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f-HARMONIC VECTORS FIELDS WITH POTENTIAL

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Abstract. *The problem studied in this paper is related to the f-harmonicity with potential $H = f^v$ of a vector field from a Riemannian manifold (M, g) to its tangent bundle TM equipped with the Sasaki metric G^s. We show that an f-harmonic vector field with potential f^v is parallel if and only if f is constant function. We also characterize the f-harmonic vector fields with potential f^v on some three Riemannian Lie groups.*

Keywords: f-harmonic maps with potential, Riemannian manifold, vector fields

Mathematics Subject Classification (2020): 53C15, 53C25, 58E30

1. Introduction

f-harmonic maps between two Riemannian manifolds, which generalize harmonic maps, were first introduced by A. Lichnerowicz [6] in 1969. Moreover, F-harmonic maps between Riemannian manifolds were first introduced by M. Ara [1] in 1999, which could be considered as the special cases of f-harmonic maps.

The concept of harmonic maps with potential, was initially suggested by A. Ratto in [7] and recently developed by several authors: V. Branding [2], R. Jiang [4] and others.

The notion of f- harmonic map with potential was recently studied by Z. Kaddour [5]. A vector field X on a Riemannian manifold M is a section of the tangent bundle, and in particular it is a map of M into TM. In this paper,we studied the f-harmonicity with potential f^v of a vector field from a Riemannian manifold (M, g) to its tangent bundle TM equipped with the Sasaki metric G^s. We show that an f-harmonic vector field with potential f^v is parallel if and only if f is constant function. We also characterize the f-harmonic vector fields with potential f^v on some Three Riemannian Lie groups.

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2. Preliminaries

2.1. f-harmonic map with potential H

Consider a smooth map $\varphi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds, let H be a smooth function on N and let f be a smooth positive function on M. For any compact domain D of M the $H - f$-energy functional of φ is defined by

$$E_{H,f}(\varphi) = \int_D \big(fe(\varphi) - H(\varphi)\big)v_g,$$

where $e(\varphi)$ is the energy density of φ defined by $e(\varphi) = \dfrac{1}{2}\displaystyle\sum_{i=1}^{m} h(d\varphi(e_i), d\varphi(e_i))$, v_g is the volume element and $\{e_i\}_{i=1}^m$ an orthonormal frame on (M^m, g).

Definition 1. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a map between Riemannian manifolds of dimensions m and n. A map φ is called f-harmonic map with potential H if it is a critical point of the $H - f$-energy functional over any compact subset D of M, i/e*

$$\frac{d}{dt}E_{H,f}(\varphi_t)\bigg|_{t=0} = 0,$$

where $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D.

Theorem 1. *[5] Let $\varphi : (M^m, g) \to (N^n, h)$ be a map between Riemannian manifolds, H be a smooth function on N and let f be a smooth positive function on M, then*

$$\frac{d}{dt}E_{H,f}(\varphi_t)\bigg|_{t=0} = -\int_D h(\tau_{H,f}(\varphi), v)v_g,$$

such that:

$$\tau_{H,f}(\varphi) = \tau_f(\varphi) + (grad^N H) \circ \varphi,$$

where $\tau_f(\varphi) = f\tau(\varphi) + d\varphi(\mathrm{grad}^M f)$ is the f-tension field of φ.

$\tau(\varphi) = \mathrm{tr}_g \nabla d\varphi$ is the tension field of φ. $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D, $v = \dfrac{\partial\varphi_t}{\partial t}\bigg|_{t=0}$ denotes the variation vector field of φ.

Corollary. *A smooth map $\varphi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds is f-harmonic with potential H if and only if $\tau_{H,f}(\varphi) = 0$.*

2.2. Some result on the lift

Definition 2. *Let $f : M \longrightarrow \mathbb{R}$ be a smooth function. The vertical lift of f is the unique map f^v defined by:*

$$\begin{aligned} f^v :\ & TM \longrightarrow \mathbb{R}, \\ & (x,u) \longmapsto f^v(x,u) = f \circ \pi(x,u) = f(x), \end{aligned}$$

where $\pi : TM \longrightarrow M$ which denotes the natural projection.

Let (M, g) be an n-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=1.\cdots.n}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^j)$ $i = 1.\cdots.n$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g. We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by

$$\mathcal{V} = \left\{ a^i \frac{\partial}{\partial y^i}, a^i \in \mathbb{R} \right\}; \quad \mathcal{H} = \left\{ b^i \frac{\partial}{\partial \tilde{x}^i} - b^j u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}, b^i \in \mathbb{R} \right\}.$$

Definition 3. *Let $X \in \Gamma(TM)$. The vertical lift of X on M to TM is the unique vector field X^v satisfy*

$$X^v f^v = 0 \text{ for all } f \in C^\infty(M).$$

Definition 4. *Let $X \in \Gamma(TM)$. The vertical horizontal of X on M to TM is the unique vector field X^h satisfy*

$$d\pi \circ X^h = X \circ \pi.$$

Proposition 1. *[3] For all $X, Y \in \Gamma(TM), f : M \longrightarrow \mathbb{R}$ a smooth function*

1. $X^v f^v = 0,$
2. $(X + Y)^v = X^v + Y^v,$
3. $(fX)^v = f^v X^v,$
4. $(X + Y)^h = X^h + Y^h,$
5. $(fX)^h = f^v X^h,$
6. $(X^h f^v) = (Xf)^v.$

3. f-Harmonicity with Potential f^v of $X : (M, g) \longrightarrow (TM, G^s)$

Lemma 1. *Let (M, g) be a Riemannian manifold of dimension n, (TM, G^s) the tangent bundle endowed with the Sasaki metric G^s and $f : M \longrightarrow \mathbb{R}$ a smooth function, then*

$$\operatorname{grad}^{TM} f^v = \left(\operatorname{grad}^M(f) \right)^h.$$

Proof. We have

$$
\begin{aligned}
\operatorname{grad}^{TM}(f^v) &= \sum_{i,j}^{n} G^s\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)\frac{\delta f^v}{\delta x^i}\frac{\delta}{\delta x^j} + \sum_{i,j}^{n} G^s\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)\frac{\partial f^v}{\partial y^i}\frac{\partial}{\partial y^j} \\
&= \sum_{i,j}^{n} G^s\left(\left(\frac{\partial}{\partial x^i}\right)^h, \left(\frac{\partial}{\partial x^j}\right)^h\right)\left(\frac{\partial}{\partial x^i}\right)^h (f^v)\left(\frac{\partial}{\partial x^j}\right)^h + \\
&\quad \sum_{i,j}^{n} G^s\left(\left(\frac{\partial}{\partial x^i}\right)^v, \left(\frac{\partial}{\partial x^j}\right)^v\right)\left(\frac{\partial}{\partial x^i}\right)^v (f^v)\left(\frac{\partial}{\partial x^j}\right)^v \\
&= \sum_{i,j} g^{ij}\left(\frac{\partial f}{\partial x^i}\right)^v\left(\frac{\partial}{\partial x^j}\right)^h = \sum_{i,j} g^{ij}\left(\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}\right)^h \\
&= \left(\sum_{i,j} g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}\right)^h = \left(\operatorname{grad}^M(f)\right)^h.
\end{aligned}
$$

◀

Lemma 2. *Let (M^n, g) be a Riemannian manifold, X, Y two vectors fields on M, then for all $x \in M$*

$$d_x X(Y_x) = Y_x^h + (\nabla_Y X)_x^v.$$

Theorem 2. *Let (M, g) be a Riemannian manifold of dimension n, (TM, G^s) the tangent bundle endowed with the Sasaki metric G^s and $X : M \longrightarrow TM$ a vector field on M. Then X define a f-harmonic map with potential f^v if and only if*

$$
\begin{cases}
f\tau_h(X) + 2\operatorname{grad}^M f = 0, \\
f\tau_v(X) + \nabla_{\operatorname{grad}^M f} X = 0.
\end{cases}
$$

Proof. We have

$$
\begin{aligned}
\tau_{f^v,f}(X) &= f\tau(X) + dX(\operatorname{grad}^M f) + \operatorname{grad}^{TM} f^v \\
&= f\left(\tau_h(X)\right)^h + f\left(\tau_v(X)\right)^v + \left(\operatorname{grad}^M f\right)^h + \left(\nabla_{\operatorname{grad}^M f} X\right)^v \\
&\quad + \left(\operatorname{grad}^M f\right)^h = \left(f\tau_h(X)\right)^h + \left(f\tau_v(X)\right)^v \\
&+ \left(\operatorname{grad}^M f\right)^h + \left(\nabla_{\operatorname{grad}^M f} X\right)^v + \left(\operatorname{grad}^M f\right)^h \\
&= \left(f\tau_h(X) + 2\operatorname{grad}^M f\right)^h + \left(f\tau_v(X) + \nabla_{\operatorname{grad}^M f} X\right)^v
\end{aligned}
$$

with

$$\tau_h(X) = tr_g\Bigg(R(X, \nabla X)\Bigg) \quad \text{and} \quad \tau_v(X) = tr_g(\nabla^2 X). \qquad ◀$$

Remark 1. A parallel vector define an f-harmonic map with potential f^v on a Riemannian manifold M if and only if f is constant function on M.

Lemma 3. *A vector fields* $X = h^1(x,y,z)\dfrac{\partial}{\partial x} + h^2(x,y,z)\dfrac{\partial}{\partial y} + h^3(x,y,z)\dfrac{\partial}{\partial z}$ *on* $\mathbb{R}^3$ *is* f*-harmonic with potential* f^v *if and only if* f *is constant function and* h^1, h^2, h^3 *are harmonic finctions on* $\mathbb{R}^3$*.*

Proof. Let $X = h^1(x,y,z)\dfrac{\partial}{\partial x} + h^2(x,y,z)\dfrac{\partial}{\partial y} + h^3(x,y,z)\dfrac{\partial}{\partial z}$ be a vector field on $\mathbb{R}^3$ we have

$$\begin{aligned}
\operatorname{grad} f &= f_x(x,y,z)\frac{\partial}{\partial x} + f_y(x,y,z)\frac{\partial}{\partial y} + f_z(x,y,z)\frac{\partial}{\partial z}, \\
\nabla_{\operatorname{grad} f} X &= (f_x h^1_x + f_y h^1_y + f_z h^1_z)\frac{\partial}{\partial x} + (f_x h^2_x + f_y h^2_y + f_z h^2_z)\frac{\partial}{\partial y} \\
&\quad + (f_x h^3_x + f_y h^3_y + f_z h^3_z)\frac{\partial}{\partial z}, \\
\tau_v(X) &= \Delta h^1 \frac{\partial}{\partial x} + \Delta h^2 \frac{\partial}{\partial y} + \Delta h^3 \frac{\partial}{\partial z}, \\
\tau_h(X) &= 0.
\end{aligned}$$

Hence using , X is a harmonic if and only if $\begin{cases} f_x = f_y = f_z = 0, \\ f\Delta h^1 = 0, \\ f\Delta h^2 = 0, \\ f\Delta h^3 = 0, \end{cases}$

that is equivalently to $\begin{cases} f_x = f_y = f_z = 0, \\ \Delta h^1 = 0, \\ \Delta h^2 = 0, \\ \Delta h^3 = 0. \end{cases}$

◀

Proposition 2. *A vector fields* $X = h^1(x,y,z)e_1 + h^2(x,y,z)e_2 + h^3(x,y,z)e_3$ *on* Nil_3 *define an* f*-harmonic map with potential* f^v *if and only if* f *is constant function and*

$$\begin{cases} \Delta h^1 + h^3_y - \frac{1}{2}h^1 + h^2_z + x h^2_y = 0, \\ \delta h^2 + h^3_x - \frac{1}{2}h^2 - h^1_z - x h^1_y = 0, \\ \delta h^3 - h^2_x - \frac{1}{2}h^3 - h^1_y = 0, \end{cases}$$

where Δ *is a Laplacian on* $\mathbb{R}^3$*.*

Proof. Let $G = Nil_3$ be the Heisenberg group of real 3×3 upper-triangular matrices of the form

$$A = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

endowed with the metric given by $dx^2 + \big(dy - xdz\big)^2 + (dz)^2$. We may thus identify $\mathbb{H}^3$ with $\mathbb{R}^3$ endowed with this metric. The vector fields

$$e_1 = \frac{\partial}{\partial x},\ e_2 = \frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z} + x\frac{\partial}{\partial y},$$

constitute an orthonormal frame field of $(\mathbb{H}^3, g)$. The corresponding Levi-Civita connection is determined by

$$\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = -\frac{1}{2} e_3,$$

$$\nabla_{e_1} e_3 = -\nabla_{e_3} e_1 = \frac{1}{2} e_2,$$

$$\nabla_{e_2} e_3 = \nabla_{e_3} e_2 = \frac{1}{2} e_1,$$

where the remaining covariant derivatives vanish. We also have

$$R(e_2, e_1)e_3 = R(e_2, e_3)e_1 = R(e_3, e_1)e_2 = 0,$$

$$\operatorname{grad} f = f_x e_1 + f_y e_2 + (f_z - x f_y) e_3,$$

$$\nabla_{e_1} X = e_1(h^1)e_1 + \left(e_1(h^2) + \frac{1}{2}h^3\right)e_2 + \left(e_1(h^3) - \frac{1}{2}h^2\right)e_3,$$

$$\nabla_{e_2} X = \left(e_2(h^1) + \frac{1}{2}h^3\right)e_1 + e_2(h^2)e_2 + \left(e_2(h^3) - \frac{1}{2}h^1\right)e_3,$$

$$\nabla_{e_3} X = \left(e_3(h^1) + \frac{1}{2}h^2\right)e_1 + \left(e_3(h^2) - \frac{1}{2}h^1\right)e_2 + e_3(h^3)e_3,$$

$$\begin{aligned} \nabla_{\operatorname{grad} f} X &= e_1(f)\nabla_{e_1} X + e_2(f)\nabla_{e_2} X + e_3(f)\nabla_{e_3} X \\ &= f_x\left(h^1_x + h^1_y + h^1_z + x h^1_y + \frac{1}{2}h^3 + \frac{1}{2}h^2\right)e_1 \\ &\quad + f_y\left(h^2_x + h^2_y + h^2_z + x h^2_y - \frac{1}{2}h^1 + \frac{1}{2}h^3\right)e_2 \\ &\quad + (f_z + x f_y)\left(h^3_x + h^3_y + h^3_z + x h^3_y - \frac{1}{2}h^1 - \frac{1}{2}h^2\right)e_3, \end{aligned}$$

$$\tau_v(X) = \nabla_{e_1}\nabla_{e_1} X + \nabla_{e_2}\nabla_{e_2} X + \nabla_{e_3}\nabla_{e_3} X$$

$$= \nabla_{e_1}\Big(e_1(h^1)e_1 + (e_1(h^2) + \frac{1}{2}h^3)e_2 + (e_1(h^3) - \frac{1}{2}h^2)e_3\Big)$$

$$+\nabla_{e_2}\Big((e_2(h^1) + \frac{1}{2}h^3)e_1 + e_2(h^2)e_2 + (e_2(h^3) - \frac{1}{2}h^1)e_3\Big)$$

$$+\nabla_{e_3}\Big((e_3(h^1) + \frac{1}{2}h^2)e_1 + (e_3(h^2) - \frac{1}{2}h^1)e_2 + e_3(h^3)e_3\Big)$$

$$= e_1e_1(h^1)e_1 + (e_1(h^2) + \frac{1}{2}h^3)\nabla_{e_1}e_2 + \Big(e_1e_1(h^2) + \frac{1}{2}e_1(h^3)\Big)$$

$$+\Big(e_1e_1(h^3) - \frac{1}{2}e_1(h^2)\Big)e_3 + \Big(e_1(h^3) - \frac{1}{2}h^2\Big)\nabla_{e_1}e_3$$

$$+\Big(e_2(h^1) + \frac{1}{2}h^3\Big)\nabla_{e_2}e_1 + \Big(e_2e_2(h^1) + \frac{1}{2}e_2(h^3)\Big)e_1$$

$$+e_2(h^2)\nabla_{e_2}e_2 + e_2e_2(h^2)e_2 + \Big(e_2(h^3) - \frac{1}{2}h^1)\Big)\nabla_{e_2}e_3$$

$$+\Big(e_2e_2(h^3) - \frac{1}{2}e_2(h^1)\Big)e_3 + \Big(e_3(h^1) + \frac{1}{2}h^2\Big)\nabla_{e_3}e_1$$

$$+\Big(e_3e_3(h^1) + \frac{1}{2}e_3(h^2)\Big)e_1 + \big(e_3(h^2) - \frac{1}{2}h^1\Big)\nabla_{e_3}e_2$$

$$+\Big(e_3e_3(h^2) - \frac{1}{2}e_3(h^2)\Big)e_2 + e_3(h^3)\nabla_{e_3}e_3 + e_3e_3(h^3),$$

$$\tau_v(X) = \Big(\Delta h^1 + e_2(h^3) - \frac{1}{2}h^1 + e_3(h^2)\Big)e_1 + \Big(\delta h^2 + e_1(h^3) - \frac{1}{2}h^2 - e_3(h^1)\Big)e_2$$

$$+\Big(\delta h^3 - e_1(h^2) - \frac{1}{2}h^3 - e_2(h^1)\Big)e_3 = \Big(\Delta h^1 + h^3_y - \frac{1}{2}h^1 + h^2_z + xh^2_y\Big)e_1$$

$$+\Big(\delta h^2 + h^3_x - \frac{1}{2}h^2 - h^1_z - xh^1_y\Big)e_2 + \Big(\delta h^3 - h^2_x - \frac{1}{2}h^3 - h^1_y\Big)e_3,$$

$$\tau_h(X) = 0.$$

Hence a vector field X define an f-harmonic map potential f^v if and only if

$$\begin{cases} f_x = f_y = f_z = 0, \\ f\big(\Delta h^1 + h^3_y - \frac{1}{2}h^1 + h^2_z + xh^2_y\big) = 0, \\ f\big(\delta h^2 + h^3_x - \frac{1}{2}h^2 - h^1_z - xh^1_y\big) = 0, \\ f\big(\delta h^3 - h^2_x - \frac{1}{2}h^3 - h^1_y\big) = 0, \end{cases}$$

that is equivalently to

$$\begin{cases} f_x = f_y = f_z = 0, \\ \Delta h^1 + h^3_y - \frac{1}{2}h^1 + h^2_z + xh^2_y = 0, \\ \delta h^2 + h^3_x - \frac{1}{2}h^2 - h^1_z - xh^1_y = 0, \\ \delta h^3 - h^2_x - \frac{1}{2}h^3 - h^1_y = 0. \end{cases}$$

◀

Proposition 3. *A vector fields* $X = u(x,y,z)\dfrac{\partial}{\partial x} + v(x,y,z)\dfrac{\partial}{\partial y} + w(x,y,z)\dfrac{\partial}{\partial z}$ *on* H^3 *define an* f*-harmonic map potential* f^v *if and only if*

$$\begin{cases} 2zf_x + zc^2(uv_z - vu_z + wv_y - vw_y) + c^2uv = 0, \\ 2zf_y + zc^2(uw_z - vw_z + vw_x - wv_x) + c^2uv = 0, \\ 2zf_z + zc^2(vy_x - uv_x + wu_y - uv_y) + c^2(v^2 + w^2 + 2u^2) = 0, \\ c^2z^2(\Delta u + f_xu_x + f_yu_y + f_zu_z) + zc^2(2v_x + 2w_y - u_z) - 2uc^2 = 0, \\ c^2z^2(\Delta v + f_xv_x + f_yv_y + f_zv_z) - zc^2(uf_x + v_z + u_x) - c^2v = 0, \\ c^2z^2(\Delta w + f_xw_x + f_yw_y + f_zw_z) - zc^2(w_z + u_x + uf_y) - c^2w = 0, \end{cases}$$

where Δ *is a Laplacian on* $\mathbb{R}^3$.

Proof. The hyperbolic space (H^3, g) of constant sectional curvature $-c^2$

$$H^3 = \Big\{(x,y,z) \in \mathbb{R}^n, z > 0\Big\},$$

$$g = \frac{1}{(cy)^2}\Big((dx)^2 + (dy)^2 + (dz)^2\Big).$$

The vector fields $V = cz\dfrac{\partial}{\partial z}, e_1 = cz\dfrac{\partial}{\partial x}, e_2 = cz\dfrac{\partial}{\partial y}$ constitute an orthonormal frame field of (H^3, g) and

$$[V, e_i] = ce_i, \quad [e_i, e_j] = 0, \quad 1 \le i, j \le n-1.$$

Let ∇ be the Levi-Civita connection of (H^3, g) we have

$$\nabla_{e_i}V = -ce_i, \quad \nabla_V V = 0 = \nabla_V e_i, \quad \nabla_{e_i}e_j = c\delta_{ij}V \text{ for any } 1 \le i, j \le 2.$$

We also have

$$\begin{aligned} R(e_1,e_2)V = R(e_1,V)e_2 = R(e_2,V)e_1 &= 0, \\ R(e_1,V)e_1 = R(e_2,V)e_2 &= c^2V, \\ R(e_2,V)V = -R(e_1,e_2)e_1 &= -c^2e_2, \\ R(e_1,V)V = R(e_1,e_2)e_2 &= -c^2e_1. \end{aligned}$$

Let $X = u(x,y,z)V + v(x,y,z)e_1 + w(x,y,z)e_2$ be a vector field on H^3.

We have

$$\nabla_V X = V(u)V + V(v)e_1 + V(w)e_2,$$

$$R(X, \nabla_V X)V = \Big(uV(v) - vV(u)\Big)c^2 e_1 + \Big(uV(w) - wV(u)\Big)c^2 e_2,$$

$$\nabla_{e_1} X = (e_1(u) + cv)V + (e_1(v) - cu)e_1 + e_1(w)e_2,$$

$$R(X, \nabla_{e_1} X)e_1 = c^2\Big(ve_1(u) + v^2c + u^2c - ue_1(v)\Big)V + c^2\Big(ve_1(w) - we_1(v) + cvu\Big)e_2,$$

$$\nabla_{e_2} X = (e_2(u) + cw)V + e_2(v)e_1 + (e_2(w) - cu)e_2,$$

$$R(X, \nabla_{e_2} X)e_2 = c^2\Big(we_2(u) + w^2c + u^2c - ue_2(v)\Big)V + c^2\Big(we_2(v) - ve_2(w) + cvu\Big)e_1.$$

We obtain

$$\begin{aligned}\tau_h(X) = {} & c^2\Big(ve_1(u) + v^2c + u^2c - ue_1(v) + we_2(u) + w^2c + u^2c - ue_2(v)\Big)V \\ & + \Big(uV(v) - vV(u) + we_2(v) - ve_2(w) + cvu\Big)c^2 e_1 \\ & + c^2\Big(- we_1(v) + ve_1(w) + cvu + uV(w) - wV(u)\Big)e_2.\end{aligned}$$

Also

$$\begin{aligned}\tau_v(X) = {} & \Big(\nabla_{e_1}\nabla_{e_1}X + \nabla_{e_2}\nabla_{e_2}X + \nabla_{e_3}\nabla_{e_3}X\Big) \\ & - \Big(\nabla_{\nabla_{e_1}e_1}X + \nabla_{\nabla_{e_2}e_2}X + \nabla_{\nabla_{e_3}e_3}X\Big) \\ = {} & \Big(e_1e_1(u) + e_2e_2(u) + VV(u) - 2cV(u) + 2ce_1(v) + 2ce_2(w) - 2c^2u\Big)V \\ & + \Big(e_1e_1(v) + e_2e_2(v) + VV(v) - 2cV(v) - ce_1(u) - c^2v\Big)e_1 \\ & + \Big(e_1e_1(w) + e_2e_2(w) + VV(w) - 2cV(w) - ce_1(u) - c^2w\Big)e_2\end{aligned}$$

and

$$\begin{aligned}\nabla_{\operatorname{grad} f}X &= e_1(f)\nabla_{e_1}X + e_2(f)\nabla_{e_2}X + e_3(f)\nabla_{e_3}X \\ &= \Big(e_1(f)e_1(u) + e_2(f)e_2(u) + V(f)V(u)\Big)V \\ &\quad + \Big(e_1(f)e_1(v) + e_2(f)e_2(v) + V(f)V(v) - ce_1(f)u\Big)e_1 \\ &\quad + \Big(e_1(f)e_1(w) + e_2(f)e_2(w) + V(f)V(w) - ce_2(f)u\Big)e_2.\end{aligned}$$

Hence a vector field X define an f-harmonic map potential f^v if and only if

$$\begin{cases} 2e_1(f)+\Big(uV(v)-vV(u)+we_2(v)-ve_2(w)+cvu\Big)fc^2=0,\\ 2e_2(f)+fc^2\Big(-we_1(v)+ve_1(w)+cvu+uV(w)-wV(u)\Big)=0,\\ 2V(f)+fc^2\Big(ve_1(u)+v^2c+2u^2c-ue_1(v)+we_2(u)+w^2c-ue_2(v)\Big)=0,\\ e_1(f)e_1(u)+e_2(f)e_2(u)+V(f)V(u)+f\Big(e_1e_1(u)+e_2e_2(u)+VV(u)-2cV(u)\\ +2ce_1(v)+2ce_2(w)-2c^2u\Big)=0,\\ e_1(f)e_1(v)+e_2(f)e_2(v)+V(f)V(v)-ce_1(f)u+f\Big(e_1e_1(v)+e_2e_2(v)+VV(v)\\ -2cV(v)-ce_1(u)-c^2v\Big)=0,\\ e_1(f)e_1(w)+e_2(f)e_2(w)+V(f)V(w)-ce_2(f)u+f\Big(e_1e_1(w)+e_2e_2(w)+VV(w)\\ -2cV(w)-ce_1(u)-c^2w\Big)=0. \end{cases}$$

That is equivalently to

$$\begin{cases} 2zf_x+zfc^2(uv_z-vu_z+wv_y-vw_y)+fc^2uv=0,\\ 2zf_y+zfc^2(uw_z+vw_x-wv_x-wzu_z)+fc^2uv=0,\\ 2zf_z+zfc^2(vu_x-uv_x+wu_y-uv_y)+fc^2(v^2+w^2+2u^2)=0,\\ c^2z^2(f\mathit{\Delta}u+f_xu_x+f_yu_y+f_zu_z)+zfc^2(2v_x+2w_y-u_z)-2fuc^2=0,\\ c^2z^2(f\mathit{\Delta}v+f_xv_x+f_yv_y+f_zv_z)-zfc^2(v_z+u_x)-zc^2uf_x-c^2v=0,\\ c^2z^2(f\mathit{\Delta}w+f_xw_x+f_yw_y+f_zw_z)-zfc^2(w_z+u_x)-zc^2uf_y-c^2w=0. \end{cases}$$

◀

Remark 2. A left-invariant unit vector fields on H^3 define an f-harmonic map potential f^v if and only if f verify $2zf_z+c^2f=0$.

4. Conclusion

This work studied the f-harmonicity with potential f^v of a vector field from a Riemannian manifold (M,g) to its tangent bundle TM equipped with the Sasaki metric G^s. We show that an f-harmonic vector field with potential f^v is parallel if and only if f is constant function. We also characterize the f-harmonic vector fields with potential f^v on some Three Riemannian Lie groups.

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