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THE UNIQUENESS OF RECONSTRUCTION OF DIFFUSION OPERATORS WITH BOUNDARY CONDITION DEPENDING QUADRATICALLY ON THE SPECTRAL PARAMETER

I.M. NABIEV*, L.I. MAMMADOVA, G.S. MAMMEDZADEH

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Abstract. *The article considers diffusion operators with non-separated boundary conditions, one of which contains a quadratic function of the spectral parameter. Some properties of the spectrum of the operators under consideration are given, uniqueness theorems for the solution of inverse problems of reconstruction of the corresponding boundary value problems are proved.*

Keywords: diffusion operator, non-separated boundary conditions, spectrum, inverse problem

Mathematics Subject Classification (2020): 34A55, 34B24, 34L05, 47E05

1. Introduction

Inverse spectral problems for differential operators have been actively studied since the last century in connection with numerous applications [1], [8], [23], [30], [32], [34]. They consist of reconstructing operators from their spectral data. A huge number of papers have been published on the theory of inverse problems. The most complete review of

* Corresponding author.

Ibrahim M. Nabiev
Baku State University, Baku, Azerbaijan;
Institute of Mathematics and Mechanics, Baku, Azerbaijan;
Azerbaijan Technical University, Baku, Azerbaijan;
Khazar University, Baku, Azerbaijan
E-mail: nabievim@yahoo.com

Leyla I. Mammadova
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
E-mail: leylaimae@yahoo.com

Guldane S. Mammedzadeh
Azerbaijan State University of Economics (UNEC), Baku, Azerbaijan
E-mail: guldanemammedzade6@gmail.com

the results obtained in this direction can be found in the works [4], [14], [20], [24]–[26], [33], and [36], which also contain extensive lists of literature in this area. Since the seventies of the last century, researchers have been intensively studying boundary value problems with non-separated boundary conditions. Currently, the range of such tasks is expanding in various directions. Among them, problems with spectral parameters in boundary conditions occupy a special place.

The aim of this paper is to prove uniqueness theorems related to the inverse spectral problem for the diffusion equation with non-separated boundary conditions, one of which contains a quadratic function of the spectral parameter. Note that some versions of inverse problems for the Sturm–Liouville and diffusion equations with boundary conditions depending on the spectral parameter were studied in [2]–[7], [9]–[12], [15], [16], [19], [21], [26]–[29], [31], [35].

2. Asymptotics of the Spectrum of Boundary Value Problems

Consider the boundary value problems generated on the interval $[0,\pi]$ by the diffusion differential equation

$$y''(x) + \left[\lambda^2 - 2\lambda p\ (x) - q\ (x)\right]\ y(x) = 0 \tag{1}$$

and boundary conditions

$$\begin{aligned} &a(\lambda)y(0) + y'(0) + \omega y(\pi) = 0, \\ &-\overline{\omega}y(0) + \gamma y(\pi) + y'(\pi) = 0 \end{aligned} \tag{2}$$

or

$$\begin{aligned} &y(0) - \ y(\pi) = 0, \\ &y'(0) - a(\lambda)y(\pi) - y'(\pi) = 0, \end{aligned} \tag{3}$$

where $p(x) \in W_2^1[0,\pi]$, $q(x) \in L_2[0,\pi]$ are real functions, $a(\lambda) = m\lambda^2 + \alpha\lambda + \beta$, λ is the spectral parameter, ω is complex, $m, \alpha,\ \beta,\ \gamma$ are real numbers. We denote by $W_2^n[0,\pi]$ the Sobolev space of complex-valued functions on the interval $[0,\pi]$ which have $n-1$ absolutely continuous derivatives and square-summable nth derivative on $[0,\pi]$. In the future, the boundary value problem (1), (2) will be denoted by P, and the problem (1), (3) – by B.

Let $c(x,\lambda)$, $s(x,\lambda)$ be the fundamental system of solutions of equation (1), determined by the initial conditions

$$c(0,\lambda) = s'(0,\lambda) = 1, \quad c'(0,\lambda) = s(0,\lambda) = 0.$$

The Wronskian of these solutions is identically equal to one:

$$\begin{vmatrix} c(x,\lambda) & s(x,\lambda) \\ c'(x,\lambda) & s'(x,\lambda) \end{vmatrix} = c(x,\lambda)s'(x,\lambda) - s(x,\lambda)c'(x,\lambda) \equiv 1. \tag{4}$$

Using this identity, it is easily established that the characteristic functions whose zeros are the eigenvalues of boundary value problems P and B respectively, have the form

$$\Delta(\lambda) = 2\mathrm{Re}\,\omega - \eta(\lambda) + |\omega|^2 s(\pi,\lambda) + a(\lambda)\sigma(\lambda), \tag{5}$$

$$\delta(\lambda) = c(\pi,\lambda) + a(\lambda)s(\pi,\lambda) + s'(\pi,\lambda) - 2, \tag{6}$$

where

$$\eta(\lambda) = c'(\pi,\lambda) + \gamma c(\pi,\lambda),\ \sigma(\lambda) = s'(\pi,\lambda) + \gamma s(\pi,\lambda). \tag{7}$$

It is known [13] that the following representations are valid for the functions $c(\pi,\lambda)$, $c'(\pi,\lambda)$, $s(\pi,\lambda)$ and $s'(\pi,\lambda)$:

$$c(\pi,\lambda) = \cos\pi(\lambda-a) - c_1\frac{\cos\pi(\lambda-a)}{\lambda} + \pi a_1\frac{\sin\pi(\lambda-a)}{\lambda} + \frac{1}{\lambda}\int_{-\pi}^{\pi}\psi_1(t)e^{i\lambda t}dt,$$

$$c'(\pi,\lambda) = -\lambda\sin\pi(\lambda-a) - c_0\sin\pi(\lambda-a) + \pi a_1\cos\pi(\lambda-a) + \frac{1}{\lambda}\int_{-\pi}^{\pi}\psi_2(t)e^{i\lambda t}dt,$$

$$s(\pi,\lambda) = \frac{\sin\pi(\lambda-a)}{\lambda} + c_0\frac{\sin\pi(\lambda-a)}{\lambda^2} - \pi a_1\frac{\cos\pi(\lambda-a)}{\lambda^2} + \frac{1}{\lambda^2}\int_{-\pi}^{\pi}\psi_3(t)e^{i\lambda t}dt,$$

$$s'(\pi,\lambda) = \cos\pi(\lambda-a) - c_1\frac{\cos\pi(\lambda-a)}{\lambda} + \pi a_1\frac{\sin\pi(\lambda-a)}{\lambda}\frac{1}{\lambda}\int_{-\pi}^{\pi}\psi_4(t)e^{i\lambda t}dt,$$

where $a = \frac{1}{\pi}\int_0^\pi p(t)\mathrm{d}t$, $a_1 = \frac{1}{2\pi}\int_0^\pi [q(t)+p^2(t)]\mathrm{d}t$,

$$c_0 = \frac{1}{2}\left[p\ (0) + p\ (\pi)\right],\ c_1 = \frac{1}{2}\left[p\ (0) - p\ (\pi)\right],\ \psi_m(t) \in L_2[-\pi,\pi],\ m = 1,2,3,4.$$

By virtue of these representations and the Paley–Wiener theorem for characteristic functions (5) and (6), we obtain the following representations:

$$\begin{aligned}\Delta(\lambda) = m\lambda^2\cos\pi(\lambda-a) + \lambda\,[(1+m\pi\alpha_1+m\gamma)\sin\pi(\lambda-a)+\\ +(mc_1+\alpha)\cos\pi(\lambda-a)+g_1(\lambda)]+\\ +(\alpha\pi a_1 - c_0 + mc_0\gamma + \alpha\gamma)\sin\pi(\lambda-a)+\\ +(\alpha c_1 - \pi a_1 - \gamma + \beta - m\gamma\pi a_1)\cos\pi(\lambda-a) + g_2(\lambda) + 2\mathrm{Re}\,\omega,\end{aligned} \tag{8}$$

$$\begin{aligned}\delta(\lambda) = m\lambda\sin\pi(\lambda-a) + (2 - ma_1\pi)\cos\pi(\lambda-a)+\\ +(mc_0+\alpha) + \sin\pi(\lambda-a) + g_3(\lambda) - 2,\end{aligned} \tag{9}$$

where $g_l(\lambda) = \int_{-\pi}^{\pi}\tilde{g}_l(t)e^{i\lambda t}\mathrm{d}t$, $\tilde{g}_l(t) \in L_2[-\pi;\pi]$, $l = 1,2,3$. According to the work [22], the eigenvalues μ_k, γ_k $(k = \pm 0, \pm 1, \pm 2, ...)$ of boundary value problems P and B at $|k| \to \infty$ obey the asymptotics

$$\mu_k = k - \frac{1}{2}\mathrm{sign}k + a + \frac{A}{m\pi k} + \frac{m_k}{k}, \tag{10}$$

$$\gamma_k = k + a + \frac{a_1}{k} + \frac{2\left[(-1)^k - 1\right]}{\pi m k} + \frac{\tau_k}{k}, \tag{11}$$

where $A = 1 + m\,(\pi a_1 + \gamma)$, $\{m_k\}, \{\tau_k\} \in l_2$.

Along with problems P and B, boundary value problems generated by the same equation (1) and boundary conditions are also considered

$$y(0) = y'(\pi) + \gamma\, y(\pi) = 0, \tag{12}$$

$$y(0) = \ y(\pi) = 0. \tag{13}$$

Let $\{\theta_k\}$ and $\{\lambda_k\}$ $(k = \pm 1, \pm 2, ...)$ be the spectra of problems (1), (12) and (1), (13), i.e. the sequences of zeros of functions $\sigma(\lambda)$ (see (7)) and $s(\pi, \lambda)$, respectively. For these zeros at $|k| \to \infty$, the following asymptotic formulas hold [13]

$$\theta_k = k - \frac{1}{2}\text{sign } k + a + \frac{a_1\pi + \gamma}{k\pi} + \frac{\xi_k}{k}, \quad \{\xi_k\} \in l_2. \tag{14}$$

$$\lambda_k = k + a + \frac{a_1}{k} + \frac{\zeta_k}{k}, \ \{\zeta_k\} \in l_2. \tag{15}$$

Hereinafter, we will assume that the number ω and the terms of the sequences $\{\mu_k\}$, $\{\gamma_k\}$, $\{\theta_k\}$ and $\{\lambda_k\}$ are real (the conditions for the realness of eigenvalues are given in [18], [22]).

3. Statements of Inverse Problems. Uniqueness Theorem

Let us denote $\sigma_k = \text{sign }\left[1 - |\omega\ s\,(\pi,\ \theta_k)|\right]$. First, let us consider the following inverse problem.

Inverse problem 1. Given sequences $\mu_k\,(k = \pm 0, \pm 1, \pm 2, ...)\,, \theta_k, \sigma_k\;(k = \pm 1, \pm 2, ...)$ and numbers β, ω, construct coefficients $p(x)$ and $q(x)$ of equation (1) and parameters m, α, γ of boundary conditions (2).

The following uniqueness theorem is valid.

Theorem 1. *If $p(0) = p(\pi)$, then the assignment of values $\mu_k\,(k = \pm 0, \pm 1, \pm 2, ...)\,, \theta_k$, $\sigma_k\ (k = \pm 1, \pm 2, ...)$, β, ω uniquely determines functions $p(x)$ and $q(x)$ and parameters m, α, γ.*

Proof. From the asymptotic formula (14) we find

$$a = \lim_{k\to+\infty}\left(\theta_k - k + \frac{1}{2}\right), \quad a_1\pi + \gamma = \pi \lim_{k\to+\infty} k\left(\theta_k - k + \frac{1}{2} - a\right).$$

By virtue of (10), for $k > 0$ we have

$$\frac{1}{m\pi k} = \mu_k - k + \frac{1}{2} - a - \frac{a_1\pi + \gamma}{\pi k} - \frac{m_k}{k}.$$

Knowing the values μ_k, a and $a_1\pi+\gamma$, we can recover parameter m using the formula

$$m = \frac{1}{-a_1\pi - \gamma + \pi \lim\limits_{k\to\infty} k\left(\mu_k - k + \frac{1}{2} - a\right)}.$$

Given sequence $\{\mu_k\}$ and parameter m, function $\Delta(\lambda)$ can be uniquely reconstructed in the form of an infinite product as follows:

$$\Delta(\lambda) = m(\mu_{-0}-\lambda)(\mu_{+0}-\lambda)\prod_{\substack{k=-\infty\\k\neq0}}^{\infty}\frac{\mu_k-\lambda}{k-\frac{1}{2}\operatorname{sign}k}.$$

For $\lambda = 2k+a$ from (8) we get

$$\Delta(2k+a) = m(2k+a)^2 + (2k+a)[mc_1+\alpha+g_1(2k+a)]+$$

$$+\alpha c_1 - \pi a_1 - \gamma + \beta - m\gamma\pi a_1 + g_2(2k+a) + 2\omega.$$

Therefore, parameterα is determined by the formula

$$\alpha = \lim_{k\to\infty}\frac{\Delta(2k+a) - m(2k+a)^2}{2k},$$

since by virtue of the condition of the theorem $c_1 = 0$ and the Riemann-Lebesgue lemma $\lim\limits_{k\to\infty} g_j(2k+a) = 0,\ j=1,2,$

Denote

$$v_+(\lambda) = -\eta(\lambda) + \omega^2 s(\pi,\lambda), \tag{16}$$

$$v_-(\lambda) = -\eta(\lambda) - \omega^2 s(\pi,\lambda). \tag{17}$$

By virtue of (5), function$v_+(\lambda)$ is recovered by the formula

$$v_+(\lambda) = \Delta(\lambda) - a(\lambda)\sigma(\lambda) - 2\omega,$$

where $\sigma(\lambda) = \prod\limits_{\substack{k=-\infty\\k\neq0}}^{\infty}\frac{\theta_k-\lambda}{k-\frac{1}{2}\operatorname{sign}k}$. According to relations (4) and (7) we have

$$c(\pi,\lambda)\sigma(\lambda) - s(\pi,\lambda)\eta(\lambda) = 1.$$

From here, when $\lambda = \theta_k$, we get

$$s(\pi,\theta_k)\,\eta(\theta_k) = -1. \tag{18}$$

Taking this equality into account, from relations (16), (17) it is easy to obtain that

$$v_-^2(\theta_k) - v_+^2(\theta_k) = -\ 4\omega^2$$

and therefore

$$v_-(\theta_k) = \operatorname{sign}\ v_-(\theta_k)\sqrt{v_+^2(\theta_k) - 4\omega^2}. \tag{19}$$

According to (17) and (18)

$$v_-(\theta_k) = \frac{1}{s(\theta_k,\pi)} - \omega^2 s(\theta_k,\pi) = \frac{1-\omega^2 s^2(\theta_k,\pi)}{s(\theta_k,\pi)}. \tag{20}$$

Using the relative positions of the zeros of functions $s(\lambda,\pi)$ and $\sigma(\lambda)$, it is easy to prove that sign $s(\theta_k,\pi) = (-1)^{k+1}$. Then, by virtue of (19) and (20), we have

$$v_-(\theta_k) = (-1)^{k+1}\sigma_k\sqrt{v_+^2(\theta_k) - 4\omega^2}.$$

Consider the function

$$g(\lambda) = v_+(\lambda) - v_-(\lambda) - 2\omega^2\frac{\sin\pi(\lambda-a)}{\lambda}. \tag{21}$$

It is easy to see that $\sigma(\lambda)$ is an entire sine-type function and $\{g(\theta_k)\} \in l_2$. Then, according to Theorem 1 in the book [17, p. 165], the following interpolation formula is valid for the function $g(\lambda)$

$$g(\lambda) = \sigma(\lambda)\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\frac{g(\theta_k)}{(\lambda-\theta_k)\sigma'(\theta_k)}, \tag{22}$$

where $g(\theta_k) = v_+(\theta_k) - (-1)^{k+1}\sigma_k\sqrt{v_+^2(\theta_k) - 4\,\omega^2} - 2\omega^2\,\frac{\sin\theta_k\pi}{\theta_k}$. Therefore, as can be seen from (22), in addition to spectra $\{\mu_k\}$, $\{\theta_k\}$ and numbers β, ω, it is sufficient to specify sequence $\{\sigma_k\}$ in order to determine function $v_-(\lambda)$ from (21).

Functions $s(\pi,\lambda)$ and $s'(\pi,\lambda)$ and parameter γ are recovered as in the proof of Theorem 2.1 in [13]. It is known [13], [14] that the functions $p(x)$ and $q(x)$ are uniquely determined by the zeros of these functions.

Thus, from sequences $\{\mu_k\}$, $\{\theta_k\}$ and numbers β, ω, both coefficients $p(x)$ and $q(x)$ of equation (1) and parameters m, α, γ of boundary conditions (2) are uniquely recovered. The theorem is proven. ◄

Let us now consider the inverse problem of reconstructing the boundary value problem B. Denote $\rho_k = \text{sign}\,[1-|s'(\pi,\lambda_k)|]$, $k=\pm1,\pm2,\ldots$.

Inverse problem 2. Knowing sequences $\{\gamma_k\}$, $\{\lambda_k\}$, $\{\rho_k\}$ and number β, recover boundary value problem B.

The following uniqueness theorem is valid.

Theorem 2. *If $p(0) = -p(\pi)$, then the assignment of values $\gamma_k\,(k=\pm0,\pm1,\pm2,\ldots)$, $\lambda_k,\rho_k\,(k=\pm1,\pm2,\ldots)$, β uniquely determines functions $p(x)$ and $q(x)$* and parameters *m,α.*

Proof. According to the asymptotic formula (11) we have

$$\gamma_{2k} = 2k + a + \frac{a_1}{2k} + \frac{\tau_{2k}}{2k},$$

$$\gamma_{2k+1} = 2k + 1 + a + \frac{a_1}{2k} - \frac{2}{\pi mk} + \frac{\eta_k}{k}, \quad \{\eta_k\} \in l_2.$$

From here, parameter m can be determined by the formula

$$m = \frac{2}{\pi} \lim_{k\to\infty} \frac{1}{k(\gamma_{2k} + \gamma_{2k+1} + 1)}.$$

Knowing the spectrum $\{\gamma_k\}$ and the parameter m, the characteristic function $\delta(\lambda)$ can be recovered in the form of an infinite product as follows:

$$\delta(\lambda) = m\pi(\gamma_{-0} - \lambda)(\gamma_{+0} - \lambda) \prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\gamma_k - \lambda}{k}$$

(see [18]).

By virtue of the asymptotic formula (15), $a = \lim\limits_{k\to+\infty} (\lambda_k - k)$ takes place. From the representation (9) for $\lambda = 2k + \frac{1}{2} + a$ we obtain

$$\delta\left(2k + \frac{1}{2} + a\right) = \left(2k + \frac{1}{2} + a\right) m + \mathrm{mc}_0 + \alpha + g_3\left(2k + \frac{1}{2} + a\right) - 2. \tag{23}$$

Since $p(0) + p(\pi) = 0$ by the condition of the theorem, then $c_0 = 0$. Then from (23) by virtue of the Riemann-Lebesgue lemma it follows that

$$\alpha = \lim_{k\to\infty} \left[\delta\left(2k + \frac{1}{2} + a\right) - \left(2k + \frac{1}{2} + a\right) m + 2\right].$$

Consider the function

$$u_+(\lambda) = c(\pi, \lambda) + s'(\pi, \lambda), \tag{24}$$

$$u_-(\lambda) = c(\pi, \lambda) - s'(\pi, \lambda). \tag{25}$$

We find function $u_+(\lambda)$ from relation (6):

$$u_+(\lambda) = \delta(\lambda) - a(\lambda)s(\pi, \lambda) + 2,$$

where $s(\pi, \lambda)$ is recovered by the formula

$$s(\pi, \lambda) = \pi \prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\lambda_k - \lambda}{k}.$$

Since $\lambda_k, k = \pm1, \pm2, ...$ are the zeros of the function $s(\pi, \lambda)$, it follows from identity (4) that

$$c\,(\pi, \lambda_k)\, s'\,(\pi, \lambda_k) = 1. \tag{26}$$

Reasoning as in the proof of Theorem 1 and using relations (24)–(26) and Theorem 1 in [17, p. 165], we obtain

$$u_-\,(\lambda_k) = (-1)^k \rho_k\,\sqrt{u_+^2\,(\lambda_k) - 4},$$

$$u_{-}(\lambda) = s(\pi,\lambda) \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{u_{-}(\lambda_k)}{(\lambda-\lambda_k)\frac{\partial s(\pi,\lambda_k)}{\partial \lambda}}. \tag{27}$$

The uniqueness of the constructed function $u_{-}(\lambda)$ follows from the fact that the interpolation formula (27) establishes a one-to-one correspondence between l_2 and the space of entire functions of exponential type not exceeding π, belonging to $L_2(-\infty,\infty)$.

Therefore, the characteristic function $s'(\pi,\lambda)$ of the boundary value problem generated by equation (1) and boundary conditions $y(0) = y'(\pi) = 0$ is recovered by the formula

$$s'(\pi,\lambda) = \frac{1}{2}\left[u_{+}(\lambda) - u_{-}(\lambda)\right].$$

It is known [13], [14] that the functions $p(x)$ and $q(x)$ are uniquely determined from the zeros of this function and the sequence $\{\lambda_n\}$.

Thus, given sequences $\{\gamma_k\}$, $\{\lambda_k\}$, $\{\rho_k\}$ and number β, coefficients $p(x)$ and $q(x)$ of equation (1) and parameters α, m of boundary conditions (3) are uniquely determined, i.e., boundary value problem B is completely reconstructed. The theorem is proven.

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