} + b_1c_1^{*(2)} + b_1^{*(2)}l_{1,0} - 2a_1^{*(2)}l_{1,1}.$$

Additionally we will use asymptotic expansion as $t\to\infty$ for the 4$^{\text{th}}$ moment of renewal-reward process [2]:

$$D_4(t) = D_{4,4}t^4 + D_{4,3}t^3 + D_{4,2}t^2 + D_{4,1}t + D_{4,0} + o(1), \tag{9}$$

where

$$D_{4,4} = 24a_1^{*(4)},$$

$$D_{4,3} = 24b_1^{*(4)} + 36a_1^{*(3)},$$

$$D_{4,2} = 24c_1^{*(4)} + 36b_1^{*(3)} + 6a_2^{*(2)} + 8a_{1*3},$$

$$D_{4,1} = 24d_1^{*(4)} + 36c_1^{*(3)} + 6b_2^{*(2)} + 8b_{1*3} + a_4,$$

$$D_{4,0} = 24e_1^{*(4)} + 36d_1^{*(3)} + 6c_2^{*(2)} + 8c_{1*3} + b_4,$$

$$a_1^{*(4)} = \frac{1}{4}a_1 a_1^{*(3)},\ b_1^{*(4)} = \frac{1}{3}a_1 b_1^{*(3)} + a_1^{*(3)} b_1,\ c_1^{*(4)} = \frac{1}{2}a_1 c_1^{*(3)} + b_1 b_1^{*(3)} + 3a_1^{*(3)} l_{1,0},$$

$$d_1^{*(4)} = a_1 d_1^{*(3)} + b_1 c_1^{*(3)} + 2b_1^{*(3)} l_{1,0} - 6a_1^{*(3)} l_{1,1},$$

$$e_1^{*(4)} = a_1 l_1^{*(3)} + d_1^{*(3)} b_1 + c_1^{*(3)} l_{1,0} - 2b_1^{*(3)} l_{1,1} + 3a_1^{*(3)} l_{1,2}.$$

In order to derive the asymptotic expansion $\alpha_3(t)$, we substitute relations (6)-(8) into expression (4). After some calculations, asymptotic expansion (2) can be obtained.

Similarly, substituting relations (6)-(9) into expression (5), we obtain the asymptotic expansion of $\alpha_4(t)$, leading to expansion (3).

Theorem 1 is proved. ◄

Proposition. *([3]) Let the conditions of Theorem 1 be satisfied. Then as $t \to \infty$*

$$\sigma_S^2(t) = Var\ \left(S_{\nu(t)}\right) = B_1 t + B_2 + o(t^{-2}), \tag{10}$$

where $B_1 = a_2 + 2a_1 b_1,\ B_2 = b_2 + 4a_1 l_{1,0} + b_1^2$ and $a_k = \frac{\lambda_k}{\mu_1}, b_k = \frac{\lambda_k \mu_2}{2\mu_1^2} - \frac{n_{1,k}}{\mu_1}$.

Next main result of present paper can be formulated in following theorem.

Theorem 2. *Let us the conditions of Theorem 1 be satisfied. Then the following asymptotic equivalences as $t \to \infty$ are true for the skewness coefficient and excess kurtosis of the process $S_{\nu(t)}$:*

$$\gamma_3(t) = \frac{A_{3,2}}{B_1^{\frac{3}{2}}} t^{\frac{1}{2}} + \left(\frac{A_{3,1}}{B_1^{\frac{3}{2}}} - \frac{3A_{3,2}B_2}{2B_1^{\frac{5}{2}}} \right) t^{-\frac{1}{2}} + o\left(t^{-\frac{1}{2}}\right),$$

$$\gamma_4(t) = \frac{A_{4,3}}{B_1^2} t + \frac{A_{4,2} - 2B_2 C_1}{B_1^2} + o(1).$$

Proof. By using asymptotic expansions (2) and (10) for $\gamma_3(t)$ as $t \to \infty$ we have:

$$\gamma_3(t) = \frac{\alpha_3(t)}{\sigma_S^3(t)} = \frac{A_{3,2}t^2 + A_{3,1}t + o(t)}{(B_1 t + B_2 + o(t^{-2}))^{\frac{3}{2}}} =$$

$$\frac{A_{3,2}t^2 + A_{3,1}t + o(1)}{B_1^{\frac{3}{2}} t^{\frac{3}{2}}} \cdot \frac{1}{\left(1 + \frac{B_2}{B_1} t^{-1} + o(t^{-3})\right)^{\frac{3}{2}}} =$$

$$\left(\frac{A_{3,2}}{B_1^{\frac{3}{2}}}t^{\frac{1}{2}}+\frac{A_{3,1}}{B_1^{\frac{3}{2}}}t^{-\frac{1}{2}}+o\left(t^{-\frac{1}{2}}\right)\right)\cdot\left(1+\frac{B_2}{B_1}t^{-1}+o\left(t^{-3}\right)\right)^{-\frac{3}{2}}=$$

$$\left(\frac{A_{3,2}}{B_1^{\frac{3}{2}}}t^{\frac{1}{2}}+\frac{A_{3,1}}{B_1^{\frac{3}{2}}}t^{-\frac{1}{2}}+o\left(t^{-\frac{1}{2}}\right)\right)\cdot\left(1-\frac{3B_2}{2B_1}t^{-1}+o\left(t^{-1}\right)\right)=$$

$$\frac{A_{3,2}}{B_1^{\frac{3}{2}}}t^{\frac{1}{2}}+\left(\frac{A_{3,1}}{B_1^{\frac{3}{2}}}-\frac{3A_{3,2}B_2}{2B_1^{\frac{5}{2}}}\right)t^{-\frac{1}{2}}+o\left(t^{-\frac{1}{2}}\right).$$

By using asymptotic expansions (3) and (10) for $\gamma_4(t)$ as $t\to\infty$ we have:

$$\gamma_4(t)=\frac{\alpha_4(t)}{\sigma_S^4(t)}=\frac{A_{4,3}t^3+A_{4,2}t^2+o(t^2)}{\left(B_1t+B_2+o(t^{-2})\right)^2}=$$

$$\frac{A_{4,3}t^3+A_{4,2}t^2+o(t^2)}{B_1^2t^2}\left(1+\frac{B_2}{B_1}t^{-1}+o(t^{-3})\right)^{-2}=$$

$$\left(\frac{A_{4,3}}{B_1^2}t+\frac{A_{4,2}}{B_1^2}+o(1)\right)\left(1-\frac{2B_2}{B_1}t^{-1}+o(t^{-1})\right)=$$

$$\frac{A_{4,3}}{B_1^2}t+\frac{A_{4,2}-2B_2A_{4,3}}{B_1^2}+o(1).$$

Theorem 2 is proved. ◀

3. Conclusions

In this study, we explored the asymptotic behaviour of the skewness coefficient and excess kurtosis of the renewal-reward process with dependent components. The obtained show that higher-order characteristics of the process, such as skewness and kurtosis, play an important role in describing the deviation of the distribution from normality. Therefore, the analysis of skewness and excess kurtosis not only contributes to the theoretical development of renewal theory, but also provides useful tools for practical applications, such as insurance mathematics, reliability theory and risk analysis in complex systems.

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