Baku Mathematical Journal
2026, Vol. 5, No X, PP. XX-XX
https://doi.org/10.32010/j.bmj.2026.05

THE BOUNDEDNESS OF MAXIMAL OPERATORS IN CALDERON WEIGHTED B-MORREY SPACES

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Received: 18.09.2025 / Revised: 17.11.2025 / Accepted: 24.11.2025

Abstract. *In this paper the boundedness of the B-maximal operator M_γ from the B-Morrey space $L_{p,\lambda,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\lambda,\omega,\gamma}(\mathbb{R}^n_{k,+})$ for all $1 < p < \infty$ when $\omega \in C_{p,\gamma}(\mathbb{R}^n_{k,+})$ is proved.*

Keywords: B-maximal operator, Laplace-Bessel differential operator, generalized shift operator, weighted Lebegue space, Calderon weighted Morrey space

Mathematics Subject Classification (2020): 42B20, 42B25, 42B35

1. Introduction

The maximal operator, potentials, singular integrals and related topics associated with the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{k} \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad \gamma_1 > 0, \ldots, \gamma_k > 0,$$

have been investigated by many researchers, see B. Muckenhoupt and E. Stein [21], I. Kipriyanov [17], K. Triméche [25], L. Lyakhov [19], K. Stempak [24], A. D. Gadjiev and I. A. Aliev [8], I. A. Aliev and S. Bayrakci [2], V. S. Guliyev [10], [11], V. S. Guliyev and J. J. Hasanov [12], [13], J. J. Hasanov and Z. V. Safarov [15], A. Şerbetçi and I. Ekincioğlu [22], E. L. Shishkina [23] and others.

B-Riesz potentials $I_{\alpha,\gamma}$ generated by the generalized shift operator play important tool in Fourier-Bessel harmonic analysis and applications. These spaces have been investigated by lots of mathematicians [1], [2]-[5], [10], [11], [14], [16], [22]. In the present article

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we give the necessary and sufficient conditions for the boundedness of the commutators of the B-Riesz potentials,$[b, I_{\alpha,\gamma}]$ where b is a locally integrable function on $\mathbb{R}^n_{k,+}$, from the spaces $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \frac{n+|\gamma|}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+|\gamma|}$ and from the spaces $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$, $1 - \frac{1}{q} = \frac{\alpha}{n+|\gamma|}$. Also, It is proved that B-maximal commutators $M_{b,\gamma}$ and commutators of B-singular integral operators $[b, A_\gamma]$ are bounded from the B-Morrey space $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ for all $1 < p < \infty$, $b \in BMO_\gamma(\mathbb{R}^n_{k,+})$.

Morrey spaces, $L_{p,\lambda}(\mathbb{R}^n)$ introduced by C. Morrey in 1938 [20] play an important role in the theory of partial differential equations. These are defined by the following norm, $f \in L_p^{loc}(\mathbb{R}^n)$

$$\|f\|_{L_{p,\lambda}} \equiv \|f\|_{L_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\ r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty,$$

where $0 \le \lambda \le n$, $1 \le p < \infty$. If $\lambda = 0$ and $\lambda = n$ then $L_{p,\lambda}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ and $L_{p,\lambda}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$ respectively. If $\lambda < 0$ or $\lambda > n$ then $L_{p,\lambda}(\mathbb{R}^n) = \Theta$ where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

These spaces appeared to be quite useful in the study of the local behaviour of the solutions to elliptic partial differential equations, apriori estimates and other topics in the theory of partial differential equations.

F. Chiarenza and M. Frasca [7] studied the boundedness of the maximal operator in Morrey spaces.

Also we denote by $WL_{p,\lambda}(\mathbb{R}^n)$, the weak Morrey space, is the space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WL_{p,\lambda}} \equiv \|f\|_{WL_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\ r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(\mathbb{R}^n)$ denotes the weak L_p-space.

2. Preliminaries

Let $\mathbb{R}^n_{k,+} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : \ x_1 > 0, \dots, x_k > 0,\ 1 \le k \le n\}$ and let $E(x,r) = \{y \in \mathbb{R}^n_{k,+}\ ;\ |x - y| < r\}$, $\gamma = (\gamma_1, \dots, \gamma_k)$, $\gamma_1 > 0, \dots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \ldots + \gamma_k$, $(x')^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$. For a measurable set $E \subset \mathbb{R}^n_{k,+}$ suppose $|E|_\gamma = \int_E (x')^\gamma dx$, then $|E(0,r)|_\gamma = \omega(n,k,\gamma) r^Q$, $Q = n + |\gamma|$, where

$$\omega(n,k,\gamma) = \int\limits_{E(0,1)} (x')^\gamma dx = \frac{\pi^{\frac{n-k}{2}}}{2^k}\ \Gamma^{-1}\left(\frac{Q+2}{2}\right) \prod_{i=1}^{k} \Gamma\left(\frac{\gamma_i + 1}{2}\right).$$

Let $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ be the space of measurable functions on $\mathbb{R}^n_{k,+}$ with finite norm

$$\|f\|_{L_{p,\omega,\gamma}} = \|f\|_{L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})} = \left(\int\limits_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega^p(x) (x')^\gamma dx \right)^{1/p}, \quad 1 \le p < \infty,$$

and for $p=\infty$ the space $L_{\infty,\omega}(\mathbb{R}^n_{k,+})$ is defined by means of the usual modification

$$\|f\|_{L_{\infty,\omega}} = \underset{x\in\mathbb{R}^n_{k,+}}{ess\ sup}\ \omega(x)|f(x)|.$$

Even though the $C_{p,\gamma}$ class is well-known, for completeness, we offer the definition of $C_{p,\gamma}$ weight functions.

Definition 1. *The weight function ω belongs to the class $C_{p,\gamma}(\mathbb{R}^n_{k,+})$ for $1<p<\infty$, if the following statement*

$$\sup_{x\in\mathbb{R}^n_{k,+}, r>0} \left\| \frac{\omega}{|\cdot|^Q} \right\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(x,r))} \left\|\omega^{-1}\right\|_{L_{p',\gamma}(E(x,r))} < \infty$$

and

$$\sup_{x\in\mathbb{R}^n_{k,+}, r>0} \|\omega\|_{L_{p,\gamma}(E(x,r))} \left\| \frac{1}{\omega\cdot|\cdot|^Q} \right\|_{L_{p',\gamma}(\mathbb{R}^n_{k,+}\setminus E(x,r))} < \infty$$

are finites.

The generalized shift operator T^y is defined by (see, for example [17], [18])

$$T^y f(x) = C_{\gamma,k} \int_0^\pi \dots \int_0^\pi f\left((x',y')_\beta, x''-y''\right)\ d\nu(\beta),$$

where $d\nu(\beta) = \prod\limits_{i=1}^k \sin^{\gamma_i-1}\beta_i\ d\beta_1\dots d\beta_k$, $x'=(x_1,\dots,x_k)\in\mathbb{R}^k$,
$x''=(x_{k+1},\dots,x_n)\in\mathbb{R}^{n-k}$, $(x_i,y_i)_{\beta_i} = (x_i^2-2x_iy_i\cos\beta_i+y_i^2)^{\frac{1}{2}}$, $1\le i\le k$,
$(x',y')_\beta = ((x_1,y_1)_{\beta_1},\dots,(x_k,y_k)_{\beta_k})$ and

$$C_{\gamma,k} = \pi^{-\frac{k}{2}}\ \Gamma^{-1}\left(\frac{|\gamma|}{2}\right) \prod_{i=1}^k \Gamma\left(\frac{\gamma_i+1}{2}\right) = \frac{2^{k-1}|\gamma|}{\pi}\left(\frac{|\gamma|}{2}+1\right)\omega(2,k,\gamma).$$

It is well known that T^y is closely related to the Laplace-Bessel differential operator $\Delta_B = \sum\limits_{i=1}^k B_i + \sum\limits_{i=k+1}^n \frac{\partial^2}{\partial x_i^2}$, where $B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$, $i=1,\dots,k$.

Lemma 1. *For all $x\in\mathbb{R}^n_{k,+}$ the following equality*

$$\int\limits_{E(x,t)} g(y)(y')^\gamma dy = C_{\gamma,k}^{-1} \int\limits_{B((x,0),t)} g\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2}, z''\right) d\nu(z,\overline{z'})$$

*holds, where $B((x,0),t) = \{(z,\overline{z'})\in\mathbb{R}^n\times(0,\infty)^k : |(x_1-\sqrt{z_1^2+\overline{z}_1^2},...,$
$x_k-\sqrt{z_k^2+\overline{z}_k^2}, x''-z'')|<t\}$, $d\nu(z,\overline{z'}) = (\overline{z'})^{\gamma-1}dzd\overline{z'}$.*

Lemma 2. *For all $x \in \mathbb{R}^n_{k,+}$, $0 < \theta < \infty$,*

$$\int\limits_{\mathbb{R}^n_{k,+}} T^y g(x)\varphi(y)\left(M_\gamma \chi_{E_r}\right)^\theta (y)(y')^\gamma dy =$$

$$\int\limits_{\mathbb{R}^n\times(0,\infty)^k} g\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z''\right)\varphi(z,\overline{z'})\left(M_\nu \chi_{E((x,0),r)}\right)^\theta (z,\overline{z'})d\nu(z,\overline{z'}),$$

holds where $E((x,0),t) = \{(z,\overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k \ : \ \left|\left(x-z,\overline{z'}\right)\right| < t\}$.

The proofs of Lemmas 1 and 2 are done straightforwardly via the following substitutions

$$z'' = y'', z_i = y_i \cos\alpha_i, \quad \overline{z_i} = y_i \sin\alpha_i, \quad 0 \le \alpha_i < \pi, \ i = 1,\ldots,k,$$
$$y \in \mathbb{R}^n_{k,+}, \ \overline{z'} = (\overline{z}_1,\ldots,\overline{z}_k), \ (z,\overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k, \ 1 \le k \le n.$$

Definition 2. *[10] Let $1 \le p < \infty$, $0 \le \lambda \le Q$. $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, B-Morrey space and $L_{p,\lambda,\omega,\gamma}(\mathbb{R}^n_{k,+})$, weighted B-Morrey space associated with the Laplace-Bessel differential operator defined as the space of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_{k,+}$, with the finite norm*

$$\|f\|_{L_{p,\lambda,\gamma}} = \sup_{t>0,\, x\in\mathbb{R}^n_{k,+}} \left(t^{-\lambda} \int\limits_{E(0,t)} T^y \left|f(x)\right|^p (y')^\gamma dy \right)^{1/p},$$

$$\|f\|_{L_{p,\lambda,\omega,\gamma}} = \sup_{t>0,\, x\in\mathbb{R}^n_{k,+}} \left(t^{-\lambda} \int\limits_{E(0,t)} T^y \left|f(x)\right|^p \omega^p(y)(y')^\gamma dy \right)^{1/p}.$$

Note that

$$L_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\gamma}(\mathbb{R}^n_{k,+}),$$

and if $\lambda < 0$ or $\lambda > Q$, then $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \Theta$.

3. Maximal Operator in $L_{p,\lambda,\omega,\gamma}(\mathbb{R}^n_{k,+})$

Now we define the B-maximal operator by

$$M_\gamma f(x) = \sup_{r>0} |E(0,r)|_\gamma^{-1} \int\limits_{E(0,r)} T^y |f(x)| (y')^\gamma dy.$$

Homogeneous type maximal operator defined by

$$M_\nu f(x) = \sup_{r>0} \nu(E(x,r))^{-1} \int_{E(x,r)} \left|f(y)\right| d\nu(y).$$

Also, in the work [9] it was proved:

Proposition. *Let $1 \leq p < \infty$ and $\phi \in C_p(Y)$. Then, M_ν is bounded from $L_{p,\phi}(Y)$ to $L_{p,\phi}(Y)$, where (Y, d, ν) homogeneous type space.*

The following theorem is on the boundedness of B-maximal operator M_γ in B-Morrey spaces.

Theorem. *If $f \in L_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, $1 < p < \infty$,$0 \leq \lambda < Q$, $\omega \in C_{p,\gamma}(\mathbb{R}^n_{k,+})$. Then, M_γ is bounded from $L_{p,\lambda,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\lambda,\omega,\gamma}$.*

Proof. We need to introduce the maximal operator defined on a space of homogeneous type (Y, d, ν). By this, we mean a topological space $Y = \mathbb{R}^n \times (0, \infty)^k$ equipped with a continuous pseudometric d and a positive measure ν satisfying

$$\nu(E((x, \overline{x'}), 2r)) \leq C_1 \nu(E((x, \overline{x'}), r))$$

with a constant C_1 independent of $(x, \overline{x'})$ and $r > 0$. Here $E((x, \overline{x'}), r) = \{(y, \overline{y'}) \in Y \ : \ d(((x, \overline{x'}), (y, \overline{y'})) < r\}$ ($d\nu(y, \overline{y'})$ is given in Lemma 1), $d((x, \overline{x'}), (y, \overline{y'})) = |(x, \overline{x'}) - (y, \overline{y'})| \equiv (|x - y|^2 + |\overline{x'} - \overline{y'}|^2)^{\frac{1}{2}}$. Let (Y, d, ν) be a space of homogeneous type. Define

$$M_\nu \overline{f}(x, \overline{x'}) = \sup_{r>0} \nu(E((x, \overline{x'}), r))^{-1} \int_{E((x,\overline{x'}),r)} \left|\overline{f}(y, \overline{y'})\right| d\nu(y),$$

where $\overline{f}(x, \overline{x'}) = f\left(\sqrt{x_1^2 + \overline{x}_1^2}, \ldots, \sqrt{x_k^2 + \overline{x}_k^2}, x''\right)$.

It can be proved that

$$M_\gamma f\left(\sqrt{z_1^2 + \overline{z}_1^2}, \ldots, \sqrt{z_k^2 + \overline{z}_k^2}, z''\right) = M_\nu \overline{f}\left(\sqrt{z_1^2 + \overline{z}_1^2}, \ldots, \sqrt{z_k^2 + \overline{z}_k^2}, z'', 0\right) \tag{1}$$

and

$$M_\gamma f(x) = M_\nu \overline{f}(x, 0). \tag{2}$$

Note that if $y \in E_r = E(0, r)$, $r > 0$, then

$$M_\nu \chi_{E_r}(y) = \sup_{t>0} \frac{1}{\nu(E(y, t))} \int_{E(y,t)} \chi_{E_r}(z) d\nu(z) = \sup_{t>0} \frac{1}{\nu(E(y, t))} \int_{E(y,t)\cap E_r} d\nu(z) \leq$$

$$C \sup_{t>0} \frac{1}{\nu(E(y, t))} \int_{E(y,t)} d\nu(z) \leq C.$$

The following estimate is known (see [6])

$$M_\gamma \chi_{E_r}(y) \approx \frac{r^Q}{(|y| + r)^Q}, \ y \in \mathbb{R}^n_{k,+}, \ r > 0.$$

It is obvious from this that $(M_\nu \chi_{E((x,0),r)})^\theta \in C_p(Y)$ for any $0 < \theta < 1$ is obtained.

By the Proposition and Lemma 2, taking into account $\phi^p(y) = \omega^p(y)(M_\nu \chi_{E((x,0),r)})^\theta(y)$, $\phi \in C_p(Y)$ and equalities (1) and (2), we have

$$\left(\int_{E_r} T^y \left(M_\gamma f(x)\right)^p \omega^p(y)(y')^\gamma dy\right)^{\frac{1}{p}} =$$

$$\left(\int_Y \left(M_\nu \overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right)^p \times\right.$$

$$\left.\omega^p(y,\overline{y'})(M_\nu \chi_{E((x,0),r)})^\theta(y,\overline{y'})\ d\nu(y,\overline{y'})\right)^{\frac{1}{p}} =$$

$$\left(\int_Y (M_\nu \overline{f}(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y'',0))^p \phi(y,\overline{y'})\ d\nu(y,\overline{y'})\right)^{\frac{1}{p}} \le$$

$$C_2\left(\int_Y \left|\overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right|^p \phi(y,\overline{y'})\ d\nu(y,\overline{y'})\right)^{\frac{1}{p}} =$$

$$C_2\left(\int_Y \left|\overline{f}(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y'',0)\right|^p \times\right.$$

$$\left.\omega^p(y,\overline{y'})(M_\nu \chi_{E((x,0),r)})^\theta(y,\overline{y'})\ d\nu(y,\overline{y'})\right)^{\frac{1}{p}} =$$

$$C_2\left(\int_Y \left|f\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y''\right)\right|^p \times\right.$$

$$\left.\omega^p(y,\overline{y'})(M_\nu \chi_{E((x,0),r)})^\theta(y,\overline{y'})\ d\nu(y,\overline{y'})\right)^{\frac{1}{p}} =$$

$$C_2\ \left(\int_{\mathbb{R}^n_{k,+}} T^y[|f|\ ]^p(x)\omega^p(y)(M_\gamma \chi_{E_r})^\theta(y)(y')^\gamma dy\right)^{\frac{1}{p}} \le$$

$$C_2\ \left(\int_{E_r} T^y[|f|\ ]^p(x)\omega^p(y)(M_\gamma \chi_{E_r})^\theta(y')^\gamma dy+\right.$$

$$\left.\int_{\mathbb{R}^n_{k,+}\setminus E_r} T^y[|f|\ ]^p(x)\omega^p(y)(M_\gamma \chi_{E_r})^\theta(y)(y')^\gamma dy\right)^{\frac{1}{p}} \le$$

$$C_2\ \left(\int_{E_r} T^y[|f|\ ]^p(x)\omega^p(y)(y')^\gamma dy+\right.$$

$$\left.\sum_{j=1}^\infty \int_{E_{2^{j+1}r}\setminus E_{2^j r}} T^y[|f|\ ]^p(x)\omega^p(y)(M_\gamma \chi_{E_r})^\theta(y)(y')^\gamma dy\right)^{\frac{1}{p}} \le$$

$$C_2\ \left(\int_{E_r} T^y[|f|\ ]^p(x)\omega^p(y)(y')^\gamma dy+\right.$$

$$\left.\sum_{j=1}^\infty \int_{E_{2^{j+1}r}\setminus E_{2^j r}} T^y[|f|\ ]^p(x)\omega^p(y)\frac{r^{Q\theta}}{(|y|+r)^{Q\theta}}(y')^\gamma dy\right)^{\frac{1}{p}} \le$$

$$C_3\ \|f\|_{L_{p,\lambda,\omega,\gamma}}\left(r^\lambda+\sum_{j=1}^\infty \frac{1}{(2^j+1)^{Q\theta}}(2^{j+1}r)^\lambda\right)^{\frac{1}{p}} \le C_4\ r^{\frac{\lambda}{p}}\|f\|_{L_{p,\lambda,\omega,\gamma}}.$$

Then, we get

$$\|M_\gamma f\|_{L_{p,\lambda,\omega,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} t^{-\frac{\lambda}{p}} \|T^{\cdot}(M_\gamma f(x))\|_{L_{p,\omega,\gamma}(E_t)} \le C_4 \|f\|_{L_{p,\lambda,\omega,\gamma}}.$$

◄

Corollary. *If $1 < p < \infty$, $f \in L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, $\omega \in C_{p,\gamma}(\mathbb{R}^n_{k,+})$. Then, M_γ is bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\omega,\gamma}$.*

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