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BOUNDEDNESS OF THE DISCRETE AHLFORS-BEURLING TRANSFORM ON DISCRETE WEIGHTED LEBESGUE SPACES

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Abstract. *The Ahlfors-Beurling transform is one of the important operators in complex analysis. This transform plays a crucial role in applications to the theory of quasiconformal mappings and to the Beltrami equation with discontinuous coefficients. The Ahlfors-Beurling transform has been studied on classical Lebesgue, Morrey, Sobolev, Besov, etc. spaces. However, its discrete version has not been well studied. In this paper, we prove that the discrete Ahlfors-Beurling transform is the bounded operator in discrete weighted Lebesgue spaces.*

Keywords: discrete Ahlfors-Beurling transform, discrete weighted Lebesgue space, bounded operator

Mathematics Subject Classification (2020): 44A15, 31C20

1. Introduction

The Ahlfors-Beurling transform of a function $f \in L_p(C), 1 \leq p < \infty$ is defined as the following singular integral:

$$(Bf)(z) = -\frac{1}{\pi} \lim_{\epsilon \to 0} \int_{\{w \in C: |z-w| \geq \epsilon\}} \frac{f(w)}{(z-w)^2} dm(w).$$

The Ahlfors-Beurling transform is one of the important operators in complex analysis. It has been shown in [1], [8], [11], [23], [29] that this transform plays a crucial role in applications to the theory of quasiconformal mappings and to the Beltrami equation with discontinuous coefficients.

From the theory of singular integrals [10], [26] it is known that the Ahlfors-Beurling transform is a bounded operator in the space $L_p, 1 < p < \infty$, that is, if $f \in L_p(\mathbb{C})$, then $Bf \in L_p(\mathbb{C})$ and

$$\|Bf\|_{L_p} \leq C_p \cdot \|f\|_{L_p}.$$

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In the case $f \in L_1(\mathbb{C})$ only the weak inequality holds:

$$m\{z \in \mathbb{C} : |(Bf)(z)| > \lambda\} \leq \frac{C_1}{\lambda} \cdot \|f\|_{L_1}.$$

where m stands for the Lebesgue measure. C_p, C_1 are constants independent of f.

In [7], [9], [12]-[15], [17], [19], [20], [25], [27], [28], the boundedness of the Ahlfors-Beurling transform in other function spaces (in the spaces of Sobolev, Besov, Campanato, Morrey, weighted Lebesgue, weighted Morrey etc.) was studied. When approximating singular integral transforms, it is necessary to study the properties of discrete analogs of these transforms (see [4]-[6], [21]). Therefore, it is necessary to study the discrete analog of the Ahlfors-Beurling transform in various spaces. In [2], [3] the boundedness of the discrete Ahlfors-Beurling transform was investigated in discrete Lebesgue spaces. In this paper, we prove that the discrete Ahlfors-Beurling transform is a bounded operator in discrete weighted Lebesgue space.

2. Definitions and Auxiliary Lemmas

Definition 1. *Let w be a weight function, that is, w is a locally integrable function that takes values in $(0, \infty)$ almost everywhere. The weighted Lebesgue spaces $L_{p,w} := L_{p,w}(\mathbb{C}), 1 \leq p < \infty$, consist of functions for which the following norm is finite*

$$\|f\|_{L_{p,w}} = \left(\int_{\mathbb{C}} |f(z)|^p \cdot w(z) dm(z) \right)^{\frac{1}{p}}.$$

If $w(z) \equiv 1$, then $L_{p,w}(\mathbb{C}) = L_p(\mathbb{C})$.

For any $\lambda > 0$ and for any cube $B \subset \mathbb{C}$, denote by λB the cube with the same center as B whose side length is λ times that of B.

Definition 2. *[20] If there exists a constant $D > 0$ such that for any cube $B \subset C$ we have*

$$w(2B) \leq Dw(B),$$

then we say that w satisfies the doubling condition and we denote $w \in \Delta_2$, where $w(E) = \int_E w(z)dm(z)$.

Definition 3. *[18] A weight function w is in the Muckenhoupt class A_p with $1 < p < \infty$ if there exists $C > 0$ such that for any cube $B \subset \mathbb{C}$*

$$\left(\frac{1}{|B|} \int_B w(z)dm(z) \right) \left(\frac{1}{|B|} \int_B w(z)^{-\frac{1}{p-1}} dm(z) \right)^{p-1} \leq C, \tag{1}$$

and the infimum of C satisfying the inequality (1) is denoted by $[w]_{A_p}$. We define $A_\infty = \cup_{1<p<\infty} A_p$.

We will need the following lemmas:

Lemma 1. *[15] If $w \in A_p$ for some $1 \le p < \infty$, then $w \in \Delta_2$.*

Lemma 2. *[20] If $w \in \Delta_2$, then there exists a constant $D_1 > 1$ such that for any cube B*

$$w(2B) \ge D_1 w(B).$$

Definition 4. *[16], [22] A discrete weight w on $\mathbb{Z}_{\mathbb{C}}$ is a sequence $w = \{w_k\}_{k\in\mathbb{Z}_{\mathbb{C}}}$ of positive real numbers. We define the discrete weighted Lebesgue spaces $l_{p,w} := l_{p,w}(\mathbb{Z}_{\mathbb{C}}), 1 \le p < \infty$ the class of sequences $h = \{h_n\}_{n\in\mathbb{Z}_{\mathbb{C}}}$ satisfying the condition*

$$\|h\|_{l_{p,w}} = \left(\sum_{n\in\mathbb{Z}_{\mathbb{C}}} |h_n|^p \cdot w_n\right)^{\frac{1}{p}} < \infty,$$

where $\mathbb{Z}_{\mathbb{C}} := \{m + i\cdot n \in \mathbb{C} : m, n \in \mathbb{Z}\}$ and $\mathbb{Z}$ is the set of integers.

If $w \equiv 1$, then $l_{p,w} = l_p$.

For $m \in \mathbb{Z}_{\mathbb{C}}$ and $n \in \mathbb{N}\cup\{0\}$ define $S_{m,n} = \{k \in \mathbb{Z}_{\mathbb{C}} : ||k - m|| \le n\}$, where $||k|| := \max\{|\Re(k)|, |\Im(k)|\}$. Following standard conventions, we denote the cardinality of a set S by $|S|$. Then we have $|S_{m,n}| = (2n+1)^2$ for all $m \in \mathbb{Z}_{\mathbb{C}}$ and each $n \in \mathbb{N}\cup\{0\}$.

Definition 5. *[18] A discrete weight $w = \{w_k\}_{k\in\mathbb{Z}_{\mathbb{C}}}$ belongs to $\tilde{A}_p$ for $1 < p < \infty$, if it satisfies the following condition:*

$$[w]_{\tilde{A_p}} := \sup_{m\in\mathbb{Z}_{\mathbb{C}}} \sup_{n\in\mathbb{N}\cup\{0\}} \frac{(\sum_{k\in S_{m,n}} w_k)(\sum_{k\in S_{m,n}} w_k^{-\frac{1}{p-1}})^{p-1}}{(2n+1)^{2p}} < \infty.$$

For any $n \in \mathbb{Z}_{\mathbb{C}}$ and $\delta > 0$ define

$$P(n,\delta) = \{w \in \mathbb{C} : -\delta \le \Re(w - n) < \delta, -\delta \le \Im(w - n) < \delta\}.$$

Definition 6. *If there exists $D > 0$, such that for any $m \in \mathbb{Z}_{\mathbb{C}}$ and for any $\delta > 0$*

$$\sum_{k\in P(m,2\delta)} w_k \le D \sum_{k\in P(m,\delta)} w_k,$$

then we say that the discrete weight $w = \{w_k\}_{k\in\mathbb{Z}_{\mathbb{C}}}$ satisfies the doubling condition, and we denote $w \in \tilde{\Delta}_2$.

Lemma 3. *If $\{w_k\}_{k\in\mathbb{Z}_{\mathbb{C}}} \in \tilde{\Delta}_2$, then there exists $D_1 > 1$, such that the following condition satisfies :*

$$\sum_{k\in S_{m,2n+1}} w_k \ge D_1 \sum_{k\in S_{m,n}} w_k.$$

Proof. To prove the lemma, we define the function $w(z) = w_k$ for $z \in P(k, \frac{1}{2})$, $k \in \mathbb{Z}_{\mathbb{C}}$. Then for any $k \in \mathbb{Z}_{\mathbb{C}}$ we have $w_k = \int_{P(k,\frac{1}{2})} w(z)dm(z)$. Let $\{w_k\} \in \tilde{\Delta}_2$. Let us prove $w \in \Delta_2$. For this we take any $P(b,\delta)$. Denote accordingly by k_1 and k_2 the integer parts of the numbers $\Re(b)+\frac{1}{2}$ and $\Im(b)+\frac{1}{2}$. Let $k=k_1+ik_2$. Then by definition $w(b)=w_k$. If $\delta\le 1$, then it follows from $\{w_k\}\in\tilde{\Delta}_2$ that

$$w(2P)=\int_{P(b,2\delta)}w(z)dm(z)\le 16\delta^2\sum_{m\in S_{k,2}}w_m\le 16\delta^2\cdot D^2\cdot w_k\le 16D^2\cdot w(P).$$

If $\delta\in[r,r+1)$ for $r\in\mathbb{N}$

$$w(2P)=\int_{P(b,2\delta)}w(z)dm(z)\le\sum_{m\in\overline{P(k,2r+2)}}w_m\le D^3\sum_{m\in\overline{P(k,\frac{r+1}{4})}}w_m\le D^3\sum_{m\in\overline{P(k,\frac{r}{2})}}w_m$$

$$\le D^3\int_{P(b,\delta)}w(z)dm(z)=D^3w(P).$$

Therefore, $w\in\Delta_2$. Then it follows from Lemma 2 that there exists a constant $D_1>1$ such that for any cube P

$$w(2P)\ge D_1w(P).$$

If we take $P=P(m,n+\frac{1}{2})$, then we have

$$D_1\sum_{k\in S_{m,n}}w_k=D_1\int_{P(m,n+\frac{1}{2})}w(z)dm(z)\le\int_{P(m,2n+1)}w(z)dm(z)\le\sum_{k\in S_{m,2n+1}}w_k.$$

◀

Lemma 4. *If $\{w_k\}_{k\in\mathbb{Z}_{\mathbb{C}}}\in\tilde{A}_p$, for some $1\le p<\infty$, then $\{w_k\}_{k\in\mathbb{Z}_{\mathbb{C}}}\in\tilde{\Delta}_2$.*

Proof. Let $m\in\mathbb{Z}_{\mathbb{C}}$ and $n\in\mathbb{N}\cup\{0\}$. For any numbers $\{b_k\}_{k\in S_{m,n}}$ and $\delta>0$ applying Holder's inequality and condition $\{w_k\}\in\tilde{A}_p$, we obtain

$$\left(\sum_{k\in\overline{P(m,\delta)}}|b_k|\right)^p=\left(\sum_{k\in\overline{P(m,\delta)}}|b_k|w_k^{\frac{1}{p}}w_k^{-\frac{1}{p}}\right)^p\le\left(\sum_{k\in\overline{P(m,\delta)}}|b_k|^pw_k\right)\left(\sum_{k\in\overline{P(m,\delta)}}w_k^{-\frac{1}{p-1}}\right)^{p-1}$$

$$\le\frac{\left(\sum_{k\in\overline{P(m,\delta)}}|b_k|^pw_k\right)\cdot[w]_{\tilde{A_p}}\cdot(2[\delta]+1)^{2p}}{\left(\sum_{k\in\overline{P(m,\delta)}}w_k\right)},$$

where $[\delta]$ is the integer part of the number δ. Hence,

$$\left(\sum_{k\in\overline{P(m,\delta)}}|b_k|\right)^p\le\frac{\left(\sum_{k\in\overline{P(m,\delta)}}|b_k|^pw_k\right)\cdot[w]_{\tilde{A_p}}\cdot(2[\delta]+1)^{2p}}{\left(\sum_{k\in\overline{P(m,\delta)}}w_k\right)}. \qquad (2)$$

Putting $2P$ for P and $b_k=1$, for $k\in\overline{P(m,\delta)}$, $b_k=0$, for $k\notin\overline{P(m,\delta)}$ in (2), we get

$$1\le\frac{\left(\sum_{k\in\overline{P(m,\delta)}}w_k\right)\cdot[w]_{\tilde{A_p}}}{\left(\sum_{k\in\overline{P(m,2\delta)}}w_k\right)}.$$

Therefore,

$$\sum_{k \in \overline{P(m,2\delta)}} w_k \leq [w]_{\tilde{A}_p} \cdot \sum_{k \in \overline{P(m,\delta)}} w_k.$$

◀

Corollary. *If $\{w_k\} \in \tilde{A}_p$ for some $1 \leq p < \infty$, then there exists $D_1 > 1$ such that for any $n \in \mathbb{Z}_{\mathbb{C}}$ and $i \in \mathbb{N}$*

$$\sum_{k \in S_{n,2^i-1}} w_k \geq D_1^{i-1} \sum_{k \in S_{n,1}} w_k.$$

3. Boundedness of the Discrete Ahlfors-Beurling Transform on Discrete Weighted Lebesgue Spaces

Let $h = \{h_n\}_{n \in Z_C} \in l_p, 1 \leq p < \infty$. Namely, the sequence $B(h) = \{(Bh)_n\}_{n \in Z_C}$ is called the Ahlfors-Beurling transform of the sequence h , where

$$(Bh)_n = \sum_{m \in Z_C, m \neq n} \frac{h_m}{(n-m)^2}, n \in Z_C.$$

Note that if $h \in l_p, 1 \leq p < \infty$, then from the Hölder inequality it follows that the series $\sum\limits_{m \in Z_C, m \neq n} \frac{h_m}{(n-m)^2}$ absolutely converges and therefore the Ahlfors-Beurling transform of the sequence h exists. In [10] A. P. Calderon and A. Zygmund noted that the discrete Ahlfors-Beurling transform is of special interest among discrete analogs of singular integrals, also it was noted that the discrete analogs of singular integrals, including the discrete Ahlfors - Beurling transform, are bounded in l_p (discrete Lebesgue space). In [2] the summability properties of the discrete Ahlfors-Beurling transform are studied on discrete Lebesgue spaces. In this section, we study the boundedness of the discrete Ahlfors-Beurling transform on discrete weighted Lebesgue spaces.

Theorem 1. *If $\{w_k\}_{k \in \mathbb{Z}_{\mathbb{C}}} \in \tilde{A}_p$ with $1 < p < \infty$, then for all $\{h_m\}_{m \in \mathbb{Z}_{\mathbb{C}}} \in l_{p,w}$, the sequence $\{(Bh)_n\}_{n \in \mathbb{Z}_{\mathbb{C}}}$ is well defined.*

Proof. Let $h = \{h_n\}_{n \in \mathbb{Z}_{\mathbb{C}}} \in l_{p,w}$, $1 < p < \infty$. Then for any $n \in \mathbb{Z}_{\mathbb{C}}$

$$|(Bh)_n| \leq \sum_{m \neq n, m \in \mathbb{Z}_{\mathbb{C}}} \frac{|h_m|}{|n-m|^2} = \sum_{i=1}^{\infty} \sum_{2^{i-1} \leq ||n-m|| < 2^i} \frac{|h_m|}{|n-m|^2} \leq \sum_{i=1}^{\infty} \frac{1}{4^{i-1}} \sum_{||n-m|| < 2^i} |h_m|$$

$$\leq \sum_{i=1}^{\infty} \frac{1}{4^{i-1}} \left(\sum_{||n-m|| < 2^i} |h_m|^p w_m \right)^{\frac{1}{p}} \cdot \left(\sum_{||n-m|| < 2^i} w_m^{-\frac{1}{p-1}} \right)^{\frac{p-1}{p}}$$

$$\leq \|h\|_{l_{p,w}} \cdot [w]_{\tilde{A}_p}^{\frac{1}{p}} \cdot \sum_{i=1}^{\infty} \frac{(2^{i-1}-1)^2}{4^{i-1}} \cdot \left(\sum_{||n-m|| < 2^i} w_m \right)^{-\frac{1}{p}}.$$

Denoting

$$\delta = \log_2 D_1 > 0,$$

and using Corollary, we have

$$\sum_{m \in S_{n,2^i-1}} w_m \geq D_1^{i-1} \sum_{m \in S_{n,1}} w_m = 2^{(i-1)\delta} \sum_{m \in S_{n,1}} w_m.$$

Then

$$|(Bh)_n| \leq 16 \cdot ||h||_{l_{p,w}} \cdot [w]_{\tilde{A}_p}^{\frac{1}{p}} \cdot \frac{2^{\frac{\delta}{p}}}{\left(2^{\frac{\delta}{p}} - 1\right)\left(\sum_{m \in S_{n,1}} w_m\right)^{\frac{1}{p}}} < \infty.$$

◀

Theorem 2. *Let* $1 < p < \infty$ *. If* $\{w_k\} \in \tilde{A}_p$*, then for any* $h = \{h_m\}_{m \in \mathbb{Z}_{\mathbb{C}}} \in l_{p,w}$ *we have* $B(h) \in l_{p,w}$ *and there exists* $C_{p,w} > 0$*, such that*

$$\|B(h)\|_{l_{p,w}} \leq C_{p,w} \|h\|_{l_{p,w}}$$

for any $h \in l_{p,w}$*.*

Proof. We define $f(z)$ as $4h_k$ for $z \in P(k, \frac{1}{4})$, $k \in \mathbb{Z}_{\mathbb{C}}$, and 0 elsewhere, and $w(z)$ as w_k for $z \in P(k, \frac{1}{2})$. Firstly, let us show that if $\{w_k\} \in \tilde{A}_p$,then $w(z) \in A_p$. We take any cube $B = B(z_0, r_0) \subset \mathbb{C}$. Denote accordingly by m_1, m_2 and n the integer parts of the numbers $\Re(z_0) + \frac{1}{2}$, $\Im(z_0) + \frac{1}{2}$ and $r_0 + 1$. Let $m = m_1 + im_2$. Then we have $B \subset P\left(m, n + \frac{1}{2}\right)$. Denote

$$\delta_k = m\left(B \cap P(k, \frac{1}{2})\right), k \in S_{m,n}, m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\},$$

and

$$\delta = \max_{k \in S_{m,n}} \delta_k.$$

It follows from $\{w_k\} \in \tilde{A}_p$, that

$$\frac{1}{|B|^p} \cdot \left(\int_B w(z) dm(z)\right) \cdot \left(\int_B w(z)^{-\frac{1}{p-1}} dm(z)\right)^{p-1}$$

$$= \frac{1}{\left(\sum_{\|k-m\| \leq n} \delta_k\right)^p} \cdot \left(\sum_{\|k-m\| \leq n} \delta_k \cdot w_k\right) \cdot \left(\sum_{\|k-m\| \leq n} \delta_k \cdot w_k^{-\frac{1}{p-1}}\right)^{p-1}$$

$$\leq \frac{\delta^p}{\left(\sum_{\|k-m\| \leq n} \delta_k\right)^p} \cdot \left(\sum_{\|k-m\| \leq n} w_k\right) \cdot \left(\sum_{\|k-m\| \leq n} w_k^{-\frac{1}{p-1}}\right)^{p-1}$$

$$\leq 25^p \cdot \frac{\left(\sum_{\|k-m\| \leq n} w_k\right) \cdot \left(\sum_{\|k-m\| \leq n} w_k^{-\frac{1}{p-1}}\right)^{p-1}}{(2n+1)^{2p}} \leq 25^p \cdot [w]_{\tilde{A}_p} < \infty.$$

Therefore, $w(z) \in A_p$. It follows from $h = \{h_m\}_{m \in \mathbb{Z}_\mathbb{C}} \in l_{p,w}$ that $f \in L_{p,w}$ and

$$\|f\|_{L_{p,w}} = \left(\sum_{k \in \mathbb{Z}_\mathbb{C}} \int_{P\left(k, \frac{1}{4}\right)} |4h_k|^p \, w_k dm(z) \right)^{1/p}$$

$$= 4^{1-\frac{1}{p}} \cdot \left(\sum_{k \in \mathbb{Z}_\mathbb{C}} |h_k|^p \, w_k \right)^{1/p} = 4^{1-\frac{1}{p}} \cdot \|h\|_{l_{p,w} w}.$$

We denote

$$(Sf)(z) = \sup_{\epsilon > 0} \left| \int_{|z-w|>\epsilon} \frac{f(w)}{(z-w)^2} dm(w) \right|,$$

$$(Mf)(z) = \sup_{r>0} \frac{1}{|B(z,r)|} \int_{B(z,r)} |f(w)| dm(w).$$

For any $z \in P(j, \frac{1}{4}), j \in \mathbb{Z}_\mathbb{C}$

$$\left| \int_{|z-w|>\frac{1}{2}} \frac{f(w)}{(z-w)^2} dm(w) - \sum_{k \neq j} \frac{h_k}{(j-k)^2} \right|$$

$$= \left| \sum_{k \neq j} \int_{P(k,\frac{1}{4})} \frac{f(w)}{(z-w)^2} dm(w) - \sum_{k \neq j} \int_{P(k,\frac{1}{4})} \frac{f(w)}{(j-k)^2} dm(w) \right|$$

$$= \left| \sum_{k \neq j} \int_{P(k,\frac{1}{4})} f(w) \cdot \left(\frac{1}{(z-w)^2} - \frac{1}{(j-k)^2} \right) dm(w) \right|$$

$$\leq \sum_{k \neq j} \int_{P(k,\frac{1}{4})} |f(w)| \cdot \left| \frac{1}{(z-w)^2} - \frac{1}{(j-k)^2} \right| dm(w).$$

If $\frac{1}{2} < |z-w| \leq 1$ then $1 \leq |j-k| < 2$. Therefore,

$$\left| \frac{1}{(z-w)^2} - \frac{1}{(j-k)^2} \right| = \frac{|z-w+j-k| \cdot |z-w-j+k|}{|z-w|^2 \cdot |j-k|^2}$$

$$\leq \frac{(|z-w|+2) \cdot \frac{\sqrt{2}}{2}}{|z-w|^2} < \frac{3\sqrt{2}}{2 \cdot |z-w|^2} \leq \frac{3\sqrt{2}}{2 \cdot |z-w|^3}.$$

If $|z-w| > 1$, then $|z-w| - \frac{\sqrt{2}}{2} \leq |j-k| \leq |z-w| + \frac{\sqrt{2}}{2}$. Therefore,

$$\left| \frac{1}{(z-w)^2} - \frac{1}{(j-k)^2} \right| = \frac{|z-w+j-k| \cdot |z-w-j+k|}{|z-w|^2 \cdot |j-k|^2}$$

$$\leq \frac{(2 \cdot |z-w| + \frac{\sqrt{2}}{2}) \cdot \frac{\sqrt{2}}{2}}{|z-w|^2 \cdot (|z-w| - \frac{\sqrt{2}}{2})^2} < \frac{\sqrt{2}+1}{1,5-\sqrt{2}} \cdot \frac{1}{|z-w|^3} < \frac{35}{|z-w|^3}.$$

Hence,

$$\left|\int_{|z-w|>\frac{1}{2}} \frac{f(w)}{(z-w)^2}dm(w) - \sum_{k\neq j} \frac{h_k}{(j-k)^2}\right| < 35 \cdot \int_{|z-w|>\frac{1}{2}} \frac{|f(w)|}{|z-w|^3}dm(w). \tag{3}$$

Notice the right-hand side of the inequality (3)

$$\int_{|z-w|>\frac{1}{2}} \frac{|f(w)|}{|z-w|^3}dm(w) = \frac{3}{14\pi}\int_{|z-w|>\frac{1}{2}} |f(w)| \cdot \left(\int_{|z-w|\leq|t|\leq 2\cdot|z-w|} \frac{1}{|t|^5}dm(t)\right) dm(w)$$

$$= \frac{3}{14\pi}\int_{\frac{1}{2}<|t|<\infty} \frac{1}{|t|^5}\left(\int_{\frac{|t|}{2}<|z-w|\leq|t|} |f(w)|dm(w)\right) dm(t)$$

$$\leq \frac{3}{14\pi}\int_{\frac{1}{2}<|t|<\infty} \frac{|B(z,|t|)|}{|t|^5}\left(\frac{1}{|B(z,|t|)|}\int_{|z-w|\leq|t|} |f(w)|dm(w)\right) dm(t)$$

$$\leq \frac{3}{14} \cdot \left(\int_{\frac{1}{2}<|t|<\infty} \frac{1}{|t|^3}dm(t)\right) \cdot (Mf)(z) = \frac{6\pi}{7} \cdot (Mf)(z).$$

From (3), we have

$$|(Bh)_j| \leq (Sf)(z) + 30\pi \cdot (Mf)(z). \tag{4}$$

Multiplying $w(z)$ and integrating in $P(j,\frac{1}{4})$ in (4) we have

$$\sum_{j\in\mathbb{Z}_C} |(Bh)_j|^p \cdot w_j \leq \sum_{j\in\mathbb{Z}_C} 4 \cdot \int_{P(j,\frac{1}{4})} |(Sf)(z) + 30\pi \cdot (Mf)(z)|^p \cdot w(z)dm(z)$$

$$= 4 \cdot \int_{\mathbb{C}} |(Sf)(z) + 30\pi \cdot (Mf)(z)|^p \cdot w(z)dm(w).$$

Since $(Sf)(z) \in L_{p,w}$ (see [15]) and $(Mf)(z) \in L_{p,w}$ (see [24]), we get

$$\left(\sum_{j\in Z_{\mathbb{C}}} |(Bh)_j|^p \cdot w_j\right)^{1/p} \leq 4^{\frac{1}{p}} \cdot \|Sf + 30\pi \cdot Mf\|_{L_{p,w}} \leq 4^{\frac{1}{p}} \cdot (\|Sf\|_{L_{p,w}} + 30\pi \cdot \|Mf\|_{L_{p,w}})$$

$$\leq 4^{\frac{1}{p}} \cdot \left(\|Sf\|_{L_{p,w}\to L_{p,w}} + 30\pi \cdot \|Mf\|_{L_{p,w}\to L_{p,w}}\right) \cdot \|f\|_{L_{p,w}}$$

$$= 4\pi \cdot \left(\|Sf\|_{L_{p,w}\to L_{p,w}} + 30\pi \cdot \|Mf\|_{L_{p,w}\to L_{p,w}}\right) \cdot \|h\|_{l_{p,w}}.$$

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