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APPLICATION OF THE FINITE INTEGRAL TRANSFORMATION METHOD TO THE SOLUTION OF MIXED PROBLEMS FOR ANTIPARABOLIC EQUATIONS

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Abstract. *In the paper we study a mixed problem for an antiparabolic equation. Applying the finite integral transformation to the solution of the mixed problem under consideration, we obtain an appropriate spectral problem. Under certain conditions we study the spectral problem and establish some expansion formulas. Applying the inverse integral transformation, we obtain the analytic representation of the mixed problem under consideration.*

Keywords: antiparabolic equation, finite integral transformation method, transformation formulas, analytical expression for the solution

Mathematics Subject Classification (2020): 35K10, 35K20

1. Introduction

In all its theoretical and applied importance, the mixed problems for integration (of some) partial linear differential equations for the given boundary and initial conditions (Problem 1) are one of the current problems of mathematics. For solving Problem 1, one of the convenient but mathematically unreasonable apparatus, was symbolic calculus and the method of separation of variables, and also the integral methods

$$\tilde{f}(\lambda) = \int_0^\infty \exp(-\lambda t) f(t) dt, \tag{1}$$

the Laplace transform,

$$\tilde{f}(\lambda) = \int_0^\infty \exp(-\lambda^2 t) f(t) dt, \tag{2}$$

the Rasulov transform (see [20]) and others. whose application leads to a parametric (often called the spectral problem) problem (Problem 2).

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Integral transformations (1) or (2) does not allow a strictly mathematical justification for transition from Problem 1 to Problem 2 in the sense that Problem 2 is studied along the entire complex plane (for all complex numbers λ), and for $Re(-\lambda) > 0$ the integral (1) and for $Re(-\lambda^2) > 0$ the integral (2) “generally speaking” diverge. At the eighties of the XX century we have proposed a method for applying the finite (limits of integration of finite numbers) integral transformation [5]

$$K_f = \int_{\varepsilon}^{t} \exp(-\lambda\tau) f(\tau) d\tau, \tag{3}$$

to the solution of a wide range of (regular and irregular, integro-differential, etc) Problems 1.

We have successfully applied finite integral transformations (3) to the problems of conjugation of different order hyperbolic systems [10], mixed problems of conjugation of different order parabolic systems with nonlocal boundary conditions [7], a mixed problem of conjugation of hyperbolic and parabolic systems mixed [4], problems with integro-differential conditions for non-classic equations [6], [8], [9], mixed problems with more general (irregular, integro-differential, order of boundary conditions exceeding the order of the equation and so on) boundary conditions . In [5] we have offered some parabolic potentials that allow to solve mixed problems (when the higher derivatives are involved in the boundary conditions) for second order parabolic equations in the domains with curvilinear lateral boundaries.

We suggested some generalization of the method of separation of variables and showed its application to the solution of mixed problems with irregular boundary conditions.

If we study a not self-adjoint mixed problem with separating variables for the equitation

$$Z_1\left(\frac{\partial}{\partial t}\right) u = Z_2\left(\frac{\partial}{\partial x}\right) u, \tag{4}$$

then by the Fourier method we look for the non-trivial particular solution of the equation (4) in the form

$$u(x,t) = T(t)X(x), \tag{5}$$

and to determine $X(x)$ we obtain some spectral problem dependent on the parameter λ. Let $\Delta(\lambda)$ be the denominator of the Green function of this spectral problem. Denote by m the greatest multiplicity of iteration of the roots of the equation $\Delta(\lambda) = 0$. We can show that for $m \geq 2$ by the Fourier method it is impossible to solve this mixed problem. For $m \geq 2$ the particular non-trivial solution of the equation (4), unlike (5), is sought in the form

$$u(x,t) = \exp(\lambda^n t)\left[Z_0(x) + tZ_1(x) + ... + t^{m-1} Z_{m-1}(x)\right],$$

where n is the order of the equation, λ is a parameter, $Z_0(x), Z_1(x), ..., Z_{m-1}(x)$ are some functions satisfying the given boundary conditions of the mixed problem. In what follows, we can find a number of particular non-trivial solutions $u = u_k(x, t, \lambda_k)$ and then the solution $u = u(x,t)$ of the mixed problem is represented in the form

$$u(x,t) = \sum_k C_k u_k(x, t, \lambda_k),$$

where the unknown coefficients C_k are determined from the initial condition of the mixed problem.

As we have noted, for solving problems we need to solve the appropriate spectral problem and obtain an expansion formula of an arbitrary function (from some class) in eigen (and adjoint) functions of the appropriate spectral problem. Step by step the study of the spectral problem (in bounded and in unbounded domains) has an independent character, and a number of works have been devoted (see for example [1], [3], [12]-[14], [16]-[32]) to this study. To study a spectral problem, it is necessary to obtain the asymptotic representation of the system of fundamental (independent) particular solutions of a homogeneous parametric equation that corresponds to the equations of the spectral problem. A number of works (see for example [2], [18], [20], [27]) have been devoted to finding the asymptotics of the system of fundamental (independent) particular solutions of a homogeneous parametric equation.

In [11] we have studied the asymptotics of the system of fundamental (independent) particular solutions of the parametric equation

$$\sum_{j=0}^{2} a_j(x) \frac{d^j y}{dx^j} - \lambda^2 y = 0, \quad x \in [0,1], \tag{6}$$

$$a_j(x) \in C^{k+j}\left([0,1]\right),$$

$$a_2(x) = a(x) + \sqrt{-1}b(x), \quad a(x) > 0,$$

under the constraints

$$\frac{b(x)}{a(x)} \neq const. \tag{7}$$

The constraint (7) is the main constraint that distinguishes the work [11] from [2], [18], [20], [27]. In [11] (in some part of the λ -plane) the asymptotic representation is obtained for the system of fundamental particular solutions of equation (6), and they are successfully used for solving mixed problems for parabolic equations.

Let us consider the heat-conductivity equation

$$\frac{\partial u}{\partial t} = q \frac{\partial^2 u}{\partial x^2}, \; x \in (0,1), \; t \in (0,T).$$

In ordinary materials, the coefficient q is always positive (energy dissipates). The negative coefficient ($q < 0$) would mean that heat is spontaneously concentrated in hot spots that violates the second low of thermodynamics. Such effects are encountered only in specific composite materials whose properties are determined by resonance properties of its constituent elements, so called meteoatoms, but not by a periodic structure that determines the functional of photonic crystals or active systems with external influx of energy [15]

2. Main Results

1. We study mixed problem (8)-(10) for antiparabolic equation (8). The significant difference of an antiparabolic equation from parabolic equations (in the sense of I. G. Petrovsky) is that the heat conductivity factor on multiparabolic equations is negative , that make it quite difficult to solve such problems.

2. Boundary conditions involve higher derivatives $\left(\frac{\partial u}{\partial t}\right)$ and the derivatives exceeding the order of the equation $\left(\frac{\partial^2 u}{\partial t \partial x}\right)$.

3. It is impossible to solve the considered mixed problem for antiparabolic equations by the Rasulov contour integral method [20] (by integral transformation (2) and the Laplace integral transformation (1).

4. For solving the considered mixed problem (8)-(10), by using the integral transformation (11), the method of finite integral transformation [5] is applied.

5. For all complex numbers λ because of finiteness of the integration limits in (11) make it possible to strictly mathematically justify the transitions from the mixed problem (8)-(10) to parametric problem (12), (13) (or (15), (16)).

6. Under certain conditions, it is proved that if the function $u = u(x,t)$ is a classic solution of problem (8)-(10), it satisfies the integro-parametric equation (21) (see Theorem 1).

7. Theorems 2, 3, 4, 5 and 6 are established for the parametric (spectral) problem (15), (16).

8. Using the results of Theorems 2-6, under certain conditions it is proved that if for the antiparabolic equation the mixed problem (8)-(10) has a classic solution $u = u(x,t)$, then

i) this solution is unique;

ii) this solution is represented by formula (29);

iii) if the limit involved in (29) does not exist, or it exists but the function $u = u(x,t)$ determined by formula (29) is not a class solution of problem (8)-(10), this problem does not have a classic solution (see Theorem 7).

9. Imposing definite constraints on the data of problem (8)-(10) it is proved that the function $u = u(x,t)$ determined by formula (29) (or (30)) (for antiparabolic equation (8), in fact is a classic solution of the considered mixed problem (8)-(10) (see Theorem 8).

3. Problem Statement

Find the classic solution of the following antiparabolic equation

$$\frac{\partial u}{\partial t} = -lu + F(x,t), \quad 0 < x < 1, \quad 0 < t < T, \tag{8}$$

satisfying the boundary conditions

$$S_i\left(\frac{\partial}{\partial t}\right) u \equiv \left(\alpha_{i0} u + \alpha_{i1}\frac{\partial u}{\partial x} + \alpha_{i2}\frac{\partial u}{\partial t} + \alpha_{i3}\frac{\partial^2 u}{\partial x \partial t}\right)_{x=0} +$$

$$+\left(\beta_{i0}u+\beta_{i1}\frac{\partial u}{\partial x}+\beta_{i2}\frac{\partial u}{\partial t}+\beta_{i3}\frac{\partial^2 u}{\partial x\partial t}\right)_{x=1}=\gamma_i(t),\quad 0<t<T,\ i=1,2, \tag{9}$$

and the initial condition

$$u|_{t=0}=f(x),\quad x\in(0,1), \tag{10}$$

where

$$lu=a_2(x)\frac{\partial^2 u}{\partial x^2}+a_1(x)\frac{\partial u}{\partial x}+a_0(x)u,$$

$T\ (T>0), \alpha_{is}, \beta_{is}$ are known numbers, $a_2(x), a_1(x), a_0(x), F(x,t), \gamma_i(t), f(x)$ are known functions, $u=u(x,t)$ is the desired classic solution.

1^0. Let $a_2(x)=p(x)\left(c_1+\sqrt{-1}c_2\right), p(x)>0,\ p(x)\in C^2([0,1])$; $c_1\ (c_1>0)$, c_2 be constant numbers; $a_1(x)\in C^1([0,1]), a_0(x)\in C([0,1])$.

2^0. Let $f(x)\in C^1([0,1])$, and the functions $F(x,t), \gamma_1(t), \gamma_2(t)$ be continuous for $0\le x\le 1, 0\le t\le T$.

Denote by U a class of functions $u=u(x,t)$ satisfying the constraints:

i) $u(x,t), \frac{\partial u(x,t)}{\partial x}$ are continuous for $0\le x\le 1,\ 0\le t<T$;

ii) $\frac{\partial u(x,t)}{\partial t}, \frac{\partial^2 u(x,t)}{\partial x^2}, \frac{\partial^2 u(x,t)}{\partial x\partial t}$ are continuous for $0\le x\le 1,\ 0<t<T$.

Definition. *We say that the function $u=u(x,t)$ is a classic solution of the mixed problem* (8)-(10) *if:*

1. $u(x,t)\in U$,
2. *the function $u=u(x,t)$ satisfies the equalities* (8)-(10) *in the usual sense.*

4. Solution

Let us assume a priori that the problem (8)-(10) has a classic solution and this solution $u=u(x,t)$ is compared in (8)-(10). Then, using the finite integral transformation

$$K_\varepsilon(\varphi)=\int_\varepsilon^t \exp(\lambda^2\tau)\varphi(\tau)d\tau, \tag{11}$$

(where $\varepsilon\ (\varepsilon>0)$ is some positive number, λ is any complex number) to (8)-(10), we have

$$\left(l-\lambda^2\right)\int_\varepsilon^t \exp(\lambda^2\tau)u(x,\tau)d\tau=\exp(\lambda^2\varepsilon)u(x,\varepsilon)-\exp(\lambda^2 t)u(x,t)+$$

$$+\int_\varepsilon^t \exp(\lambda^2\tau)F(x,\tau)d\tau,\ 0<x<1,\ 0<\varepsilon<t<T, \tag{12}$$

$$S_i(-\lambda^2)\int_\varepsilon^t \exp(\lambda^2\tau)u(x,\tau)d\tau=\tilde{\gamma}_i(\lambda,t,\varepsilon),\ \ 0<\varepsilon<t<T,\quad i=1,2, \tag{13}$$

where

$$\tilde{\gamma}_i(\lambda,t,\varepsilon)=\tilde{\gamma}_{i1}(\lambda,t,\varepsilon)-\exp(\lambda^2 t)v_i(t),$$

$$\tilde{\gamma}_{i1}(\lambda,t,\varepsilon)=\int_\varepsilon^t \exp(\lambda^2\tau)\gamma_i(\tau)d\tau+\exp(\lambda^2\varepsilon)\{\alpha_{i2}u(x,\ \varepsilon)/_{x=0}+$$

$$+\alpha_{i3}\frac{\partial u(x,\varepsilon)}{\partial x}\Big/_{x=0}+\beta_{i2}u(x,\varepsilon)/_{x=1}+\beta\ _{i3}\frac{\partial u(x,\varepsilon)}{\partial x}\Big/_{x=1}\Bigg\},$$

$$v_i(t)=\alpha_{i2}u(x,t)/_{x=0}+\alpha_{i3}\frac{\partial u(x,t)}{\partial x}\Big/_{x=0}+\beta_{i2}u(x,t)/_{x=1}+\beta_{i3}\frac{\partial u(x,t)}{\partial x}\Big/_{x=1}. \quad (14)$$

For solving the problem (12), (13) at first we consider the following appropriate parametric problem:

$$(l-\lambda^2)y=\psi(x), \quad x\in(0,1), \quad (15)$$

$$S_i(-\lambda^2)y=\chi_i, \quad i=1,2, \quad (16)$$

where $\psi(x)\in C([0,1])$ and χ_i are arbitrary numbers.

According [1]-[3], [12], [32] under constraints 1^0 of the homogeneous equation, corresponding to (15) for $|\lambda|\geq R$ (R is a rather large positive number) has a system of independent particular solutions $y_j(x,\lambda)$, $(j=1,2)$ represented by the formulas

$$\frac{d^k}{dx^k}y_j(x,\lambda)=\lambda^k\exp\left(\lambda\theta_j(x)\right)\left\{g_{j0}(x)\left(\frac{d}{dx}\theta_j(x)\right)^k+\frac{1}{\lambda}E_{jk}(x,\lambda)\right\},$$

$$x\in[0,1],\ |\lambda|\geq R,\ j=1,2,\ k=0,1,$$

where

$$\theta_j(x)=(-1)^j\int_0^x 1\Big/\sqrt{a_2(\xi)}\,d\xi,$$

$$g_{j0}(x)=\left(\sqrt{-1}\right)^j\sqrt[4]{a_2(x)}\exp\left(-\frac{1}{2}\int_0^x a_1(\xi)\,/a_2(\xi)\,d\xi\right),$$

$E_{jk}(x,\lambda)$ are some functions (with respect to x are continuously differentiable for $x\in[0,1]$ and with respect to λ is analytic for $|\lambda|\geq R$) satisfying the inequalities $|E_{jk}(x,\lambda)|\leq const$, for $x\in[0,1]$, $|\lambda|\geq R$.

Denote by $\Delta(\lambda)$ a denominator of the Green function $G(x,\xi,x)$ of the parametric problem (15), (16) (see [5], [18], [20], [27]).

According to [5], for

$$\Delta(\lambda)\neq 0, \quad (17)$$

the problem (15), (16) has a unique solution and it is represented by the formula

$$y(x,\lambda)=g(x,\lambda,\chi_1,\chi_2)+\int_0^1 G(x,\xi,\lambda)\psi(\xi)d\xi \quad (18)$$

(for the expressions $g(x,\lambda,\chi_1,\chi_2)$, $G(x,\xi,\lambda)$ and $\Delta(\lambda)$ (see [5]) in which explicit representations of these functions are given by the system of independent particular solutions $y_1(x,\lambda)$ and $y_2(x,\lambda)$).

Remark 1. Under condition (17) if for the function $y(x,\lambda)$ $(y(x,\lambda)\in C^2([0,1])$, $\psi(x)$, $(\psi(x)\in C([0,1]))$ and for the numbers χ_1,χ_2 the equalities (15), (16) are fulfilled, then the equality (18) is also fulfilled, and vice-versa.

Consequently, under condition (17), from (12), (13) according to formula (18) we obtain:

$$\int_{\varepsilon}^{t} \exp(\lambda^2\tau)u(x,\tau)d\tau = g(x,\lambda,\tilde{\gamma}_1(\lambda,t,\varepsilon),\tilde{\gamma}_2(\lambda,t,\varepsilon))+$$
$$+\int_0^1 G(x,\xi,\lambda)\left\{\exp(\lambda^2\varepsilon)u(\xi,\varepsilon)-\exp(\ \lambda^2 t)u(\xi,t)+\right.$$
$$\left.+\int_{\varepsilon}^{t}\exp(\lambda^2\tau)F(\xi,\tau)d\tau\right\}d\xi,\quad 0\le x\le 1,\quad 0<\varepsilon<t<T. \tag{19}$$

In (19) as $\varepsilon\to+0$ passing to limit and using the initial condition (10), we obtain

$$\int_0^t \exp(\lambda^2\tau)u(x,\tau)d\tau = -\exp(\lambda^2 t)g(x,\lambda,v_1(t),v_2(t))-$$
$$-\exp(\lambda^2 t)\int_0^1 G(x,\xi,\lambda)u(\xi,t)d\xi+\mathcal{F}(x,t,\lambda), 0\le x\le 1,\ 0<t<T, \tag{20}$$

where

$$\mathcal{F}(x,t,\lambda)=g(x,\lambda,\mu_1(\lambda,t),\mu_2(\lambda,t))+$$
$$+\int_0^1 G(x,\xi,\lambda)\left\{f(\xi)+\int_0^t\exp(\lambda^2\tau)F(\xi,\tau)d\tau\right\}d\xi$$

$$\mu_i(\lambda,t)=\int_0^t\exp(\lambda^2\tau)\gamma_i(\tau)d\tau+\alpha_{i2}f(0)+\alpha_{i3}f'(0)+\beta_{i2}f(1)+\beta_{i3}f'(1);(i=1,2).$$

Multiplying the both hand sides of (20) by $\lambda\exp(-\lambda^2 t)$ and grouping the addends, we obtain

$$\lambda\int_0^t\exp(-\lambda^2(t-\tau))u(x,\tau)d\tau+\lambda g(x,\lambda,v_1(t),v_2(t))+$$
$$+\lambda\int_0^1 G(x,\xi,\lambda)u(\xi,t)d\xi=\lambda\exp(-\lambda^2 t)\mathcal{F}(x,t,\lambda),\ 0\le x\le 1,\quad 0\le t<T. \tag{21}$$

Thus, we established the following theorem.

Theorem 1. *Under constraints 1^0, 2^0 and* (17) *if the function $u=u(x,t)$ is a classic solution to the mixed problem* (8)-(10)*, then it satisfies the integro-parametric equation* (21).

3^0. Let boundary conditions (16) for the equation (15) be regular [18], [20], [27].

According to [20], [27] we have the following theorem.

Theorem 2. *Under constraints 1^0 and 3^0, the characteristical determinant $\Delta(\lambda)$ of the Green function $G(x,\xi,\lambda)$ of problem* (15), (16) *has a denumerable set of zeros that can be arranged into 2μ groups. The values of the s-th group lie in the finite width strip Π_s, containing the ray d_s, and all these rays are arranged in the sectors*

$$\frac{\pi}{4}<\arg\lambda<3\frac{\pi}{4},$$

$$5\frac{\pi}{4} < \arg\lambda < 7\frac{\pi}{4}.$$

If from the λ-plane we throw out the interior of small circles of radius δcentered at the zeros of the determinant $\Delta(\lambda)$ (the set of all poles $G(x,\xi,\lambda)$ is the subset of all zeros $\Delta(\lambda)$), then in the remaining part we have the inequality

$$\left|\lambda^{-d}\Delta(\lambda)\exp(-\lambda\omega)\right| \geq N_\delta,$$

where N_δ is a positive number dependent only on δ.

The number of zeros approaching as they move away from the origin of coordinates of the λ-plane, is limited, moreover, the zeros $\lambda_n^{(s)}$ (of the s-th group) of the function $\Delta(\lambda)$ allow the asymptotic representation

$$\left|\lambda_n^{(s)}\right| = \frac{2n\pi}{m_{\sigma s} - m_{1s}}\left(1 + O\left(\frac{1}{n}\right)\right).$$

The zeros of the characteristical determinant $\Delta(\lambda)$ are the poles of the solution $y(x,\lambda)$ of the problem (15), (16).

Using the results of Theorem 2 in [5], [18], [20], [27] it is shown that all the zeros of the characteristical determinant $\Delta(\lambda)$ can be surrounded by the circles O_δ of small radius δ centered in the zeros of this determinant, and we can chose a sequence of closed expanding contours of non-intersecting circles O_δ (this time $mes\Gamma_\nu = O(r_\nu)$, r_ν is the distance of the nearest point of the contour Γ_ν from the beginning of the λ plane), the following expansion theorem is valid.

Theorem 3. *Under constraints 1^0 and 3^0 for any constraints and Hölder continuous (except finitely many points in which it can have discontinuities of first kind) $\psi(x)$ with respect to $x \in [0,1]$ for $0 < x < 1$ we have the following expansion formula:*

$$-\frac{1}{2\pi\sqrt{-1}}\lim_{\nu\to\infty}\int_{\Gamma_\nu}\lambda d\lambda\int_0^1 G(x,\xi,\lambda)\psi(\xi)d\xi = \frac{1}{2}\left(\psi(x-0)+\psi(x+0)\right). \qquad (22)$$

Remark 2. The Green function $G(x,\xi,\lambda)$ can be represented in the form [5]

$$G(x,\xi,\lambda) = P(x,\xi,\lambda) + Q(x,\xi,\lambda),$$

where $P(x,\xi,\lambda)$ is a fundamental solution of equation (15) (see for example [5]) and it is completely unrelated to the boundary conditions (16). For example, for the equation

$$y'' - \lambda^2 y = \psi(x),\ x \in (0,1),$$

the fundamental solution $P(x,\xi,\lambda)$ will be

$$P(x,\xi,\lambda) = \begin{cases} -\frac{1}{2\lambda}\exp(-\lambda\left|x-\xi\right|) & for \quad Re\lambda \geq 0, \\ \frac{1}{2\lambda}\exp(\lambda\left|x-\xi\right|) & for \quad Re\lambda \leq 0. \end{cases}$$

Let r_ν be an arbitrary positive number satisfying the condition $0 < r_1 < r_2 < ...$; and $\lim\limits_{\nu\to\infty} r_\nu = \infty$. In what follows, let $O_\nu = \{\lambda : |\lambda| = r_\nu\}$.

We have the following lemma on inversion formula.

Lemma 1. *Under constraints 1^0 (if desired this constraint can be relaxed) for any constraints and Hölder continuous (except finitely many points in which it may have first order discontinuity) $\psi(x)$ with respect to $x \in [0,1]$, for $0 < x < 1$ we have the following inversion formula:*

$$-\frac{1}{2\pi\sqrt{-1}} \lim_{\nu\to\infty} \int_{O_\nu} \lambda d\lambda \int_0^1 P(x,\xi,\lambda)\psi(\xi)d\xi =$$

$$= -\frac{1}{2\pi\sqrt{-1}} \lim_{\nu\to\infty} \int_{\Gamma_\nu} \lambda d\lambda \int_0^1 P(x,\xi,\lambda)\psi(\xi)d\xi = \frac{1}{2}\left(\psi(x-0)+\psi(x+0)\right).$$

Consequently, satisfiability of Theorem 3 (on the formula of expansion of the arbitrary function in series thorough the function $G(x,\xi,\lambda)$) is equivalent to the satisfiability of the following lemma for an arbitrary function on the formula of expansion of zero in a series through the function $Q(x,\xi,\lambda)$.

Lemma 2. *Under constraints 1^0 and 3^0, for any constraints and Hölder continuous (except finitely many points in which it can have first order discontinuity) which respect to $\psi(x)$ for $x \in [0,1]$, $0 < x < 1$ we have the following formula on expansion in zero:*

$$0 = \lim_{\nu\to\infty} \int_{\Gamma_\nu} \lambda d\lambda \int_0^1 Q(x,\xi,\lambda)\psi(\xi)d\xi.$$

Further, in [5], it is shown that there exists a rather large positive number R and a rather small positive number δ that in the domain

$$R_\delta = \left\{\lambda : |\lambda| \geq R, |\arg\lambda| \leq \frac{\pi}{4} + \delta\right\},$$

the inequality (17) is fulfilled, and consequently, the functions $G(x,\xi,\lambda)$ and $g(x,\lambda,\chi_1,\chi_2)$ are analytic functions in the domain R_δ.

The following theorem was proved in [5].

Theorem 4. *Under constraints 1^0 and 3^0 (if desired, this constraint can be relaxed, i.e. the constraint "regularity" can be replaced by the constraint "well-posedness" [5]) for any bounded and Hölder continuous (except finitely number of points in which it may have discontinuities of first kind) $\psi(x)$ with respect to $x \in [0,1]$ for $0 < x < 1$ we have the following inversion formula:*

$$\frac{2\sqrt{-1}}{\pi + 4\delta} \int_L \lambda d\lambda \int_0^1 G(x,\xi,\lambda)\psi(\xi)d\xi = \frac{1}{2}\left(\psi(x-0)+\psi(x+0)\right),$$

where L is an infinite smooth line in R_δ a sufficiently distant part of which coincides with continuations of the rays $\arg\lambda = \pm\left(\delta + \frac{\pi}{4}\right)$, moreover, in (14) the integral along the lines L is understood in the sense of the main meaning.

Using the results of Theorem 2, we easily prove the following theorem.

Theorem 5. *Under constraints 1^0 and 3^0 , for any numbers χ_1 and χ_2 at $0 < x < 1$ we have the equality*

$$\lim_{\nu\to\infty}\int_{\Gamma_\nu}\lambda g(x,\lambda,\chi_1,\chi_2)d\lambda = 2\int_L \lambda g(x,\lambda,\chi_1,\chi_2)d\lambda. \qquad (23)$$

In [5] the following vanishing theorem was proved.

Theorem 6. *Under constraints 1^0 and 3^0 (constraint 3^0 can be relaxed, i.e. the "regularity" condition can be replaced by the "well -posedness" condition [5]) for any numbers χ_1, χ_2 and for any integers m, n, p and for $0 < x < 1$ we have the following vanishing formula:*

$$\int_L \lambda^m g(x,\lambda,\lambda^n\chi_1,\lambda^p\chi_2)d\lambda = 0. \qquad (24)$$

Under constraints 1^0 and 3^0 using formulas (23) and (24) for $0 < x < 1$ we have

$$\lim_{\nu\to\infty}\int_{\Gamma_\nu}\lambda g(x,\lambda,\chi_1,\chi_2)d\lambda = 0, \qquad (25)$$

where χ_1 and χ_2 are arbitrary numbers, and consequently,

$$\lim_{\nu\to\infty}\int_{\Gamma_\nu}\lambda g(x,\lambda,v_1(t),v_2(t))d\lambda = 0,\ 0 < x < 1. \qquad (26)$$

According to the expansion formula (22) we have

$$-\frac{1}{2\pi\sqrt{-1}}\lim_{\nu\to\infty}\int_{\Gamma_\nu}\lambda d\lambda\int_0^1 G(x,\xi,\lambda)u(\xi,t)d\xi = u(x,t), \quad 0 < x < 1. \qquad (27)$$

Further, due to analyticity of the integrand function we obtain

$$\int_{\Gamma_\nu}\lambda d\lambda\int_0^t \exp(-\lambda^2(t-\tau))u(x,\tau)d\tau = 0. \qquad (28)$$

Taking into account formulas (26), (27) and (28) in (25) we have

$$u(x,t) = -\frac{1}{2\pi\sqrt{-1}}\lim_{\nu\to\infty}\int_{\Gamma_\nu}\lambda\exp(-\lambda^2 t)\mathcal{F}(x,t,\lambda)d\lambda, \quad 0 < x < 1, \quad 0 < t < T. \qquad (29)$$

Thus, we established the following theorem.

Theorem 7. *Under constraints 1^0, 2^0 and 3^0, if for the antiparabolic equation the mixed problem* (8)-(10) *has a classic solution $u(x,t)$, then:*

1. *it is unique;*
2. *this solution is represented by the formula* (29)*;*
3. *if the limit contained in* (29) *does not exist, or it exists, but the function $u(x,t)$ determined by formula* (29) *is not a classic solution of the problem* (8)-(10)*, then this problem does not have a classic solution.*

Imposing certain constraints on the data of problem (8)-(10), by the method stated in [5], we can be convinced that the function $u(x,t)$, determined by formula (29), in fact is a classic solution of problem (8)-(10).

We explain what has been said using the following model case.

4^0. Let for $0 \le x \le 1, 0 \le t \le T$ $a_2(x) \equiv a^2$, $a_1(x) \equiv 0$, $a_0(x) \equiv 0$, $F(x,t) \equiv 0$, $\gamma_i(t) \equiv 0$ $(i = 1,2)$, where a^2 is a constant number.

5^0. Let

$$S_1\left(\frac{\partial}{\partial t}\right) u \equiv u/_{x=0}, \quad S_2\left(\frac{\partial}{\partial t}\right) u \equiv u/_{x=1}.$$

6^0. Let

$$f(x) = \sum_{k=1}^{\infty} b_k \exp(-a^2k^2\pi^2T) \sin k\pi x,$$

where b_k are some numbers satisfying the inequalities

$$|b_k| \le Ak^m, k = 1,2,...,$$

A $(A > 0), m$ are some numbers.

We have the following theorem.

Theorem 8. *For* $Rea^2 > 0$, $T > 0$ *if constraints* 4^0, 5^0 *and* 6^0 *are fulfilled (if desired, this constraint can be relaxed), then for an antiparabolic equation the mixed problem* (8)-(10) *has a unique classic solution and it is represented by the following formula:*

$$u(x,t) = \sum_{k=1}^{\infty} \left(2\int_0^1 f(\xi) \sin k\pi\xi d\xi\right) \exp(a^2k^2\pi^2t) \sin k\pi x, \quad 0 \le x \le 1, \quad 0 \le t < T. \quad (30)$$

Proof. Under constraints 4^0 and 5^0, calculating the residues contained in (29) for solving problem (8)-(10) we have formula (30). Further, taking into account constraint 6^0 in (30), we can easily be convinced that the function $u(x,t)$ determined by formula (30) is a classic solution of problem (8)-(10). Under constraints 4^0,5^0 and 6^0 the uniqueness of the solution of the mixed problem (8)-(10) follows from the above Theorem 7. ◄

References

1. Aliev A.R., Eyvazov E.H. The variation of eigenvalues of the two-dimensional magnetic Schrödinger operator with respect to the angle of a flat sector. *Turkish J. Math.*, 2026, **50** (1), pp. 164-172.
2. Birkhoff G.D. On the asymptotic character of the solutions of certain linear differential equations containing a parameter. *Trans. Amer. Math. Soc.*, 1908, **9** (2), pp. 219-231.
3. Bondarenko N.P., Chitorkin E.E. Inverse Sturm-Liouville problem with spectral pa rameter in the boundary conditions. *Mathematics*, 2023, **11** (5), Art. 1138, pp. 1-19.
4. Gasymov É.A. On the theory of mixed problems of conjugation for systems of various types. *Dokl. Akad. Nauk Azerb. SSR*, 1981, **37** (2), pp. 12-18 (in Russian).

5. Gasymov É.A. *Integral Transforms and Parabolic Potentials; Their Application to the Solution of Some Mixed Problems.* Cand. Sci. (Phys.-Math.) Dissertation, Moscow State Univ., Moscow, 1984 (in Russian).
6. Gasymov É.A. Application of an integral transformation to the solution of a mixed problem for a nonclassical equation. *Differ. Uravn.*, 1989, **25** (5), pp. 909-911 (in Russian).
7. Gasymov É.A. Mixed problems for conjugation of parabolic systems of different orders with nonlocal boundary conditions. *Differ. Equ.*, 1990, **26** (8), pp. 1003-1012.
8. Gasymov É.A. Application of integral transformations to the solution of some mixed problems. *Differ. Uravn.*, 1992, **28** (3), pp. 521-522 (in Russian).
9. Gasymov É.A. Application of the finite integral transform method to solving a mixed problem with integro-differential conditions for a nonclassical equation. *Differ. Equ.*, 2011, **47** (3), no. 3, pp. 319-332.
10. Gasymov E.A. Analysis of mixed problems of matching in hyperbolic systems of different orders. *Comput. Math. Math. Phys.*, 2014, **54** (7), pp. 1144-1158.
11. Gasymov E.A., Efendiev R.F., Orucov D.H. Inversion formula for solving parametric problems with irregular boundary conditions for variable coefficients equations. *Azerb. J. Math.*, 2025, **15** (2), pp. 264-281.
12. Gasymov M.G. On the decomposition in a series of eigenfunctions for a nonselfconjugate boundary value problem of the solution of a differential equation with a singularity at a zero point. *Dokl. Akad. Nauk SSSR*, 1965, **165** (2), pp. 261-264 (in Russian).
13. Hammou H., Labbas R., Maingot S., Medeghri A. Nonlocal general boundary value problems of elliptic type in L^p cases. *Mediterr. J. Math.*, 2016, **13** (4), pp. 1669-1683.
14. Kuchment P. On some spectral problems of mathematical physics. *Partial Differential Equations and Inverse Problems*, pp. 241-276, *Contemp. Math.*, **362**, Amer. Math. Soc., Providence, RI, 2004.
15. Lagarkov A.N., Kisel V.N., Sarychev A.K., Semenenko V.N. Electrophysics and electrodynamics of metamaterials. *High Temp.*, 2010, **48** (6), pp. 983-999.
16. Lažetić N.L. On uniform convergence on closed intervals of spectral expansions and their derivatives, for functions from $W_p^{(1)}$. *Mat. Vesnik*, 2004, **56** (3-4), pp. 91-104.
17. Marchenko V.A. *Sturm-Liouville Operators and Applications.* AMS Chelsea Publ., Providence, RI, 2011.
18. Naimark M.A. *Linear Differential Operators.* Nauka, Moscow, 1969 (in Russian).
19. Pivovarchik V. Inverse problem for the Sturm-Liouville equation on a star-shaped graph. *Math. Nachr.*, 2007, **280** (13-14), pp. 1595-1619.
20. Rasulov M.L. *The Contour Integral Method, and Its Application to the Investigation of Problems for Differential Equations.* Nauka, Moscow, 1964 (in Russian).
21. Sadovnichii V.A., Podol'skii V.E. Traces of differential operators. *Differ. Equ.*, 2009, **45** (4), pp. 477-493.
22. Sakhnovich A. Dirac type system on the axis: explicit formulae for matrix potentials with singularities and soliton-positon interactions. *Inverse Probl.*, 2003, **19**, pp. 845-854.
23. Savchuk A.M., Shkalikov A.A. Sturm-Liouville operators with distribution potentials. *Tr. Mosk. Mat. Obs.*, 2003, **64**, pp. 159-212 (in Russsian).

24. Shieh Ch.-T., Yurko V.A. Inverse nodal and inverse spectral problems for discontinuous boundary value problems. *J. Math. Anal. Appl.*, 2008, **347** (1), pp. 266-272.
25. Shin K.C. On half-line spectra for a class of non-self-adjoint Hill operators. *Math. Nachr.*, 2003, 261/262, pp. 171-175.
26. Shkalikov A.A. Basis properties of root functions of differential operators with spectral parameter in the boundary conditions. *Differ. Equ.*, 2019, **55** (5), pp. 631-643.
27. Tamarkin Ya.D. *On Certain General Problems in the Theory of Linear Ordinary Differential Equations and on Series Expansions of Arbitrary Functions.* Petrograd, 1917 (in Russian).
28. Tikhonov A.N., Samarskii A.A. *Equations of Mathematical Physics.* Izd. MGU, Moscow, 1999 (in Russian).
29. Yang C.F., Pivovarchik V.N. Inverse nodal problem for Dirac system with spectral parameter in boundary conditions. *Complex Anal. Oper. Theory*, 2013, **7** (4), pp. 1211-1230.
30. Yurko V. Inverse spectral problems for differential pencils on the half-line with turning points. *J. Math. Anal. Appl.*, 2006, **320** (1), pp. 439-463.
31. Yurko V. An inverse spectral problem for non-selfadjoint Sturm-Liouville operators with nonseparated boundary conditions. *Tamkang J. Math.*, 2012, **43** (2), pp. 289-299.
32. Yurko V.A. Inverse spectral problems for differential operators with non-separated boundary conditions. *J. Inverse Ill-Posed Probl.*, 2020, **28** (4), pp. 567-616.