Baku Mathematical Journal
2026, Vol. 5, No X, PP. XX-XX
https://doi.org/10.32010/j.bmj.2026.04

FINITE SIMPLE GROUPS IN WHICH ALL CYCLIC SUBGROUPS OF PRIME POWER ORDER ARE TI-SUBGROUPS

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Received: 29.09.2025 / Revised: 07.11.2025 / Accepted: 14.11.2025

Abstract. *Let G be a finite group. A subgroup H of G is called a TI-subgroup if $H \cap H^x \in \{1, H\}$ for all $x \in G$, and a group is called a PCTI-group if all of its cyclic subgroups of prime power order are TI-subgroups. In this paper, we prove that a finite non-abelian simple S is a PCTI-group if and only if S is* $\mathrm{PSL}(2, q)$ *or the Janko simple group J_1.*

Keywords: finite simple group, prime power order, TI-subgroup

Mathematics Subject Classification (2020): 20B15, 20B30, 20B40

1. Introduction

Throughout this paper, all groups are finite. Our notation and terminology are standard. Let G be a group and H be a subgroup of G. If for each $x \in G$ we always have $H \cap H^x \in \{1, H\}$, then we say that H is a TI-subgroup of G. The group whose cyclic subgroups are TI-subgroups is called a CTI-group. The classification of finite groups in which certain subgroups are TI-subgroups is a subject that has drawn significant attention. In [19], Walls characterized the structure of the groups in which all subgroups are TI-subgroups. In [11] Guo, Li and Flavell classified groups all of whose abelian subgroups are TI-subgroups. In [17], Mousavi described the structure of the non-nilpotent CTI-group. In [1], Abdollahi gave a classification of the nilpotent CTI-groups. The goal of this paper is to investigate finite groups in which every cyclic subgroup of the prime power order is a TI-subgroup.

Definition. *Let G be a finite group. G is called a PCTI-group if all of its cyclic subgroups of prime power order are TI-subgroups.*

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Obviously, every CTI-group is a PCTI-group. Since the PCTI condition is hereditary for subgroups and quotient groups (see Lemma 1), it is important to consider the case of simple PCTI-groups. In this paper, we classify the ?nite simple PCTI-groups and obtain the following result.

Theorem. *Let G be a finite non-abelian simple group. Then G is a PCTI-group if and only if it is one of the following types:*

(1) $\mathrm{PSL}(2, q)$, *where $q > 3$ is a power of a prime p;*

(2) the Janko simple group J_1.

2. Some Lemmas

Since every cyclic subgroups of a p-group is of order of a prime power, a p-group G is a PCTI-group if and only if G is a CTI-group. We first give some lemmas.

Corollary. *[1, Theorem 1.2] Let G be a p-group. Then G is a PCTI-group if and only if one of the following occurs:*

(i) G is a Dedekindian group, i.e., all subgroups of G are normal in G;

(ii) G is an exponent p;

(iii) $G' = \Omega_1(G^p)$ is of order p and $\Phi(G) = G^p$ is a central cyclic subgroup of G;

(iv) G is a 2-group such that $G = A\langle x\rangle$, where A is an abelian subgroup of G and x is an involution in $G \setminus A$ such that $a^x = a^{-1}$.

Note that the groups in case (iv) have more than one half of involutions.

Lemma 1. *Let G be a PCTI-group. H is a subgroup of G and N is a normal subgroup of G. Then:*

(i) H is a PCTI-group;

(ii) G/N is also a PCTI-group.

Proof. (i) is trivial. Next, we prove (ii). Let $xN \in G/N$ with $o(xN) = p^k$. Now, it suffices to prove that $\langle xN\rangle \cap \langle xN\rangle^{gN} = \langle xN\rangle$ or N for all $gN \in G/N$. We note that since $o(xN) = p^k$ we have $x^{p^k} \in N$. Suppose that $o(x) = m$, and $m = p^l n$, where $p \nmid n$. Obviously, $(n, p^k) = 1$, there exist integers μ and ν such that $n\mu - p^k\nu = 1$, thus $x^{n\mu} = x^{1+p^k\nu} \in xN$. We can easily see that the element $x^{n\mu}$ has order $\frac{m}{(m,n\mu)}$, which is a prime power. By (i), $\langle x^{n\mu}\rangle$ is a PCTI-group, it follows that for $\forall g \in G$ we have $\langle x^{n\mu}\rangle \cap \langle x^{n\mu}\rangle^g = \langle x^{n\mu}\rangle$ or 1. Thus, we conclude that $\langle xN\rangle \cap \langle xN\rangle^{gN} = \langle xN\rangle$ or N. Hence G/N is also a PCTI-group. ◄

Remark. If G is a PCTI, it follows that any subquotient of G is also PCTI.

Lemma 2. *[17, Theorem 4.2] Let G be a finite non-abelian simple CTI-group. Then* $G \cong \mathrm{PSL}(2, q)$, *where $q > 3$ is a prime power.*

Lemma 3. *Let G be an alternating simple group. Then G is a PCTI-group if and only if $G \cong A_5$ or A_6.*

Proof. With the help of GAP ([18]), A_5 and A_6 are PCTI-groups. Next, we prove that A_7 is not a PCTI-group. Since A_7 are exactly all even permutations of 7 letters, there does not exist an element of order 8 or 9. Let $H = \langle x\rangle, x \in A_7$ and $o(x) = 4$. Because all elements of order 4 in A_7 are conjugate, we let

$$H = \langle x\rangle = \langle (1234)(56)\rangle = \{(1), (1234)(56), (1432)(56), (13)(24)\}.$$

Choose $g = (567)$. Then

$$H^g = \{(1), (1234)(67), (1432)(67), (13)(24)\}.$$

So $H\cap H^g = \{(1), (13)(24)\}$, and then $H = \langle x\rangle = \langle (1234)(56)\rangle$ is not a TI-subgroup. Thus A_7 is a not PCTI-group. Moreover, since A_7 is a subgroup of A_n for $n \geq 7$, A_n $(n \geq 7)$ is not a PCTI-group. ◀

Lemma 4. *Let G be a p-group. Then $G\wr\mathbb{Z}_p$ is a PCTI-group if and only if $G\wr\mathbb{Z}_p \cong \mathbb{Z}_2\wr\mathbb{Z}_2$.*

Proof. Assume that $G \wr \mathbb{Z}_p \cong \mathbb{Z}_2 \wr \mathbb{Z}_2$. Since $\mathbb{Z}_2 \wr \mathbb{Z}_2$ is isomorphic to D_8, it is PCTI obviously.

Conversely, let $\mathbb{Z}_p = \langle\sigma\rangle$ and $\sigma = (123\cdots p)$. The wreath product $G \wr \mathbb{Z}_p$ is a semi direct product of $G^p = G \times G \times \cdots \times G$ with p times by $\langle\sigma\rangle$, in which the action of $\langle\sigma\rangle$ is a permutation of the coordinates of G^p. Next, we use Lemma 1 to check in case by case.

Case (i). Assume that $G \wr \mathbb{Z}_p$ is Dedekindian. Note that p is a prime, so $p \geq 2$. We choose the subgroup $(G, 1, \cdots , 1)$. Obviously, $(G, 1, \cdots , 1)^\sigma = (1, G, \cdots , 1)$, and so $(G, 1, \cdots , 1)$ is not normal in $G \wr \mathbb{Z}_p$. Thus $G \wr \mathbb{Z}_p$ is not a PCTI group.

Case (ii). Assume that $exp(G \wr \mathbb{Z}_p) = p$. Let $x = (g, 1, 1, \cdots , 1; \sigma) \in G \wr \mathbb{Z}_p$, where $g \neq 1$, then we have $x^p = (g, g, \cdots , g; 1)$. Since $g \neq 1$, it follows that $o(x) \geq p^2$. It follows that $exp(G \wr \mathbb{Z}_p) \neq p$, and so this case is impossible.

Case (iii). we will prove that if $|(G \wr \mathbb{Z}_p)'| = p$, then $p = 2$. By the definition of the wreath product, we know that $(G\wr\mathbb{Z}_p)' \leq G^p$. For any $g = (g_1, g_2, \cdots , g_p) \in G^p$, we have

$$\begin{aligned}[g, \sigma] &= g^{-1}g^\sigma = (g_1^{-1}, g_2^{-1}, \cdots , g_p^{-1})(g_{1^{\sigma^{-1}}}, g_{2^{\sigma^{-1}}}, \cdots , g_{p^{\sigma^{-1}}})\\ &= (g_1^{-1}g_p, g_2^{-1}g_1, \cdots , g_p^{-1}g_{p-1}).\end{aligned}$$

If $g_1 \neq 1, g_2 = \cdots = g_p = 1$, then

$$[g, \sigma] = (g_1^{-1}, g_1, 1, \cdots , 1).$$

Similarly, If $g_2 \neq 1, g_1 = g_3 = \cdots = g_p = 1$, then

$$[g, \sigma] = (1, g_2^{-1}, g_2, 1, \cdots , 1).$$

It follows that $|(G \wr \mathbb{Z}_p)'| \geq 2|G|$. Since $|(G \wr \mathbb{Z}_p)'| = p$, we have $p = 2$. So $G \wr \mathbb{Z}_p$ is $\mathbb{Z}_2 \wr \mathbb{Z}_2$.

Case (iv). Suppose that $G \wr \mathbb{Z}_2 \cong A \rtimes \mathbb{Z}_2$ for some abelian group A and involution $x \in G \wr \mathbb{Z}_2 - A$ such that $a^x = a^{-1}$ for all $a \in A$. Next, we also prove $G \wr \mathbb{Z}_2 \cong \mathbb{Z}_2 \wr \mathbb{Z}_2$. We denote by $i_2(X)$ and $c_1(X)$ the number of involutions and the set $\{x \in X|x^2 = 1\}$, respectively. Note that $i_2(A \rtimes \mathbb{Z}_2) = i_2(A) + |A|$. Let $(g_1, g_2; (12)) \in G \wr \mathbb{Z}_2$. Then

$$(g_1, g_2; (12))^2 = (g_1g_{1^{(12)}}, g_2g_{2^{(12)}}; 1) = (g_1g_2, g_2g_1; 1).$$

Therefore, if $(g_1, g_2; (12))$ is an involution if and only if $g_1g_2 = 1$, that is, $g_1 = g_2^{-1}$. Thus the number of elements of order 2 in the coset $(G \times G)(12)$ is $|G|$, and therefore

$$i_2(G \wr \mathbb{Z}_2) = (i_2(G) + 1)^2 + |G| - 1.$$

Since $G \wr \mathbb{Z}_2 \cong A \rtimes \mathbb{Z}_2$, we have

$$i_2(A) + |A| = (i_2(G) + 1)^2 + |G| - 1$$

and $|A| = |G|^2$. For the group $A \rtimes \mathbb{Z}_2$, it is obvious that

$$\frac{i_2(A) + |A|}{2|A|} > \frac{1}{2}$$

that is

$$\frac{(i_2(G) + 1)^2 + |G| - 1}{2|G|^2} > \frac{1}{2}.$$

It follows that $i_2(G) > |G| - 2$, namely, G is an elementary abelian 2-group. Meanwhile, in this case, $i_2(A) = i_2(G)$ and the maximum element order in group $G \wr \mathbb{Z}_2$ is 4. Since A is an abelian group, we can set $A = \mathbb{Z}_2^{m_1} \times \mathbb{Z}_4^{m_2}$ with $m_2 > 0$. Note that $\mathrm{c}_1(A) = 2^{m_1+m_2}$ and $|A| = |G|^2$, thus

$$\mathrm{c}_1(A) = i_2(A) + 1 = |G| = |A|^{\frac{1}{2}} = 2^{\frac{m_1}{2}+m_2}.$$

So we conclude that $m_1 = 0$, that is $A \cong \mathbb{Z}_4^{m_2}$ and $G \cong \mathbb{Z}_2^{m_2}$. Observe that if $\mathbb{Z}_4^{m_2} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2^{m_2} \wr \mathbb{Z}_2$, then $\mathbb{Z}_2^{m_2} \wr \mathbb{Z}_2$ contains a normal subgroup to $\mathbb{Z}_4^{m_2}$. Obviously, the centralizer of any element of order 4 in $\mathbb{Z}_4^{m_2} \rtimes \mathbb{Z}_2$ is $\mathbb{Z}_4$. Next, we discuss the case of elements of order 4 in $\mathbb{Z}_2^{m_2} \wr \mathbb{Z}_2$. Let $(g, 1; (12))$ be an element of order 4 , where g is an involution of $\mathbb{Z}_2^{m_2}$. If $g_1, g_2 \in \mathbb{Z}_2^{m_2}$, then

$$(g, 1; (12))(g_1, g_2; (12)) = (gg_2, g_1; 1),$$
$$(g_1, g_2; (12))(g, 1; (12)) = (g_1, g_2g; 1).$$

If $(g_1, g_2; (12))$ centralizes $(g, 1; (12))$, then $g_1 = gg_2$ and $g_2g = g_1$. By the arbitrariness of g_1, g_2, we have $g = g_1g_2$, which shows that in this case, the number of elements centralizing $(g, 1; (12))$ is 2^{m_2}. Similarly,

$$(g, 1; (12))(g_1, g_2; 1) = (gg_2, g_1; (12))$$

and

$$(g_1, g_2; 1)(g, 1; (12)) = (g_1g, g_2; (12)).$$

If $(g_1, g_2; 1)$ centralizes $(g, 1; (12))$, then $g_1 = g_2$. That is, in this case, the number of elements centralizing $(g, 1; (12))$ is 2^{m_2}. Combining the two cases, we can conclude that

$$|C_{\mathbb{Z}_2^{m_2} \wr \mathbb{Z}_2}((g_1, g_2; 1))| = 2^{m_2} + 2^{m_2} = 2^{m_2+1}.$$

According to $\mathbb{Z}_4^{m_2} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2^{m_2} \wr \mathbb{Z}_2$, it follows that $2^{m_2+1} = 4^{m_2}$, so $m_2 = 1$. Thus $G \wr \mathbb{Z}_2 \cong \mathbb{Z}_2 \wr \mathbb{Z}_2$. This completes the proof. ◀

Lemma 5. *Let $G = \langle a, b | a^{2^{n-1}} = 1, a^{2^{n-2}} = b^2, a^b = a^{-1} \rangle$ with $n \geq 4$, a semi-dihedral group. Then G is not a PCTI-group.*

Proof. Obviously, G is not a Dedekindian group and $exp(G) \neq 2$. Since $[a, b] = a^{-1}a^b = a^{-2}$ and $o(a^{-2}) = 2^{n-2}$, we have $|G'| \neq 2$. Then G does not satisfy Corollary 1(iii). By [4, Theorem 124.4], we obtain

$$i_2(G) = 2^{n-2} + 1 < \frac{1}{2}|G|.$$

According to Corollary, it follows that in case (iv) has more than one half involution, so G does not satisfy Corollary (iv). Therefore, G is not a PCTI-group. ◀

Next, we will consider the group $\mathrm{PSL}(3, q)$.

Lemma 6. *Let $G \cong \mathrm{PSL}(3, q)$, where $q \geq 3$ is a power of prime p. Then G is not a PCTI-group.*

Proof. Let P be a Sylow 2-subgroup of G. Since all subgroups of a PCTI-group are PCTI-groups, it suffices to prove that P is not a PCTI-group.

Case 1. $q = 2^n, n \geq 2$.

The Sylow 2-subgroup P of $\mathrm{PSL}(3, q)$ can be taken as the following group (see [6, Section 4]):

$$P = \left\{ \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \mid \alpha, \beta, \gamma \in GF(q) \right\} .$$

Let A and B be the subgroups for which $\beta = 0$ and $\alpha = 0$, respectively. That is,

$$A = \left\{ \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \alpha, \gamma \in GF(q) \right\}$$

and

$$B = \left\{ \begin{pmatrix} 1 & 0 & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \mid \beta, \gamma \in GF(q) \right\} .$$

Then direct computation yields the following properties of P (see [6, p.494]):

(1) P has order q^3, and $Z(P) = P' = \Phi(P) = A \cap B$ is elementary abelian of order q;

(2) A and B are elementary abelian normal subgroups of P of order q^2. If $x \notin A \cup B$, then x has order 4.

Next, we verify that P does not satisfy any type in groups of Corollary:

(i) Let $H = \langle t \rangle \leq P$ and $z \in P$ with

$$t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ z = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $H^z \neq H$, so it implies that $H \ntrianglelefteq P$. Therefore P is not a Dedekindian group.

(ii) It is clear that $\forall x \notin A \cup B$, x has order 4. So $exp(P) \neq p$.

(iii) Because $Z(P)$ is elementary abelian of order q, we have $|P'| = q \neq 2$. This is impossible.

(iv) Assume that the number of elements of order 4 in P is m. By the property (2) above, we can obtain that $m = |P| - |A \cup B|$. Since

$$m = |P| - |A \cup B| \geq |P| - |A| - |B| = q^3 - 2q^2 \geq \frac{1}{2}|P|,$$

it follows that

$$i_2(P) < \frac{1}{2}|P|,$$

contradicts $i_2(P) > \frac{1}{2}|P|$ in this case. Therefore, P is not a PCTI-group.

Case 2. $q \geq 3$ and q is odd.

By [10, p.486], the Sylow 2-subgroup P of $\mathrm{PSL}(3, q)$ with q odd are semi-dihedral if $q \equiv -1 (\mathrm{mod}\ 4)$ and are wreathed if $q \equiv 1 (\mathrm{mod}\ 4)$. For the former case, it cannot occur because a semi-dihedral group is not PCTI-group by Lemma 5. For the latter case, by using Lemma 4, it is clear that P is not a PCTI-group. Therefore, we conclude that $\mathrm{PSL}(3, q)$ is not a PCTI-group, where $q \geq 3$ is a power of prime p. ◀

By Lemma 1 (ii) and the preceding results, it follows that $SL(3, q)$ with $q \geq 3$ is also not a PCTI-group. Next, we consider the case of $SL(2, q)$ with $q > 3$ is odd.

Lemma 7. *If $q \geq 3$ is odd, then $\mathrm{SL}(2, q)$ is not a PCTI-group.*

Proof. Since q is odd, $\mathrm{SL}(2, q)$ has a unique element of order 2, say Z. Obviously, the number of elements of order 4 is more than one (otherwise, the subgroup of order 4 is normal in $\mathrm{SL}(2, q)$). We let H and H_1 of order 4 be conjugate. So $H \cap H_1 = \{1, z\}$, and then H is not a TI-subgroup of $\mathrm{SL}(2, q)$. Therefore, $\mathrm{SL}(2, q)$ is not a PCTI-group. ◀

Lemma 8. *Let $G \cong Sz(q)$, $q = 2^{2n+1}, n \geq 1$, then G is not a PCTI-group.*

Proof. Let P be a Sylow 2-subgroup of G. Since $|G| = q^2(q^2+1)(q-1)$, $|P| = q^2 = 2^{4n+2}$. By [14, Theorem 2.4], P is a Suzuki 2-group, and has an exponent 4. Moreover, its center $Z(P) = P' = \Phi(P)$ and the involutions in P together with the identity element constitute $Z(P)$, thus $|Z(P)| = q$. Next, we verify that P is not a PCTI-group by using Lemma 1. In fact, we prove that a Suzuki 2-group is not a PCTI-group.

(i) Since $P' \neq 1$, P is not abelian. By [9], we have $P \cong Q_8 \times Z_2^n$.Then $|Z(P)| = 2^{n+1}$. The Suzuki 2-group satisfies the following property: $|P| = |Z(P)|^2$ or $|P| = |Z(P)|^3$. Thus, if P is the Dedekindian group, then $n = 1$ and $|P| = |Z(P)|^2$, and so $|P| = 2^4$. This contradicts $|P| = q^2 = 2^{4n+2}$. Therefore, P is not a Dedekindian group.

(ii) is not impossible because its exponent is 4.

(iii) As $Z(P) = P'$ and $|Z(P)| = q$, we have $|P'| \neq 2$. This case is also not possible.

(iv) In this case it is also impossible because $i_2(P) = |Z(P)| - 1 = q - 1 < \frac{1}{2}|P|$.

Consequently, $G \cong Sz(q)$, $q = 2^{2n+1}, n \geq 1$, is not a PCTI-group. ◀

Next, we will continue to present the case of $PSU(3, q)$.

Proposition. *Let $G \cong PSU(3, q)$, $q \geq 3$. Then G is not a PCTI-group.*

Proof. First, let $q = 2^n, n \geq 2$ and $P \in Syl_2(G)$. Now, P is isomorphic to the following group:

$$P = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & a^q & 1 \end{pmatrix} \middle| b + b^q = a^{1+q},\ a, b \in GF(q^2) \right\}.$$

Therefore, P can be regarded as the group of pairs $\{(a,b)|b + b^q = a^{1+q},\ a, b \in GF(q^2)\}$ with multiplication $(a,b)(c,d) = (a+c, a^q + b + d)$. Then by [5, p.1] and [13], it follows that P is a Suzuki 2-group. According to the proof of Lemma 8, a Suzuki 2-group is not a PCTI-group. Next, we consider the case $q \geq 3$ with q odd. By [10, p.487], the Sylow 2-subgroup P of PSU$(3,q)$ with q odd is semi-dihedral if $q \equiv 1(\bmod\ 4)$ and is wreathed if $q \equiv -1(\bmod\ 4)$. We can conclude that in this case P is not a PCTI-group by Theorem 6. Therefore, $PSU(3,q)$, $q \geq 3$, is not a PCTI-group. ◀

3. The Proof of Theorem

Suppose that G is a non-abelian simple PCTI-group. By the classification of finite simple groups [7], G is an alternating group, a group of Lie type, or a sporadic group.

Table 1. *Non-PCTI Sporadic Simple Group*

Groups	non-PCTI-subquotient	Groups	non-PCTI-subquotient
M_{12}	M_{11}	M_{22}	A_7
M_{23}	A_8	M_{24}	M_{22}
J_2	$PSU(3,3)$	J_4	$PSU(3,11)$
HS	M_{22}	McL	M_{22}
Suz	A_7	Ly	A_{11}
He	A_7	Ru	A_8
$O'N$	A_7	Co_3	HS
Co_2	McL	Co_1	Co_2
Fi_{22}	A_{10}	Fi_{23}	A_{12}
Fi'_{24}	Fi_{23}	Th	$PSU(3,8)$
HN	A_{12}	B	Fi_{22}
M	Th		

First, we consider the sporadic simple groups. If $G = M_{11}$, then it has a semi-dihedral Sylow 2-subgroup by [2, Proposition 4]. However, a semi-dihedral group is not a PCTI-group. Hence M_{11} is not a PCTI-group. Assume that $G = J_1$. As $|J_1| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ and J_1 have an elementary abelian Sylow 2-subgroup (see [10, p.483, Theorem] or [7, p.36]). Clearly, all prime-order cyclic subgroups are TI-subgroups. Thus, J_1 is a PCTI-group. Next, assume that $G = J_3$. Let Q be the Syolw 3-subgroup of J_3. By [16, Lemma

2.1] the center $Z(Q)$ is an elementary abelian of order 9. There is an element x of Q of order 3 such that $V = \langle x, Z(Q)\rangle$ is an elementary abelian of order 27. In particular, the elements of $Q - V$ are all of the order 9 and $Q' = V$. Therefore, V is the Frattini subgroup. Now, we can verify whether Q is a PCTI-group by Corollary 1.

(i) If Q is a Dedekindian group, then Q is abelian by [9], a contradiction.

(ii) It is clear that $exp(Q) \neq 3$.

(iii) $|Q'| \neq 3$ because $Q' = V$ and $|V| = 27$.

(iv) This case cannot occur because Q is not a 2-group.

Therefore, J_3 is not a PCTI-group.

The table 1 given below shows that the remaining sporadic simple group has a proper non-PCTI-subquotient, and that, therefore, there is no sporadic simple PCTI-group except for J_1.

Suppose that G is an alternating group. Since an alternating group of degree n contains a maximal subgroup which is isomorphic to an alternating group of degree $n-1$, it follows that $G \cong A_5$ or A_6 by Lemma 3.

Now suppose that G is a group of Lie type over Galois field $GF(q)$. If G is a Chevalley group except for $\mathrm{PSL}(2,q)$ and $\mathrm{PSp}(4,q)$, then by [3, Theorem 1], it has a subgroup isomorphic to $\mathrm{SL}(3,q)$ or $\mathrm{PSL}(3,q)$. According to Lemma 6, it shows that $\mathrm{SL}(3,q)$ and $\mathrm{PSL}(3,q)$ are PCTI-group if and only if $q = 2$. So, the group G cannot be a PCTI-group except for $q = 2$. For $q = 2$, we check in case by case as follows.

(1) $G \cong \mathrm{PSL}(n,2)$, where $n \geq 3$. As G has a subgroup which is isomorphic to $\mathrm{PSL}(n-1,2)$, we have that $\mathrm{PSL}(4,2)$ is a subgroup of G. But $\mathrm{PSL}(4,2) \cong A_8$ is not a PCTI-group. Hence, G is a PCTI-group if and only if $n = 3$.

(2) $G \cong \mathrm{PSp}(2n,2)$, where $n \geq 3$. Since $A_8 \leq PSp(6,2) \leq G$, we can get that $\mathrm{PSp}(2n,2)(n \geq 3)$ can not be a PCTI-group.

(3) $G \cong P\Omega^+(2n,2)$, where $n \geq 3$. As $\mathrm{PSL}(4,2) \cong A_8 \cong P\Omega^+(6,2) \leq G$, we have that $P\Omega^+(2n,2)(n \geq 3)$ cannot be a PCTI-group.

(4) $G \cong E_6(2)$. Note that G contains a subgroup that is isomorphic to $\mathrm{PSL}(5,2)$. By the above case (1), $\mathrm{PSL}(5,2)$ is not a PCTI-group and hence G is not a PCTI-group as well.

(5) Assume that $G \cong E_7(2)$ or $E_8(2)$. Since $E_7(2)$ and $E_8(2)$ contain a subgroup which is isomorphic to $E_6(2)$, we conclude that G is not a PCTI-group.

(6) $F_4(2)'$ is not a PCTI-group since $\mathrm{PSL}(6,2) \leq F_4(2)'$.

Assume that $G \cong \mathrm{PSL}(2,q)$, where $q > 3$ is a prime power. Lemma 2 implies that G is a PCTI-group.

If $G \cong \mathrm{PSp}(4,q)$ and q is even, then G has a maximal subgroup that is isomorphic to the Suzuki group $Sz(q)$ by [15]. Applying Lemma 8, we know that G is not a PCTI-group. Since $\mathrm{PSp}(4,3) \cong \mathrm{PSU}(4,2)$, using the GAP Small Group library ([18]), we see that the Sylow 2-subgroup of $\mathrm{PSU}(4,2)$ is SmallGroup(64,138), which is not a PCTI-group. Therefore, $\mathrm{PSp}(4,3)$ is not a PCTI-group. When $q \geq 5$ and q is odd, $\mathrm{Sp}(4,q)$ have a subgroup $\mathrm{GL}(2,q)$. By Lemma 1(ii) and Lemma 7, $\mathrm{PSp}(4,q)$, where $q \geq 5$ and q is odd, is not a PCTI-group.

Next, assume that G is a Steinberg group. Suppose that G is one of the types of ${}^2D_n(q)$, $n \geq 4$, ${}^2E_6(q)$ or ${}^3D_4(q)$. By [3, Theorem 1], G has a subgroup isomorphic to

SL$(3, q)$ or PSL$(3, q)$. Hence they are not PCTI-groups when $q > 2$. Next, we deal with the case of $q = 2$.

(1) $G \cong {}^3D_4(2)$. As ${}^3D_4(2)$ has a subgroup SL$(2, 3)$ by [20, Theorem 4.3], which is not a PCTI-group by Lemma 7. Hence, ${}^3D_4(2)$ is not a PCTI-group.

(2) $G \cong {}^2E_6(2)$. Since ${}^2E_6(2)$ contains a subgroup Fi_{22} ([8, p. 7, table 3]), we get that ${}^3E_6(2)$ is not a PCTI-group by the preceding Table 3.

(3) $G \cong {}^2D_n(2)$, where $n \geq 3$. We regard ${}^2D_n(2)$ as $P\Omega^-(2n, 2)$. As $P\Omega^-(6, 2) \cong$ PSU$(4, 2)$ (see [7, p.26]). In the previous discussion, we have already proven that PSU$(4, 2)$ is not a PCTI-group. Hence ${}^2D_n(2)(n \geq 3)$ are not PCTI-groups.

In the following, we consider the remaining Steinberg groups. Assume that $G \cong {}^2A_n(q)$ where $n \geq 2$. Since ${}^2A_n(q)$ has a proper subgroup isomorphic to either SL$(3, q^2)$ or PSL$(3, q^2)$ when $n \geq 5$ (see [3, Theorem 1]). Then Lemma 6 implies that ${}^2A_n(q)$ $(n \geq 5)$ is not a PCTI-group. Next, we consider ${}^2A_n(q)$, where $n \in \{2, 3, 4\}$. We regard ${}^2A_n(q)$ as PSU$(n + 1, q)$. Since PSU$(4, q)$ and PSU$(5, q)$ contain a subgroup which is isomorphic to PSU$(3, q)$, by Lemma 2, they are not PCTI-groups except $q = 2$. Let $q = 2$. Since PSU$(4, 2)$ has a Sylow 2-subgroup $P \cong C_2 \wr C_2^2$, which is not a PCTI-group by Lemma 4. Thus PSU$(4, 2)$ is not a PCTI-group. Moreover, [7, p.73] implies that PSU$(4, 2)$ contains a subgroup PSU$(4, 2)$. So PSU$(5, 2)$ is not a PCTI-group.

Furthermore, the simple group of type ${}^2F_4(2^{2n+1})$ is not a PCTI-group, because it contains a subgroup $Sz(2^{2n+1})$. Assume that $G \cong {}^2G_2(q)$, where $q = 3^{2n+1}$. Let Q be a Sylow 3-subgroup of ${}^2G_2(q)$. By [12, Theorem 2.1], its center $Z(Q)$ is an elementary abelian subgroup of order $q = 3^{2n+1}$, $Q' = \Phi(Q)$ is an elementary abelian subgroup of order q^2 and the elements of Q' have order 9. It follows that Q clearly does not satisfy Lemma 1(ii),(iii) and (iv). Moreover, if Q is a Dedekindian group, then Q must be an abelian group by [9], which is a contradiction. So ${}^2G_2(3^{2n+1})$ is not a PCTI-group. Finally, Lemma 8 shows that the Suzuki groups $Sz(q)$, $q = 2^{2n+1}$, $n \geq 1$, are not PCTI-groups.

Acknowledgements This paper was supported by the NSF of China (Grant No. 12161035).

References

1. Abdollahi A., Mousavi H. Finite nilpotent groups whose cyclic subgroups are TI-subgroups. *Bull. Malays. Math. Sci. Soc.*, 2017, **40** (4), pp. 1577-1589.
2. Alperin J.L., Brauer R., Gorenstein D. Finite groups with quasi-dihedral and wreathed Sylow 2-subgroups. *Trans. Amer. Math. Soc.*, 1970, **151**, pp. 1-261.
3. Barry M.J.J., Ward M.B. Simple groups contain minimal simple groups. *Publ. Mat.*, 1997, **41** (2), pp. 411-415.
4. Berkovich Y., Janko Z. *Groups of Prime Power Order*, **3**. De Gruyter, Berlin, 2011.
5. Collins M.J. A characterisation of the unitary groups $U_3(2^n)$. *Bull. London Math. Soc.*, 1971, **3**, pp. 195-196.
6. Collins M.J. The characterization of finite groups whose Sylow 2-subgroups are of type $L_3(q)$, q even. *J. Algebra*, 1973, **25** (3), pp. 490-512; Erratum, ibid, 1973, **27** (1), pp. 199-203.

7. Conway J.H., Curtis R.T., Norton S.P., Parker R.A., Wilson R.A. $\mathbb{ATLAS}$ *of Finite Groups. Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford Univ. Press, Eynsham, 1985.
8. Craven D.A. The maximal subgroups of the exceptional groups $F_4(q)$, $E_6(q)$ and $^2E_6(q)$ and related almost simple groups. *Invent. Math.*, 2023, **234** (2), pp. 637-719.
9. Dedekind R. Ueber Gruppen, deren sämmtliche Theiler Normaltheiler sind. *Math. Ann.*, 1897, **48** (4), pp. 548-561 (in German).
10. Gorenstein D. *Finite Groups.* Chelsea Publ. Co., New York, 1980.
11. Guo X., Li Sh., Flavell P. Finite groups whose abelian subgroups are TI-subgroups. *J. Algebra*, 2007, **307** (2), pp. 565-569.
12. Héthelyi L., Horváth E., Petényi F. The depth of the maximal subgroups of Ree groups. *Comm. Algebra*, 2019, **47** (1), pp. 37-66.
13. Higman G. Suzuki 2-groups. *Illinois J. Math.*, 1963, **7** (1), pp. 79-96.
14. Janabi H.A., Héthelyi L., Horváth E. TI subgroups and depth 3 subgroups in simple Suzuki groups. *J. Group Theory*, 2021, **24** (3), pp. 601-617.
15. Lüneburg H. *Translation Planes.* Springer-Verlag, Berlin-New York, 1980.
16. McKay J., Wales D. The multipliers of the simple groups of order $604,800$ and $50,232,960$. *J. Algebra*, 1971, **17** (2), pp. 262-272.
17. Mousavi H., Rastgoo T., Zenkov V. The structure of non-nilpotent CTI-groups. *J. Group Theory*, 2013, **16** (2), pp. 249-261.
18. The GAP Group. *GAP: Groups, Algorithms and Programming.* Version 4.4, 2005. http://www.gap-system.org.
19. Walls G.L. Trivial intersection groups. *Arch. Math.*, 1979, **32** (1), pp. 1-4.
20. Wilson R.A. *The Finite Simple Groups.* Graduate Texts in Math., **251**. Springer Science & Business Media, 2009.