Baku Mathematical Journal
2026, Vol. 5, No 1, PP. 23-40
https://doi.org/10.32010/j.bmj.2026.03

VARIATION OF PARAMETERS AND HÖLDER-TYPE STABILITY IN UNPERTURBED MATRIX DIFFERENTIAL SYSTEMS WITH INITIAL TIME DIFFERENCE

C. YAKAR*, **D. KARSLIOĞLU**

Received: 22.08.2025 / Revised: 05.11.2025 / Accepted: 13.11.2025

Abstract. *In this study, we explore the connection between unperturbed matrix differential systems that exhibit variations in both their initial conditions and the starting time. Through the application of variation of parameters methods, we construct integral representations that reveal this relationship. To formulate Hölder-type stability criteria under the presence of an initial time shift in nonlinear matrix differential systems, it is essential to employ the variational framework derived from the unperturbed matrix model.*

Keywords: differential equations, initial time difference, Hölder stability, variation of parameters

Mathematics Subject Classification (2020): 34D10, 34D20, 34D99

1. Introduction

The variation of parameters formula (VPF) has long been recognized as a valuable analytical approach within the qualitative study of nonlinear matrix differential equations, particularly due to its effectiveness in examining the characteristics of their solutions. Recent studies, as referenced in [1]–[3], have initiated a line of research focusing on nonlinear matrix systems and initial value problems in the absence of an Initial Time Difference (ITD), along with the development of related theories involving differential inequalities. In the forthcoming sections, we derive VPF-based formulations that establish links between unperturbed matrix systems with differing initial data, as well as between unperturbed matrix models under non-identical initial conditions.

* Corresponding author.

Coşkun Yakar
Gebze Technical University, Kocaeli, Türkiye
E-mail: cyakar@gtu.edu.tr

Dilara Karslıoğlu
Yeditepe University, Istanbul, Türkiye
E-mail: dilara.karslioglu@yeditepe.edu.tr

The dynamic properties of differential equations have been the subject of substantial qualitative analysis. In particular, initial value problems influenced by spatial unperturbed matrix systems–while maintaining a fixed initial time–have been widely studied, as discussed in [1]–[3], [8], [10]. More recently, researchers have turned their attention to systems characterized by both spatial deviations and discrepancies in the initial time. This line of inquiry is commonly referred to as Initial Time Difference (ITD) stability analysis. In [6], [7], variation of parameters techniques were applied to investigate the relationships between (1) unperturbed matrix systems with varying initial states exhibiting an initial time shift. To establish the conditions for Initial Time Difference Hölder Stability (ITD HS), we employ the variational system derived from the original unperturbed matrix differential equations.

One fundamental distinction between ITD HS and classical Hölder Stability (HS) lies in their respective reference frameworks. Classical HS is typically defined with respect to the zero (null) solution, whereas ITD HS is formulated relative to an unperturbed matrix system, taking into account both initial time shifts and variations in the initial conditions. As shown in Section 4, the transformation techniques commonly used in classical HS theory produce results that differ from those obtained under the ITD framework, demonstrating that ITD HS cannot be regarded as equivalent to classical HS.

To formulate criteria for ITD HS including its variation-based and exponential asymptotic forms we use the variational system derived from the unperturbed matrix framework. Definitions for various forms of ITD HS and time-shift-based Hölder stability are introduced in detail. Our analysis shows that classical stability with respect to the null solution and ITD HS are fundamentally different, so limiting the stability analysis to the null solution is insufficient. Furthermore, while ITD HS implies ITD stability, the converse does not necessarily hold. Theoretical results are also provided for variation-based ITD HS, exponential asymptotic ITD HS, and both uniform ITD stability and uniform ITD HS of the unperturbed matrix system relative to the unperturbed system.

2. Preliminaries

We consider the following nonlinear matrix differential models characterized by different initial conditions:

$$X' = F(t, X), \quad X(t_0) = X_0, \quad t \geq t_0,\ t_0 \in \mathbb{R}_+\,, \tag{1}$$

$$X' = F(t, X), \quad X(\rho_0) = Y_0, \quad t \geq \rho_0,\ \rho_0 \in \mathbb{R}_+\,. \tag{2}$$

The operator $Vec(\cdot)$ is defined in [5], which maps an $m \times n$ matrix $A = (a_{ij})$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, onto the vector composed of the columns of A:

$$Vec(\cdot) = (a_{11}, a_{21}, ..., a_{m1}, a_{12}, a_{22}, ..., a_{m2}, ..., a_{1n}, a_{2n}, ..., a_{mn})^T\,.$$

Then the corresponding vector differential systems can be written as

$$x' = F(t, x), \quad X(t_0) = x_0 \ \ for \ \ t \geq t_0, \quad t_0 \in \mathbb{R}_+\,, \tag{3}$$

$$x' = F(t, x), \quad x(\tau_0) = y_0 \ \ for \ \ t \geq \tau_0, \quad t_0 \in \mathbb{R}_+.$$

For a fixed initial point (ρ_0, Y_0), in the rest of the paper, we shall use

$$\tilde{\mathcal{X}} = X(t, \rho_0, Y_0) \quad \text{denotes the solution of (2) passing through } (\rho_0, Y_0),$$

and $\mathcal{X} = X(t - \tilde{\zeta}, t_0, X_0)$ denotes the solution of (1) for $\tilde{\zeta} = \rho_0 - t_0 \geq 0$ and $t \geq \rho_0$.

Definition 1.
We say that $\tilde{\mathcal{X}}$ exhibits ITD stability relative to the solution $\mathcal{X}$, provided that $\forall\ \epsilon > 0$, there exists positive constants $\delta_1 = \delta_1(\rho_0, \epsilon)$ and $\delta_2 = \delta_2(\rho_0, \epsilon)$ s.t.

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| < \epsilon$$

when $\|Y_0 - X_0\| < \delta_1$ and $|\rho_0 - t_0| < \delta_2$, $\forall\ t \geq \rho_0$.
Furthermore, if the constants δ_1 and δ_2 can be chosen independently of ρ_0, then the solution $\tilde{\mathcal{X}}$ is uniformly ITD stable w.r.t. $\mathcal{X}$ for $t \geq \rho_0$.

Definition 2. *We say that $\tilde{\mathcal{X}}$ exhibits ITD Lipschitz stability relative to the solution $\mathcal{X}$ if and only if there exists a constant $\mathcal{L}^* = \mathcal{L}^*(\rho_0) > 0$ s.t.:*

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\rho_0) \left(\|Y_0 - X_0\| + |\rho_0 - t_0|\right), \quad \textit{for all} \quad t \geq \rho_0.$$

Moreover, when the Lipschitz constant $\mathcal{L}^*$ does not depend on the initial time ρ_0, the solution $\tilde{\mathcal{X}}$ is uniformly Lipschitz stable with respect to ITD, As compared with the corresponding solution $\mathcal{X}$ of system (1).

Definition 3. *We say that $\tilde{\mathcal{X}}$ possess ITD HS of order λ, where $0 < \lambda < 1$, w.r.t. the solution $\mathcal{X}$ s.t. a constant $\mathcal{L}^* = \mathcal{L}^*(\rho_0) > 0$ exists and satisfies:*

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\rho_0) \left(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda\right), \quad \textit{for all} \quad t \geq \rho_0. \tag{4}$$

If the constant $\mathcal{L}^*$ does not depend on ρ_0, then the solution $\tilde{\mathcal{X}}$ is said to be uniformly HS w.r.t. ITD, in relation to the solution $\mathcal{X}$ of (1).
In particular, when $\lambda = 1$ and $\mathcal{L}^*$ is also independent of ρ_0, this notion reduces to uniform ITD Lipschitz stability.

Definition 4. *We say that $\tilde{\mathcal{X}}$ exhibits ITD asymptotically HS of order λ, where $0 < \lambda < 1$, relative to the solution $\mathcal{X}$ there exists $\mathcal{L}^* = \mathcal{L}^*(\rho_0) > 0$ s.t.*

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\rho_0) \left(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda\right) \nu(t - \rho_0), \quad t \geq \rho_0,$$

where $\nu(t - \rho_0) \to 0$ as $t \to \infty$.
If the constant $\mathcal{L}^*$ does not depend on ρ_0, then the solution $\tilde{\mathcal{X}}$ is said to be uniformly asymptotically HS w.r.t. ITD.

Definition 5. *We say that $\tilde{\mathcal{X}}$ exhibits ITD asymptotically HS in variation of order λ (with $0 < \lambda < 1$) compared to $\mathcal{X}$ if the following two conditions are fulfilled:*
(i) *there exists a constant $\mathcal{L}^* = \mathcal{L}^*(\rho_0) > 0$ s.t.:*

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\rho_0) \left(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda\right) \nu(t - \rho_0), \quad t \geq \rho_0,$$

where $\nu(t-\rho_0) \to 0$ as $t \to \infty$.
(ii) *For every $\beta > 0$, there exists a constant $\mathcal{N}^* > 0$ s.t. the fundamental matrix solution $\gamma(t, \rho_0, Y_0)$ of the associated variational system (as defined in the following Theorem 1) satisfies:*

$$\|\gamma(t, \rho_0, Y_0)\| \leq \mathcal{N}^*, \quad \forall t \geq \rho_0, \text{ whenever } \|Y_0\| \leq \beta.$$

If the constant $\mathcal{L}^*$ does not depend on ρ_0, then the system is said to exhibit uniform ITD asymptotic HS in variation.

Additionally, when $\lambda = 1$ and $\mathcal{N}^*$ remains independent of ρ_0, the system satisfies the condition of uniform ITD asymptotic Lipschitz stability in variation.

Definition 6. *We say that $\tilde{\mathcal{X}}$ exhibits generalized exponential asymptotic HS under ITD, for $\lambda \in (0,1)$, w.r.t. the solution $\mathcal{X}$ if there exists a constant $\mathcal{L}^* = \mathcal{L}^*(\rho_0) > 0$ s.t.:*

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\rho_0) \left(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda\right) \exp\left[\mathcal{S}^*(\rho_0) - \mathcal{S}^*(t)\right], \, t \geq \rho_0,$$

where the function $\mathcal{S}^*$ belongs to the class

$$\mathcal{K}^* = \{\mathcal{S}^* \in C[\mathbb{R}_+, \mathbb{R}_+] : \mathcal{S}^* \text{ is strictly increasing and } \mathcal{S}^*(0) = 0\}$$

and $\mathcal{S}^*(t) \to \infty$ as $t \to \infty$. If the constant $\mathcal{L}^*$ is independent of ρ_0, then the solution is called uniformly generalized exponentially asymptotically HS.

Moreover, if $\lambda = 1$, then the system satisfies uniform generalized exponential asymptotic Lipschitz stability.

In the special case where $\tilde{p}(t) = \beta t$ for some $\beta > 0$, the system exhibits ITD exponential asymptotic Hölder stability.

We consider the variation of parameters formulae for differential equations relative to the initial time difference in the next section, to do these we will give well-known lemmas [8], necessary for the main results.

3. Variation of Parameters in Nonlinear Matrix Differential Systems under Initial Time Difference

In this section, we examine the stability behavior of the solutions associated with systems (1) and (2), as detailed in the subsequent results.

Throughout the remainder of this paper, we shall use the following notations:

$$\bar{\mathcal{X}} = X(t, t_0, X_0) \quad \text{denotes the solution of (1) passing through } (t_0, X_0),$$

$$\mathcal{X}^* = X(t, \rho_0, X_0) \quad \text{denotes the solution of (2) passing through } (\rho_0, X_0).$$

Theorem 1. *Consider that F and its partial derivatives are continuous w.r.t.x, denoted by $\partial F/\partial X$, in the domain $\mathbb{R}_+ \times \mathbb{R}^{n\times n}$. Consider $\tilde{\mathcal{X}}$ denoting the unique solution of (4). Define the matrix-valued function $\mathcal{H}(t, \rho_0, Y_0) := \frac{\partial F}{\partial X}(t, X(t, \rho_0, Y_0))$. Under these*

conditions, the following result holds:
(i) The derivative $\gamma(t,\rho_0,Y_0) := \frac{\partial\tilde{\mathcal{X}}}{\partial Y_0}$ *exists and satisfies the matrix differential equation*

$$Z' = \mathcal{H}(t,\rho_0,Y_0)z \tag{5}$$

for $\gamma(\rho_0,\rho_0,Y_0) = \mathcal{I}_{n\times n}$, $\mathcal{I}_{n\times n}$ *means the identity matrix.*
(ii) The partial derivative $\frac{\partial\tilde{\mathcal{X}}}{\partial\rho_0}$ *also exists and solves the same differential equation* (5), *with the initial condition*

$$\frac{\partial X(\rho_0,\rho_0,Y_0)}{\partial\rho_0} = -F(\rho_0,Y_0),$$

and satisfies the identity

$$\frac{\partial\tilde{\mathcal{X}}}{\partial\rho_0} + \gamma(t,\rho_0,Y_0)F(\rho_0,Y_0) = 0, \quad \forall t \geq \rho_0.$$

(iii) Let $\tilde{\mathcal{X}}$ *be any solution of the unperturbed matrix system* (1). *Then it satisfies the integral identity known as Alekseev's formula:*

$$\tilde{\mathcal{X}} = \bar{\mathcal{X}} + \int_{\rho_0}^{t} \gamma(t,s,X(s,\rho_0,Y_0))\,\mathcal{R}^*(s,X(s,\rho_0,Y_0))\,ds,$$

where $\gamma(t,\rho_0,Y_0) = \frac{\partial\tilde{\mathcal{X}}}{\partial Y_0}$, *and* $t \geq \rho_0$.

A detailed justification of this theorem is available in the literature; see, for instance, [9], [10] for a comprehensive exposition.

This section focuses on establishing a formal link between unperturbed matrix differential models.

Theorem 2. *Assume that* $W(t,t_0,X_0)$ *and* $Z(t,t_0,X_0)$ *are the solutions of matrix differential systems of* (6), (7), *respectively, for* $t \geq \tau_0$. *Consider the matrix differential systems*

$$W' = A(t,\tau_0,X_0)W, \quad Y(\tau_0) = I \ for \ t \geq \tau_0, \tag{6}$$

$$Z' = ZB(t,\tau_0,X_0), \quad Z(\tau_0) = I \ for \ t \geq \tau_0. \tag{7}$$

Then the solution of the matrix differential system of (7) *satisfies the relation*

$$X(t,\tau_0,X_0) = W(t,\tau_0,X_0)CZ(t,\tau_0,X_0), \qquad t \geq \tau_0.$$

For a detailed proof of this theorem, please refer to [9], [10].

Theorem 3. *Assume that* $F(t,x)$ *in* (3) *has continuous partial derivatives on* $\mathbb{R}_+ \times \mathbb{R}^{n\times n}$ *and let*

$$G(t,\tau_0,Y_0) = \frac{\partial F}{\partial X}(t,X(t,\tau_0,Y_0)).$$

If there exist $n \times n$ *matrices* $A(t,\tau_0,Y_0)$ *and* $B(t,\tau_0,Y_0)$ *as in Theorem 2 such that*

$$G = (A \otimes I) + (I \otimes B^T),$$

then
(i) $\varphi(t,\tau_0,Y_0) = \partial X(t,\tau_0,Y_0)/\partial Y_0$ *exists, exists the fundamental matrix solution of*

$$\varphi' = G(t,\tau_0,Y_0)\varphi$$

such that $\varphi(t,\tau_0,Y_0) = I$, *and therefore*

$$\varphi(t,\tau_0,Y_0) = W(t,\tau_0,Y_0) \otimes Z^T(t,\tau_0,Y_0),$$

where W *and* Z *are the solutions of* (6) *and* (7), *respectively.*
(ii) any solution of (2) *satisfies the integral equation for* $t \geq \tau_0$

$$\tilde{\mathcal{X}} = \bar{\mathcal{X}} + \int_{\tau_0}^{t} W(t,s,Y(s,\tau_0,Y_0))R(s,Y(s,\tau_0,Y_0))Z(s,Y(s,\tau_0,Y_0))ds.$$

The detailed proof of the theorem is in [4].

Theorem 4. *Suppose that the nonlinear matrix differential systems described by* (1) *and* (2) *possess unique solutions denoted by* $\mathcal{X}$ *and* $\tilde{\mathcal{X}}$. *Under these assumptions, the following integral identity holds:*

$$\tilde{\mathcal{X}} = \mathcal{X} + \int_0^1 \gamma(t,\rho_0,\tilde{\Omega}(s))\,ds \cdot (Y_0 - X_0) -$$

$$-\int_{\rho_0}^{t} \left[\frac{\partial X}{\partial \rho_0}\left(t,s,X(s-\tilde{\zeta})\right) + \gamma\left(t,s,X(s-\tilde{\zeta})\right) F\left(s-\tilde{\zeta}, X(s-\tilde{\zeta})\right)\right] ds \qquad (8)$$

for all $t \geq \rho_0 \geq 0$, *where the interpolating function is defined by* $\tilde{\Omega}(s) := sY_0 + (1-s)X_0$ *for* $s \in [0,1]$.

Proof. Assume that both systems (1) and (2) admit unique solutions: $\mathcal{X}$, associated with the initial pair (t_0, X_0), and $\tilde{\mathcal{X}}$, corresponding to (ρ_0, Y_0), with $t \geq \rho_0 \geq 0$ and $\rho_0 \in \mathbb{R}_+$.

Define the auxiliary function

$$\mathcal{E}(s) := X(t,s,X(s-\tilde{\zeta})) \quad \text{for} \quad s \in [\rho_0, t],$$

where it is assumed that $X(\rho_0 - \tilde{\zeta}, t_0, X_0) = X_0$. Then, by applying the chain rule, we obtain

$$\frac{d}{ds}\mathcal{E}(s) = \frac{\partial X}{\partial \rho_0}(t,s,X(s-\tilde{\zeta})) + \frac{\partial X}{\partial X_0}(t,s,X(s-\tilde{\zeta})) \cdot X'(s-\tilde{\zeta}). \qquad (9)$$

Integrating both sides of (9) from ρ_0 to t, we arrive at

$$\mathcal{X} = \mathcal{X}^* + \int_{\rho_0}^{t} \left[\frac{\partial X}{\partial \rho_0}(t,s,X(s-\tilde{\zeta})) + \gamma(t,s,X(s-\tilde{\zeta}))F(s-\tilde{\zeta}, X(s-\tilde{\zeta}))\right] ds. \qquad (10)$$

Next, consider the function

$$\mathcal{S}^*(s) := X(t,\rho_0,\tilde{\Omega}(s)), \quad \text{where } \tilde{\Omega}(s) := sY_0 + (1-s)X_0, \quad s \in [0,1].$$

Taking the derivative of $\mathcal{S}^*(s)$ w.r.t. s yields:

$$\frac{d}{ds}\mathcal{S}^*(s) = \frac{\partial X}{\partial Y_0}(t, \rho_0, \tilde{\Omega}(s)) \cdot (Y_0 - X_0). \tag{11}$$

Integrating (11) over $s \in [0,1]$, we obtain the relation

$$\tilde{\mathcal{X}} = \mathcal{X}^* + \int_0^1 \gamma(t, \rho_0, \tilde{\Omega}(s))\, ds \cdot (Y_0 - X_0). \tag{12}$$

Substituting the expression in (12) into (10) leads to the desired result given in (8), completed the proof. ◀

Remark 1. Suppose that the nonlinear matrix differential system (2) admits unique solutions $\mathcal{X}^*$ and $\tilde{\mathcal{X}}$, respectively. In the special case when $\tilde{\zeta} = 0$, these solutions are related by the following integral expression:

$$\tilde{\mathcal{X}} = \mathcal{X}^* + \int_0^1 \gamma(t, \rho_0, \tilde{\Omega}(s))\, ds \cdot (Y_0 - X_0), \quad t \geq \rho_0, \tag{13}$$

where the function $\tilde{\Omega}(s)$ is given by the convex combination $\tilde{\Omega}(s) = sY_0 + (1-s)X_0$, with $s \in [0,1]$.

Equation (13) thus characterizes the dependence between solutions of the same unperturbed matrix system (2) that differ only in their initial values.

Theorem 5. *Consider the system* (1) *satisfy the assumptions stated in Theorem 4, and Consider* $\bar{\mathcal{X}}$ *denote its solution. Then, for any solution* $\tilde{\mathcal{X}}$ *of* (2) *and* $\mathcal{X}$*:*

$$\tilde{\mathcal{X}} = \mathcal{X} - \int_{\rho_0}^{t} \frac{\partial X}{\partial \rho_0}\left(t, \nu, X(\nu - \tilde{\zeta})\right) d\nu \int_{\rho_0}^{t} \frac{\partial X}{\partial X_0}\left(t, \nu, X(\nu - \tilde{\zeta})\right) f\left(\nu - \tilde{\zeta}, X(\nu - \tilde{\zeta})\right) d\nu +$$

$$+ \int_0^1 \frac{\partial X}{\partial X_0}\left(t, \rho_0, \tilde{\Omega}(s)\right) ds \cdot (Y_0 - X_0),$$

where $\tilde{\Omega}(s) := sY_0 + (1-s)X_0$ *for* $s \in [0,1]$.

Proof. Consider the unique solution $\tilde{\mathcal{X}}$ to the unperturbed matrix system (2), defined for $t \geq \rho_0$, and the function $\mathcal{X}$, corresponding to system (1) initiated at (t_0, X_0), where $t_0 \in \mathbb{R}_+$. Assume further that all systems admit unique solutions in their respective domains.

Define the auxiliary function

$$\bar{\mathcal{E}}(\nu) := X(t, \nu, X(\nu - \tilde{\zeta})), \quad \text{for } \nu \in [\rho_0, t],$$

with the condition $X(\rho_0 - \tilde{\zeta}, t_0, X_0) = X_0$. By applying the chain rule, the total derivative of $\bar{\mathcal{E}}$ w.r.t. ν yields:

$$\frac{d}{d\nu}\bar{\mathcal{E}}(\nu) = \frac{\partial X}{\partial \nu}(t, \nu, x(\nu - \tilde{\zeta})) + \frac{\partial X}{\partial X}(t, \nu, x(\nu - \tilde{\zeta})) \cdot \frac{dX(\nu - \tilde{\zeta})}{d\nu} =$$

$$= \frac{\partial X}{\partial \nu}(t, \nu, x(\nu - \tilde{\zeta})) + \frac{\partial X}{\partial X}(t, \nu, x(\nu - \tilde{\zeta}))f(\nu - \tilde{\zeta}, x(\nu - \tilde{\zeta})) =$$
$$=: \widetilde{f}(\nu, x(\nu - \tilde{\zeta}), \tilde{\zeta}). \tag{14}$$

By Integrating of (14),

$$\mathcal{X} = X(t, \rho_0, X_0) + \int_{\rho_0}^{t} \widetilde{F}(\nu, x(\nu - \tilde{\zeta}), \tilde{\zeta})\, d\nu. \tag{15}$$

Now define another auxiliary function

$$\tilde{\mathcal{S}}^*(s) := X(t, \rho_0, \tilde{\Omega}(s)), \quad \text{where} \quad \tilde{\Omega}(s) := sY_0 + (1-s)X_0, \quad s \in [0,1].$$

Then, the derivative w.r.t. s is

$$\frac{d}{ds}\tilde{\mathcal{S}}^*(s) = \frac{\partial X}{\partial X_0}(t, \rho_0, \tilde{\Omega}(s)) \cdot (Y_0 - X_0). \tag{16}$$

Integrating (16) over the interval $s \in [0,1]$ yields:

$$\tilde{\mathcal{X}} = X(t, \rho_0, X_0) + \int_0^1 \frac{\partial X}{\partial X_0}(t, \rho_0, \tilde{\Omega}(s))\, ds \cdot (Y_0 - X_0). \tag{17}$$

Combining the expressions in (15) and (17), we deduce the following identity:

$$\tilde{\mathcal{X}} = \mathcal{X} + \int_0^1 \frac{\partial X}{\partial X_0}(t, \rho_0, \tilde{\Omega}(s))\, ds \cdot (Y_0 - X_0) - \int_{\rho_0}^{t} \widetilde{F}(\nu, x(\nu - \tilde{\zeta}), \tilde{\zeta})\, d\nu.$$

The proof is completed. ◄

Remark 2. Consider $\bar{\mathcal{X}}$ denote the solution of the unperturbed matrix system (1). Then, for any solution $\tilde{\mathcal{X}}$ of the unperturbed matrix system (2), the corresponding unperturbed matrix solution $\mathcal{X}$ satisfies the following integral representation:

$$\tilde{\mathcal{X}} = \mathcal{X} + \int_0^1 \frac{\partial X}{\partial X_0}(t, \rho_0, \tilde{\Omega}(s))\, ds \cdot (Y_0 - X_0) - \int_{\rho_0}^{t} \widetilde{F}(\nu, x(\nu - \tilde{\zeta}), \tilde{\zeta})\, d\nu,$$

where $\tilde{\Omega}(s) := sY_0 + (1-s)X_0$ for $s \in [0,1]$.

Remark 3. Suppose that the solution $\tilde{\mathcal{X}}$ of the nonlinear unperturbed matrix system (2) possesses continuous partial derivatives with respect to both ρ_0 and Y_0. That is, $\frac{\partial X}{\partial \rho_0}$ and $\frac{\partial X}{\partial X_0}$ exist and are continuous on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$.

Assume further that the systems (1) and (2) admit unique solutions $\tilde{\mathcal{X}}$ and $\mathcal{X}^*$, respectively, for all $t \geq \rho_0$, initialized at the same time.

For $\tilde{\zeta} = 0$, the solutions $\tilde{\mathcal{X}}$ and $\mathcal{X}^*$ satisfy the following integral relation:

$$\tilde{\mathcal{X}} - \mathcal{X}^* = -\int_{\rho_0}^{t} \left[\frac{\partial X}{\partial \rho_0}(t, \nu, X(\nu)) + \frac{\partial X}{\partial X_0}(t, \nu, x(\nu))\mathcal{F}^*(\nu, x(\nu))\right] d\nu +$$
$$+ \int_0^1 \frac{\partial X}{\partial X_0}(t, \rho_0, \tilde{\Omega}(s))\, ds \cdot (Y_0 - X_0),$$

where $\tilde{\Omega}(s) := sY_0 + (1-s)X_0$ for $s \in [0,1]$, and $t \geq \rho_0$.

This identity characterizes the deviation between (1) and (2) when the two trajectories begin at the same initial time but from different spatial positions.

Remark 4. Suppose that the unperturbed matrix vector field takes the form

$$\mathcal{F}^*(t,x) = F(t-\tilde{\zeta}, x) + \mathcal{R}^*(t-\tilde{\zeta}, x),$$

where $\mathcal{R}^*$ denotes the unperturbed system component.

Under these assumptions, the following integral representation holds:

$$\tilde{\mathcal{X}} = \mathcal{X} + \int_0^1 \frac{\partial X}{\partial X_0}(t, \rho_0, \tilde{\Omega}(s))\, ds \cdot (Y_0 - X_0) + \int_{\rho_0}^t \mathcal{R}^*(\nu, X(\nu - \tilde{\zeta}))\, d\nu,$$

where $\tilde{\Omega}(s) := sY_0 + (1-s)X_0$ for $s \in [0,1]$.

This representation highlights the contribution of both the initial condition difference and the unperturbed remainder term to the overall system evolution.

Theorem 6. *Consider the system* (1) *satisfy the assumptions stated in Theorem 2, and Consider* $\bar{\mathcal{X}}$ *be its unique solution. Then, for any solution* $\tilde{\mathcal{X}}$ *of the unperturbed matrix system* (2)*, the difference between* $\tilde{\mathcal{X}}$ *and the shifted unperturbed matrix solution* $\mathcal{X}$ *can be expressed:*

$$\tilde{\mathcal{X}} - \mathcal{X} = X(t, \rho_0, Y_0 - X_0) + \int_{\rho_0}^t \overline{\mathcal{H}}\left(s, \tilde{d}(s), \tilde{\zeta}\right) ds,$$

where $\overline{\mathcal{H}}(s, \tilde{d}(s), \tilde{\zeta})$ *represents the integrand that depends on the unperturbed matrix dynamics, and* $\tilde{d}(s)$ *is a trajectory determined by the system's state evolution.*

Proof. Assume that all involved systems admit unique solutions in their respective domains. Define

$$\mathcal{S}^*(s) := X(t, s, \tilde{d}(s)), \text{ where } \tilde{d}(s) := X(s, \rho_0, X_0) - X(s - \tilde{\zeta}, t_0, X_0),$$

$s \in (\rho_0, t)$. Differentiating $\mathcal{S}^*(s)$ w.r.t. s, we compute:

$$\frac{d}{ds}\mathcal{S}^*(s) = \frac{\partial X}{\partial s}(t, s, \tilde{d}(s)) + \frac{\partial X}{\partial z}(t, s, \tilde{d}(s)) \cdot \frac{\tilde{d}s}{ds} =$$
$$= \frac{\partial X}{\partial s}(t, s, \tilde{d}(s)) + \frac{\partial X}{\partial z}(t, s, \tilde{d}(s)) \cdot \mathcal{H}(s, \tilde{d}(s), \tilde{\zeta}) =: \overline{\mathcal{H}}(s, \tilde{d}(s), \tilde{\zeta}), \tag{18}$$

where the function $\mathcal{H}(s, \tilde{d}(s), \tilde{\zeta})$ is defined by

$$\mathcal{H}(s, \tilde{d}(s), \tilde{\zeta}) = \mathcal{F}^*(s, \tilde{d}(s) + X(s - \tilde{\zeta})) - f(s - \tilde{\zeta}, X(s - \tilde{\zeta})).$$

Integrating both sides of (18) over the interval $[\rho_0, t]$, and noting that $\tilde{d}(t) = \tilde{\mathcal{X}} - \mathcal{X}$, we obtain the following representation:

$$\tilde{\mathcal{X}} - \mathcal{X} = X(t, \rho_0, Y_0 - X_0) + \int_{\rho_0}^t \overline{\mathcal{H}}(s, \tilde{d}(s), \tilde{\zeta})\, ds.$$

This completes the proof. ◀

In recent studies, the qualitative dynamics of nonlinear matrix differential systems involving discrepancies in both initial time and position have garnered increasing attention [7], [11]. Notably, the concept of Lipschitz stability employed in this context is defined not relative to the trivial solution, but in reference to the corresponding unperturbed matrix system, which itself incorporates deviations in both the initial state and starting time. These generalized stability notions concerning ITDs are employed in Section 4 to derive further analytical results.

4. On HS Analysis of Nonlinear Unperturbed Matrix Differential Models with Initial Time Difference

In what follows, we demonstrate that, in the nonlinear matrix setting, HS under ITD provides a stronger notion of stability than ITD stability alone, regardless of whether it is uniform or non-uniform.

Theorem 7. *[Lipschitz Stability] For nonlinear matrix differential systems exhibiting an ITD, Lipschitz stability w.r.t. the reference solution implies ITD stability–whether uniform or non-uniform–for the corresponding nonlinear matrix differential system.*

Proof. Let us consider the notion of Lipschitz stability under ITD as given in Definition 1. Suppose that the solution $\tilde{\mathcal{X}}$ of the unperturbed matrix system (2), initiated at (ρ_0, Y_0), satisfies the Lipschitz stability criterion w.r.t. the reference solution $\mathcal{X}$ of the unperturbed matrix system (1), where $\tilde{\zeta} = \rho_0 - t_0 \geq 0$ and $t \geq \rho_0$.

Then, by definition, there exists a constant $\mathcal{L}^* = \mathcal{L}^*(\rho_0) > 0$ s.t.:

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\rho_0) \left(\|Y_0 - X_0\| + |\rho_0 - t_0|\right), \quad \forall t \geq \rho_0.$$

Consider $\epsilon > 0$ be arbitrary. If we select

$$\|Y_0 - X_0\| < \frac{\epsilon}{2L} =: \delta_1(\epsilon, \rho_0), \quad \text{and} \quad |\rho_0 - t_0| < \frac{\epsilon}{2L} =: \delta_2(\epsilon, \rho_0),$$

then it follows that

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| < \epsilon, \quad \forall t \geq \rho_0,$$

which confirms that $\tilde{\mathcal{X}}$ is stable w.r.t ITD to the unperturbed matrix solution $\mathcal{X}$.

Moreover, if the constant $\mathcal{L}^*$ is independent of ρ_0, then the associated thresholds δ_1 and δ_2 are also independent of ρ_0, yielding uniform stability in the sense of ITD. ◀

Theorem 8. *[Hölder Stability Implies Stability] In nonlinear matrix differential systems, ITD Hölder stability ensures ITD stability w.r.t. the reference solution of the nonlinear matrix differential system.*

Proof. Consider the notion of ITD HS as given in Definition 1. Suppose that the solution $\tilde{\mathcal{X}}$ of the unperturbed matrix system (2), initialized at (ρ_0, Y_0), satisfies this condition w.r.t. the unperturbed matrix $\mathcal{X}$. Consider the associated Hölder exponent satisfy $0 < \lambda < 1$.

By assumption, there exists a constant $\mathcal{L}^* = \mathcal{L}^*(\rho_0) > 0$ s.t. for all $t \geq \rho_0$, the following inequality holds:

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\rho_0)\left(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda\right).$$

Consider $\epsilon > 0$ be arbitrary. Define

$$\delta_1(\epsilon, \rho_0) := \left(\frac{\epsilon}{2L}\right)^{1/\lambda}, \quad \delta_2(\epsilon, \rho_0) := \left(\frac{\epsilon}{2L}\right)^{1/\lambda}.$$

Then, for any $\|Y_0 - X_0\| < \delta_1$ and $|\rho_0 - t_0| < \delta_2$, we have

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| < \epsilon, \quad \forall t \geq \rho_0.$$

This confirms that $\tilde{\mathcal{X}}$ is stable w.r.t. the unperturbed matrix solution under the ITD HS criterion.

Moreover, if the constant $\mathcal{L}^*$ is independent of ρ_0, then so are the thresholds δ_1 and δ_2, ensuring uniform HS under ITD. ◄

Theorem 9. *ITD stability in nonlinear matrix differential systems does not necessarily imply Lipschitz or Hölder stability w.r.t. initial time shifts.*

For a rigorous discussion and proof, see [10], [11]. A specific counterexample illustrating the failure of this implication in the special case $Y_0 = X_0$ and $\rho_0 = t_0$ is presented in [2], Example 1.4.

Theorem 10. *Consider the nonlinear matrix differential systems* (1) *and* (2) *admit unique solutions* $\bar{\mathcal{X}}$ *and* $\tilde{\mathcal{X}}$*, respectively, defined for* $t \geq \rho_0$*, with* $\tilde{\zeta} := \rho_0 - t_0 \geq 0$*. Suppose the following conditions hold:*
(i) There exist finite, nonnegative constants $\mathcal{K}^*(\rho_0)$ *and* $\mathcal{M}^*(\rho_0)$ *s.t.*

$$\int_{\rho_0}^{\infty} \mathcal{N}^*(s)\,ds \leq \mathcal{K}^*(\rho_0), \quad \text{and} \quad \exp\left(\int_{\rho_0}^{\infty} \tilde{\mu}(s)\,ds\right) \leq \mathcal{M}^*(\rho_0),$$

where $\mathcal{N}^*(s), \tilde{\mu}(s) \in C([0,\infty), \mathbb{R}_+)$*.*
(ii) The function f *is Hölder continuous in both time and state variables, with exponent* $0 < \lambda < 1$*, and*

$$\|\tilde{b}(t)\|^\lambda \leq \left\| f\left(t, \tilde{b}(t, \rho_0, \tilde{b}_0) + \mathcal{X}\right) - f\left(t - \tilde{\zeta}, \mathcal{X}\right)\right\| \leq$$
$$\leq \mu(t)\|\tilde{b}(t)\|^\lambda + \frac{|\tilde{\zeta}|^\lambda}{\mathcal{K}^*(\rho_0)} N(t),$$

where $\tilde{b}(t, \rho_0, \tilde{b}_0) := \tilde{\mathcal{X}} - \mathcal{X}$*, and* $\tilde{b}_0 := \tilde{b}(\rho_0) = Y_0 - X_0$*.*

Then, $\tilde{\mathcal{X}}$ *is HS under ITD w.r.t. the reference solution* $\mathcal{X}$*.*

Proof. Consider $\tilde{\mathcal{X}}$ and $\mathcal{X}$ be the unique solutions to (2) and (1), respectively. Define

$$\tilde{b}(t) := \tilde{\mathcal{X}} - \mathcal{X}, \quad \tilde{b}_0 := \tilde{b}(\rho_0) = Y_0 - X_0.$$

Raising both sides to the power $\lambda \in (0,1)$, and applying the assumptions in (i) and (ii) of Theorem 10, we obtain the inequality:

$$\|\tilde{b}(t)\|^\lambda \leq \|\tilde{b}_0\|^\lambda + \int_{\rho_0}^{t} \left[\tilde{\mu}(s)\|v(s)\|^\lambda + \frac{|\tilde{\zeta}|^\lambda}{\mathcal{K}^*(\rho_0)} \mathcal{N}^*(s) \right] ds. \tag{19}$$

Let us consider

$$\tilde{d}(t) := \|\tilde{b}(t)\|^\lambda, \quad Z(\rho_0) = \|\tilde{b}_0\|^\lambda.$$

Then inequality (19) becomes:

$$\tilde{d}(t) \leq Z(\rho_0) + |\tilde{\zeta}|^\lambda + \int_{\rho_0}^{t} \tilde{\mu}(s)\tilde{d}(s)\, ds.$$

By applying Gronwall's inequality, we deduce:

$$\tilde{d}(t) \leq \left(z(\rho_0) + |\tilde{\zeta}|^\lambda \right) \exp\left(\int_{\rho_0}^{t} \tilde{\mu}(s)\, ds \right).$$

Thus, taking the λ-th root,

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{M}^*(\rho_0)^{1/\lambda} \left(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda \right),$$

which confirms HS under ITD.

Furthermore, if $\mathcal{M}^*(\rho_0)$ is uniformly bounded w.r.t. ρ_0, then the same inequality holds with a constant independent of ρ_0, implying uniform Hölder stability. ◀

Theorem 11. *Consider the nonlinear matrix differential systems* (1), (2), *and* (2) *admit unique solutions given by:*

- $\bar{\mathcal{X}}$, *defined for* $t \geq t_0$, *corresponding to the unperturbed matrix system* (1)*;*
- $\tilde{\mathcal{X}}$, *defined for* $t \geq \rho_0$, *for the shifted unperturbed matrix system* (2)*,*

where $\tilde{\zeta} := \rho_0 - t_0 \geq 0$ *and* $\mathcal{X}$ *is well-defined on* $t \geq \rho_0$. *Assume further that the followings hold:*

(i) the solution $\tilde{\mathcal{X}}$ *is uniformly HS under ITD w.r.t.* $\mathcal{X}$*;*

(ii) there exists $\mathcal{M}^*{}_1 > 0$ *and* $\lambda_1 \in C([0,\infty), \mathbb{R}_+)$, *s.t. for the fundamental matrix* $\gamma(t, s, X(s))$ *(as defined in the following Theorem 13), the following bounds hold:*

$$\|\gamma(t, s, X(s))\| \|Z\left(t, s, X(s)\right)\| \leq \mathcal{M}^*{}_1, \|\mathcal{R}*(s, X(s))\| \leq \lambda_1(s)\|X(s)\|$$

for $\|X(s)\| \leq \beta$.

Then, $\tilde{\mathcal{X}}$ *is uniformly HS in variation w.r.t. the solution* $\mathcal{X}$*, under ITD.*

Proof. Let us consider the expression obtained from Theorem 1 and the integral representation in equation (1), along with the HS result for $\tilde{b}(t) := \tilde{\mathcal{X}} - \mathcal{X}$ given in Theorem 10.

$$\tilde{\mathcal{X}} - \mathcal{X} = \tilde{b}(t) + \int_{\rho_0}^{t} \gamma(t, s, X(s)) Z\,(t, s, X(s))\, \mathcal{R}^*(s, X(s))\, ds.$$

Taking norms and applying the triangle inequality:

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \le \mathcal{L}^*(\rho_0) \left(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda\right) +$$

$$+ \int_{\rho_0}^{t} \|\gamma(t, s, X(s))\| \|Z\,(t, s, X(s))\,\| \|\mathcal{R}^*(s, X(s))\| ds.$$

Under assumption (ii), we have $\|\gamma(t, s, X(s))\| \|Z\,(t, s, X(s))\,\| \le \mathcal{M}^*{}_1$ and $\|\mathcal{R}^*(s, X(s))\| \le \lambda_1(s) \|X(s)\|$ for $\|X(s)\| \le \beta$, so

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \le \mathcal{L}^*(\rho_0)(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda) + \mathcal{M}^*{}_1 \int_{\rho_0}^{t} \lambda_1(s) \|X(s)\|\, ds.$$

Let us consider

$$\mathcal{A}^*(t) := \|\tilde{\mathcal{X}} - \mathcal{X}\|, \quad \mathit{\Lambda}(t) := \mathcal{M}^*{}_1 \lambda_1(t).$$

Then the inequality becomes:

$$\mathcal{A}^*(t) \le C + \int_{\rho_0}^{t} \lambda(s) T(s)\, ds,$$

where

$$\mathcal{C} := \mathcal{L}^*(\rho_0)(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda) + \beta \mathcal{M}^{*2}_1 \int_{\rho_0}^{t} \lambda_1(s)\, ds.$$

By Gronwall's inequality,

$$\mathcal{A}^*(t) \le \mathcal{C} \cdot \exp\left(\int_{\rho_0}^{t} \lambda(s)\, ds\right), \quad t \ge \rho_0,$$

where

$$\mathcal{M}^* \ge [\|Y_0 - X_0\| + \|\rho_0 - t_0\|]^{-1}$$

$$\left[\left[\left(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda\right) \mathcal{L}^*\,(\rho_0) + \alpha \mathcal{M}^{*2}_1 \mathit{\Omega}(\rho_0)\right]\right] \exp\left[\mathcal{M}^*{}_1 \mathcal{N}^*{}_1\,(\tau_0)\right].$$

Hence, $\exists\ \mathcal{M}^* > 0$ s.t.

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \le \mathcal{M}^* \left(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda\right),$$

demonstrating that $\tilde{\mathcal{X}}$ is uniformly HS in variation w.r.t. the reference solution $\mathcal{X}$ under ITD. ◄

Theorem 12. *Consider the nonlinear matrix differential systems* (1) *and* (2) *admit unique solutions* $\tilde{\mathcal{X}}$ *and* $\mathcal{X}$*, respectively, defined for* $t \geq \rho_0$*, with* $\tilde{\zeta} := \rho_0 - t_0 \geq 0$*, and Consider* $\tilde{b}(t, \rho_0, \tilde{b}_0) := \tilde{\mathcal{X}} - \mathcal{X}$*. Assume the following conditions hold:*

(i) If $\mathcal{S}^*, \tilde{s} \in \bar{K}$*, where* $\bar{K} := \{\mathcal{S}^* \in C([0,\infty), \mathbb{R}_+) | \mathcal{S}^*$ *strictly increasing,* $\mathcal{S}^*(0) = 0$*,* $\lim_{t\to\infty} \tilde{s}(t) = \infty\}$*, and* $\tilde{p}$ *is an antiderivative of* $\mathcal{S}^*$*, then*

$$\lim_{h\to 0^+} \inf \frac{\|\tilde{b}(t+h)\|^\lambda - \|\tilde{b}(t)\|^\lambda}{h} \leq$$

$$\leq \lim_{h\to 0^-} \frac{\|\tilde{b}(t) + [F(t, X(t-\tilde{\zeta}) + \tilde{b}(t)) - F(t, X(t-\tilde{\zeta}))]h\| - \|\tilde{b}(t)\|}{h} \leq -\tilde{s}(t)\|\tilde{b}(t)\|^\lambda.$$

(ii) The nonlinear matrix f *is Hölder continuous, and satisfies*

$$\|F(t, X(t-\tilde{\zeta})) - F(t-\tilde{\zeta}, X(t-\tilde{\zeta}))\| \leq \frac{|\tilde{\zeta}|^\lambda}{\mathcal{L}^*{}_2(\rho_0)} \mathcal{L}^*{}_1(t),$$

where $\mathcal{L}^*{}_1 \in C([0,\infty), \mathbb{R}_+)$ *and* $\mathcal{L}^*{}_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ *are s.t.*

$$\int_{\rho_0}^{\infty} \exp(\mathcal{S}^*(u) - \mathcal{S}^*(\rho_0)) \mathcal{L}^*{}_1(u)\, du = \mathcal{L}^*{}_2(\rho_0).$$

Then, $\tilde{\mathcal{X}}$ *is generalized exponentially asymptotically HS regarding the solution* $\mathcal{X}$*.*

Proof. Define

$$\tilde{b}(t) := \tilde{\mathcal{X}} - \mathcal{X}, \quad \mathcal{A}^*(t) := \|\tilde{b}(t)\|^\lambda.$$

Under assumption (i),

$$\mathcal{A}^{*\prime}{}_+(t) \leq -\tilde{p}(t)\mathcal{A}^*(t) + \frac{|\tilde{\zeta}|^\lambda}{\mathcal{L}^*{}_2(\rho_0)} \mathcal{L}^*{}_1(t), \quad \mathcal{A}^*(\rho_0) = \|Y_0 - X_0\|^\lambda$$

is satisfied.

This is a linear inhomogeneous inequality. By variation of parameters and by multipliying with $\exp\left(\int_{\rho_0}^t \mathcal{S}^*(s)ds\right)$ to obtain

$$\mathcal{A}^*(t) \leq e^{\left(-\int_{\rho_0}^t \mathcal{S}^*(s)ds\right)} \left[T(\rho_0) + \frac{|\tilde{\zeta}|^\lambda}{\mathcal{L}^*{}_2(\rho_0)} \int_{\rho_0}^t \exp\left(\int_{\rho_0}^u \mathcal{S}^*(s)ds\right) \mathcal{L}^*{}_1(u)\, du\right].$$

Using assumption (ii), we have:

$$\int_{\rho_0}^{\infty} \exp(\mathcal{S}^*(u) - \mathcal{S}^*(\rho_0)) \mathcal{L}^*{}_1(u)\, du = \mathcal{L}^*{}_2(\rho_0),$$

$$\mathcal{A}^*(t) \leq \exp(\mathcal{S}^*(\rho_0) - \mathcal{S}^*(t)) \left(\|Y_0 - X_0\|^\lambda + |\tilde{\zeta}|^\lambda\right).$$

Hence,

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \exp\left(\frac{\mathcal{S}^*(\rho_0) - \mathcal{S}^*(t)}{\lambda}\right) \left(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda\right)^{1/\lambda}.$$

Finally, setting $\mathcal{M}^* := \exp\left(\frac{\mathcal{S}^*(\rho_0)-\mathcal{S}^*(t)}{\lambda}\right)$, we obtain:

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{M}^* \left(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda\right)^{1/\lambda},$$

where $\mathcal{M}^* = \mathcal{L}^{*\lambda} > 0$ is the resulting stability constant. Which confirms that $\tilde{\mathcal{X}}$ means generalized exponentially asymptotically Hölder w.r.t. $\mathcal{X}$. Hence, by invoking Definition 6 with $\mathcal{L}^* = 1$, we conclude that the solution $\tilde{\mathcal{X}}$ to the unperturbed matrix system (2) satisfies the criterion for generalized exponentially asymptotic HS relative to the reference trajectory $\mathcal{X}$. ◀

Remark 5. Consider the systems (2) and (1) admit unique solutions $\tilde{\mathcal{X}}$ and $\mathcal{X}$, respectively, for $t \geq \rho_0$ and $\tilde{\zeta} = \rho_0 - t_0 \geq 0$ with

$$\tilde{b}(t) := \tilde{\mathcal{X}} - \mathcal{X}.$$

(i) There exists a constant $\theta > 0$ s.t. $\tilde{b}(t)$ satisfies the following

$$\lim_{h\to 0^+} \sup \frac{\|\tilde{b}(t+h)\|^\lambda - \|\tilde{b}(t)\|^\lambda}{h} \leq$$

$$\leq \lim_{h\to 0^-} \frac{\|\tilde{b}(t) + [F(t, X(t-\tilde{\zeta}) + \tilde{b}(t)) - F(t, X(t-\tilde{\zeta}))]h\| - \|\tilde{b}(t)\|}{h} \leq -\theta\|\tilde{b}(t)\|^\lambda.$$

(ii) The nonlinear matrix f is HS in time with exponent $0 < \lambda < 1$, and satisfies

$$\|F(t, X(t-\tilde{\zeta})) - F(t-\tilde{\zeta}, X(t-\tilde{\zeta}))\| \leq \frac{|\tilde{\zeta}|^\lambda}{\mathcal{L}^*{}_2(\rho_0)} \mathcal{L}^*{}_1(t),$$

for some functions $\mathcal{L}^*{}_1 \in C([0,\infty), \mathbb{R}_+)$ and $\mathcal{L}^*{}_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfying

$$\int_{\rho_0}^{\infty} \exp\left(\theta(s-\rho_0)\right) \mathcal{L}^*{}_1(s)\, ds = \mathcal{L}^*{}_2(\rho_0).$$

Under these conditions, the solution $\tilde{\mathcal{X}}$ is exponentially asymptotically HS w.r.t $\mathcal{X}$.

Proof. Define the difference function $\tilde{\xi}(t) := \tilde{\mathcal{X}} - \mathcal{X}$, and set

$$\mathcal{A}^*(t) := \|\tilde{b}(t)\|^\lambda.$$

Under the assumptions, $\mathcal{A}^*(t)$ satisfies the inequality

$$D^+\mathcal{A}^*(t) \leq -\theta\mathcal{A}^*(t) + \frac{|\tilde{\zeta}|^\lambda}{\mathcal{L}^*{}_2(\rho_0)} \mathcal{L}^*{}_1(t), \quad T(\rho_0) = \|Y_0 - X_0\|^\lambda.$$

This leads to a linear differential inequality, whose solution yields

$$\mathcal{A}^*(t) \leq e^{-\theta(t-\rho_0)} \left[T(\rho_0) + \frac{|\tilde{\zeta}|^\lambda}{\mathcal{L}^*{}_2(\rho_0)} \int_{\rho_0}^{\infty} e^{\theta(s-\rho_0)} \mathcal{L}^*{}_1(s)\, ds\right].$$

By applying assumption (ii), namely

$$\int_{\rho_0}^{\infty} e^{\theta(s-\rho_0)}\mathcal{L}^*{}_1(s)\,ds = \mathcal{L}^*{}_2(\rho_0),$$

we obtain

$$\mathcal{A}^*(t) \le e^{-\theta(t-\rho_0)}\left[\|Y_0 - X_0\|^\lambda + |\tilde{\zeta}|^\lambda\right],$$

which implies

$$\|\tilde{\mathcal{X}} - \mathcal{X}\|^\lambda \le e^{-\theta(t-\rho_0)}\left[\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda\right].$$

Taking the λ-th root on both sides, we get

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \le e^{-\theta(t-\rho_0)/\lambda}\left[\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda\right]^{1/\lambda}.$$

Therefore, there exists a constant $\mathcal{L}^* \ge 1$ s.t.

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \le \mathcal{L}^*\left(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda\right)^{1/\lambda},$$

and the exponential decay is governed by the constant $\mathcal{M}^* := \mathcal{L}^{*\lambda} > 0$. By applying Definition 6 with $\mathcal{L}^* = 1$ and choosing $\tilde{p}(t) = \theta t$ for some constant $\theta > 0$, $\tilde{\mathcal{X}}$ is exponentially asymptotically HS under ITD, w.r.t. the reference trajectory $\mathcal{X}$. ◀

Theorem 13. *Consider the nonlinear matrix differential systems* (1), (2), *and* (2) *admit unique solutions* $\bar{\mathcal{X}}$, $\tilde{\mathcal{X}}$, *and* $\tilde{\mathcal{X}}$ *with initial values at* (t_0, X_0) *and* (ρ_0, Y_0). *Consider* $\mathcal{X}$ *be well-defined for* $t \ge \rho_0$, *where* $\tilde{\zeta} := \rho_0 - t_0 \ge 0$. *Suppose, by the assumptions of Theorem 11:*

(i) The solution $\tilde{\mathcal{X}}$ *to the unperturbed matrix system* (2) *is exponentially asymptotically HS under ITD, w.r.t. the trajectory* $\mathcal{X}$.

(ii) There exists a constant $\mathcal{N}^* > 0$ $\lambda \in C([0,\infty), \mathbb{R}_+)$ *s.t.*

$$\|\gamma(t,s,X(s))\|\|Z\,(t,s,X(s))\,\| \le \mathcal{N}^* \exp[-\beta(t-s)]$$

and

$$\|\mathcal{R}^*(s, X(s))\| \le \lambda(s)\|X(s)\|,$$

where $\|X(s)\| \le \beta$ *for some* $\beta > 0$.

Then, the solution $\tilde{\mathcal{X}}$ *of the unperturbed matrix system* (2) *is exponentially asymptotically HS in variation under ITD, w.r.t. the solution* $\mathcal{X}$.

Proof. Under the assumptions of Theorem 11, and using hypotheses (i) and (ii), we consider the decomposition:

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \le \|\tilde{b}(t,\rho_0,\tilde{b}_0)\| + \int_{\rho_0}^{t} \|\gamma(t,s,X(s))\|\|Z\,(t,s,X(s))\,\|\|\mathcal{R}^*(s,X(s))\|\,ds \le$$

$$\le \mathcal{M}^*\left(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda\right)e^{-\beta(t-\rho_0)} + \int_{\rho_0}^{t} \mathcal{N}^*\lambda(s)e^{-\beta(t-s)}\|X(s)\|\,ds,$$

where $\tilde{b}(t) = \tilde{\mathcal{X}} - \mathcal{X}$, and $\|\gamma(t,s,X(s))\|\|Z\,(t,s,X(s))\,\| \le N e^{-\beta(t-s)}$ by assumption.

Now, define

$$\mathcal{A}^*(t) := e^{-\beta(t-\rho_0)}\|\tilde{\mathcal{X}} - \mathcal{X}\|.$$

Then,

$$\mathcal{A}^*(t) \leq \mathcal{M}^* \left(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda\right) + \int_{\rho_0}^{t} \mathcal{N}^* \lambda(s)\mathcal{A}^*(s)\, ds.$$

By Gronwall's inequality, this yields

$$\mathcal{A}^*(t) \leq \left[\mathcal{M}^* \left(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda\right)\right] \cdot \exp\left(\mathcal{N}^* \int_{\rho_0}^{t} \lambda(s)\, ds\right).$$

then we have

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\beta, \rho_0) \left(\|Y_0 - X_0\|^\lambda + |\rho_0 - t_0|^\lambda\right) e^{-\beta(t-\rho_0)},$$

where the constant $\mathcal{L}^*(\beta, \rho_0)$ is given by

$$\mathcal{L}^*(\beta, \rho_0) := \mathcal{M}^* \cdot \exp\left(\mathcal{N}^* \int_{\rho_0}^{\infty} \lambda(s)\, ds\right).$$

Thus, $\tilde{\mathcal{X}}$ is exponentially asymptotically HS in variation w.r.t. the reference trajectory $\mathcal{X}$, under ITD. ◀

Theorem 14. *If a nonlinear matrix differential system exhibits exponentially asymptotic (uniform) HS under ITD, then the corresponding nonlinear matrix matrix differential system is also asymptotically (uniform) HS under ITD.*

The result follows directly from Definition 4, which characterizes asymptotic HS in the presence of an initial time shift. Since exponential decay implies asymptotic behavior, the claim is immediate. For brevity, the detailed steps are omitted.

Theorem 15. *If a nonlinear matrix differential system satisfies exponentially asymptotic (uniform) HS in variation under ITD, then it also satisfies asymptotic (uniform) HS in variation under ITD.*

This result is a direct consequence of Definition 5, which characterizes HS in variation w.r.t. initial time shifts. Since exponential decay implies asymptotic decay, the conclusion follows immediately. Detailed verification is omitted for brevity.

Acknowledgements We are grateful to the reviewers for their helpful critiques and suggestions, which have contributed to the improvement of this work.

References

1. Brauer F., Strauss A. Perturbations of nonlinear systems of differential equations. III. *J. Math. Anal. Appl.*, 1970, **31** (1), pp. 37-48.
2. Dannan F.M., Elaydi S. Lipschitz stability of nonlinear systems of differential equations. *J. Math. Anal. Appl.*, 1986, **113** (2), pp. 562-577.
3. Dannan F.M., Elaydi S. Lipschitz stability of nonlinear systems of differential equations. II. Liapunov functions. *J. Math. Anal. Appl.*, 1989, **143** (2), pp. 517-529.
4. Fausett D.W., Köksal S. *Variation of parameters formula and Lipschitz stability of nonlinear matrix differential equations.* World Congress of Nonlinear Analysts '92, **I-IV** (Tampa, FL, 1992), de Gruyter, Berlin, 1996, pp. 1415-1426.
5. Horn R.A., Johnson Ch.R. *Topics in Matrix Analysis.* Cambridge Univ. Press, Cambridge, 1991.
6. Lakshmikantham V., Leela S. *Differential and Integral Inequalities: Theory and Applications.* **I**: *Ordinary Differential Equations. Mathematics in Science and Engineering*, **55-I**. Academic Press, New York-London, 1969, pp. 131-190.
7. Shaw M.D., Yakar C. Generalized variation of parameters with initial time difference and a comparison result in term Lyapunov-like functions. *Int. J. Nonlinear Differ. Equ. Theory-Methods Appl.*, 1999, **5** (1), pp. 86-108.
8. Wang P., Liu X. Nonlinear variation of parameters formula for impulsive differential equations with initial time difference and application. *Abstr. Appl. Anal.*, 2014, **2014**, Art. ID 725832, pp. 1-6.
9. Yakar C. *Stability Analysis of Nonlinear Differential Systems with Initial Time Difference.* Thesis (Ph.D.) - Florida Institute of Technology. ProQuest LLC, Ann Arbor, MI, 2000.
10. Yakar C. *Variation-of-parameters formulae and Lipschitz stability criteria for nonlinear matrix differential equations with initial time difference.* Differential & Difference Equations and Applications, Hindawi Publ. Corp., New York, 2006, pp. 1201-1216.
11. Yakar C., Shaw M.D. Initial time difference stability in terms of two measures and a variational comparison result. *Dynamics of Continuous, Discrete and Impulsive Systems. Ser. A: Mathematical Analysis*, 2008, **15** (3), pp. 417-425.