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ON AN APPROXIMATE SOLUTION OF ONE CLASS OF SYSTEMS OF INTEGRAL EQUATIONS OF THE SECOND KIND

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Abstract. *This paper provides a justification of the collocation method for a system of integral equations arising in boundary value problems for Maxwell's equations. At specifically chosen collocation points, the system of integral equations is replaced by a system of algebraic equations, for which the existence and uniqueness of the solution are established. The convergence of the solutions of the algebraic system to the exact solution of the system of integral equations is proven, and the convergence rate of the method is indicated.*

Keywords: Maxwell's equations, systems of integral equations, collocations method

Mathematics Subject Classification (2020): 45E05, 31B10

1. Introduction and Problem Statement

It is well known that one of the methods for solving boundary value problems for Maxwell's equations is to reduce the problem to a system of second kind integral equations. Clearly, such systems admit closed-form solutions only in very rare cases. Therefore, the development of approximate methods with rigorous theoretical justification becomes a matter of primary importance.

Let $D \subset \mathbb{R}^3$ be a bounded domain with a twice continuously differentiable boundary Ω, and let $n(x) = (n_1(x), n_2(x), n_3(x))$ denote the outward unit normal at point $x \in \Omega$. Consider the following boundary value problems:

Interior Maxwell boundary value problem. Find two vector fields $E, H \in C^{(1)}(D) \bigcap C(\bar{D})$ satisfying the Maxwell's equations

$$rot\, E - ikH = 0, \quad rot\, H + ikE = 0 \ \text{ in } \ D$$

and the boundary condition

$$[n, E] = f \ \text{ on } \ \Omega,$$

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where k is the wave number such that $Imk \geq 0$, $[a, b]$ denotes the vector cross product of vectors a and b. Here, $f = (f_1, f_2, f_3)$ is a given tangential field with the additional property that its surface divergence $Div f$ exists in the sense of the limit integral definition and is of class $H_\alpha(\Omega)$, where by $H_\alpha(\Omega)$ denote the space of all continuous functions on Ω satisfying the Hölder condition with exponent $\alpha \in (0, 1]$.

Exterior Maxwell boundary value problem. Find two vector fields $E, H \in C^{(1)}\left(\mathbb{R}^3 \backslash \bar{D}\right) \bigcap C\left(\mathbb{R}^3 \backslash D\right)$ satisfying the Maxwell's equations in $\mathbb{R}^3 \backslash \bar{D}$, the Silver-Müller radiation condition

$$\left[H, \frac{x}{|x|}\right] - E = o\left(\frac{1}{|x|}\right), \ |x| \to \infty,$$

and

$$\left[E, \frac{x}{|x|}\right] + H = o\left(\frac{1}{|x|}\right), \ |x| \to \infty,$$

uniformly in all directions $x/|x|$, and the boundary condition

$$[n, E] = f \quad \text{on} \quad \Omega,$$

where the function f have the same meaning as in the interior Maxwell boundary value problem.

Let

$$\Phi_k(x, y) = \frac{\exp(ik|x-y|)}{4\pi|x-y|}, \quad x, y \in \mathbb{R}^3, \quad x \neq y,$$

be the fundamental solution of the Helmholtz equation. As shown in [2, p. 126], the electromagnetic field of a surface distribution of magnetic dipoles

$$E(x) = rot \int_\Omega \Phi_k(x, y)\, \mu(y)\, d\Omega_y,$$

$$H(x) = \frac{1}{ik} E(x), x \in \mathbb{R}^3 \backslash D$$

with tangential density $\mu = (\mu_1, \mu_2, \mu_3) \in H_\alpha(\Omega)$, $0 < \alpha < 1$, solves the interior Maxwell problem in D provided μ is a solution of the system of integral equations

$$\mu(x) - 2\int_\Omega [n(x), rot_x\{\Phi_k(x, y)\, \mu(y)\}]\, d\Omega_y = -2f(x), \quad x \in \Omega. \tag{1}$$

It solves the exterior Maxwell problem in $\mathbb{R}^3 \backslash \bar{D}$ provided μ is a solution of the system of integral equations

$$\mu(x) + 2\int_\Omega [n(x), rot_x\{\Phi_k(x, y)\, \mu(y)\}]\, d\Omega_y = 2f(x), \quad x \in \Omega. \tag{2}$$

It is worth noting that approximate methods for solving systems of integral equations arising in conjugation boundary value problems for the Helmholtz equation in both two- and three-dimensional spaces were studied in [7] and [5], respectively. In [9], solution methods for systems of integral equations related to conjugation problems for Maxwell's equations were examined. The Cauchy problem for systems of nonlinear Volterra-type integral equations was considered in [1], while in [10], numerical methods were proposed for solving systems of hypersingular integral equations associated with a certain class of boundary value problems for the Helmholtz equation. As can be seen, despite a significant number of studies on approximate solutions to various boundary value problems using the method of integral equation systems, approximate solutions to boundary value problems for Maxwell's equations using the integral equation systems (1) and (2) have not yet been investigated. This paper is devoted to addressing this gap.

2. Main Results

It is worth emphasizing that the counterexample constructed by A.M. Lyapunov (see [3, pp. 84–86]) demonstrates that, in general, the derivative of a single-layer potential with a continuous density does not exist. However, taking into account the identity $(n(y), \mu(y)) = 0$, $\forall y \in \Omega$, we obtain (see [2, p. 60])

$$[n(x), rot_x \{\Phi_k(x, y) \mu(y)\}] = (n(x) - n(y), \mu(y)) \, grad_x \Phi_k(x, y) - \mu(y) \frac{\partial \Phi_k(x, y)}{\partial n(x)},$$

where (a, b) denotes the scalar product of vectors a and b. Then it is easy to see that the system of integral equations (1) can be rewritten in operator form as

$$(I - T)\, \mu^{\mathrm{T}} = -2 f^{\mathrm{T}}, \tag{3}$$

and the system of integral equations (2) as

$$(I + T)\, \mu^{\mathrm{T}} = 2 f^{\mathrm{T}}, \tag{4}$$

here the notation "a^{T}" denotes the transpose of the vector a, I is the identity operator and

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix},$$

where

$$(T_{11}\mu_1)(x) = 2 \int_\Omega \left((n_1(x) - n_1(y)) \frac{\partial \Phi_k(x, y)}{\partial x_1} - \frac{\partial \Phi_k(x, y)}{\partial n(x)} \right) \mu_1(y)\, d\Omega_y, \quad x \in \Omega,$$

$$(T_{12}\mu_2)(x) = 2 \int_\Omega (n_2(x) - n_2(y)) \frac{\partial \Phi_k(x, y)}{\partial x_1} \mu_2(y)\, d\Omega_y, \quad x \in \Omega,$$

$$(T_{13}\mu_3)(x) = 2 \int_\Omega (n_3(x) - n_3(y)) \frac{\partial \Phi_k(x, y)}{\partial x_1} \mu_3(y)\, d\Omega_y, \quad x \in \Omega,$$

$$(T_{21}\mu_1)(x) = 2 \int_\Omega (n_1(x) - n_1(y)) \frac{\partial \Phi_k(x, y)}{\partial x_2} \mu_1(y)\, d\Omega_y, \quad x \in \Omega,$$

$$(T_{22}\mu_2)(x) = 2 \int_\Omega \left((n_2(x) - n_2(y)) \frac{\partial \Phi_k(x, y)}{\partial x_2} - \frac{\partial \Phi_k(x, y)}{\partial n(x)} \right) \mu_2(y)\, d\Omega_y, \quad x \in \Omega,$$

$$(T_{23}\mu_3)(x) = 2\int_{\Omega} (n_3(x) - n_3(y)) \frac{\partial \Phi_k(x, y)}{\partial x_2} \mu_3(y)\, d\Omega_y, \quad x \in \Omega,$$
$$(T_{31}\mu_1)(x) = 2\int_{\Omega} (n_1(x) - n_1(y)) \frac{\partial \Phi_k(x, y)}{\partial x_3} \mu_1(y)\, d\Omega_y, \quad x \in \Omega,$$
$$(T_{32}\mu_2)(x) = 2\int_{\Omega} (n_2(x) - n_2(y)) \frac{\partial \Phi_k(x, y)}{\partial x_3} \mu_2(y)\, d\Omega_y, \quad x \in \Omega,$$
$$(T_{33}\mu_3)(x) = 2\int_{\Omega} \left((n_3(x) - n_3(y)) \frac{\partial \Phi_k(x, y)}{\partial x_3} - \frac{\partial \Phi_k(x, y)}{\partial n(x)} \right) \mu_3(y)\, d\Omega_y, \quad x \in \Omega.$$

As can be seen, by taking into account the inequality (see [12, p. 400])

$$|n(y) - n(x)| \le M^1 |y - x|, \quad \forall x, y \in \Omega,$$

it follows that the expressions $(T_{mp}\varphi)(x)$, $x \in \Omega$, $m, p = \overline{1,3}$, are weakly singular integrals. Therefore, the operators T_{mp}, $m, p = \overline{1,3}$, are compact in the space $C(\Omega)$. Consequently, the solutions of the integral equation systems (3) and (4) can be studied in the broader function space $C^3(\Omega) = C(\Omega) \times C(\Omega) \times C(\Omega)$, equipped with the norm $\|\rho\|_3 = \max_{m=\overline{1,3}} \|\rho_m\|_\infty$, where $\rho = (\rho_1, \rho_2, \rho_3)$ and $\|\rho_m\|_\infty = \max_{x\in\Omega} |\rho_m(x)|$.

To justify the collocation method, we first divide the surface Ω into "regular" elementary parts $\Omega = \bigcup_{l=1}^{N} \Omega_l$. A "regular" elementary parts is defined as a set of points satisfying the following conditions:

(1) For any $l \in \{1, 2, ..., N\}$, the elementary parts Ω_l is closed, and its set of interior points relative to Ω, denoted $\overset{0}{\Omega}_l$, is non-empty. Moreover, $mes\,\overset{0}{\Omega}_l = mes\Omega_l$ and $j \in \{1, 2, ...N\}$, $j \neq l$, $\overset{0}{\Omega}_l \bigcap \overset{0}{\Omega}_j = \emptyset$ hold;

(2) For any $l \in \{1, 2, ..., N\}$, the elementary parts Ω_l is a connected portion of the surface Ω with a continuous boundary;

(3) For any $l \in \{1, 2, ..., N\}$, there exists a so-called reference point $x(l) = (x_1(l), x_2(l), x_3(l)) \in \Omega_l$ such that:

(3.1) $r_l(N) \sim R_l(N)$ ($r_l(N) \sim R_l(N) \Leftrightarrow C_1 \le \frac{r_l(N)}{R_l(N)} \le C_2$, where C_1 and C_2 are positive constants independent of N), where $r_l(N) = \min_{x\in\partial\Omega_l} |x - x(l)|$ and $R_l(N) = \max_{x\in\partial\Omega_l} |x - x(l)|$;

(3.2) $R_l(N) \le \frac{r_0}{2}$, where r_0 is the radius of the standard sphere for Ω (see [12, pp. 400–401]);

(3.3) $r_j(N) \sim r_l(N), \forall j \in \{1, 2, ..., N\}$.

It is clear that $r(N) \sim R(N)$ and $\lim_{N\to\infty} r(N) = \lim_{N\to\infty} R(N) = 0$, where $R(N) = \max_{l=\overline{1,N}} R_l(N)$ and $r(N) = \min_{l=\overline{1,N}} r_l(N)$.

Lemma. *[6]. When the surface Ω is partitioned into "regular" elementary parts $\Omega = \bigcup_{l=1}^{N} \Omega_l$, the following relation holds: $R(N) \sim \frac{1}{\sqrt{N}}$.*

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[1] From here on we will denote by M positive constants that are different in different inequalities.

Let us now consider the $3N$ dimensional matrix $T^{3N} = (t_{lj})_{l,j=1}^{3N}$ with elements:

$$t_{ll} = 0 \quad \text{for} \quad l = \overline{1,N};$$

$$t_{lj} = 2\left((n_1(x(l)) - n_1(x(j))) \frac{\partial \Phi_k(x(l)\,,\,x(j))}{\partial x_1} - \frac{\partial \Phi_k(x(l)\,,\,x(j))}{\partial n(x(l))} \right) mes\Omega_j$$

$$\text{for} \quad l,j = \overline{1,N} \quad \text{and} \quad j \neq l;$$

$$t_{l,N+l} = 0 \quad \text{for} \quad l = \overline{1,N};$$

$$t_{l,N+j} = 2(n_2(x(l)) - n_2(x(j))) \frac{\partial \Phi_k(x(l)\,,\,x(j))}{\partial x_1} mes\Omega_j \quad \text{for} \quad l,j = \overline{1,N} \quad \text{and} \quad j \neq l;$$

$$t_{l,2N+l} = 0 \quad \text{for} \quad l = \overline{1,N};$$

$$t_{l,2N+j} = 2(n_3(x(l)) - n_3(x(j))) \frac{\partial \Phi_k(x(l)\,,\,x(j))}{\partial x_1} mes\Omega_j \quad \text{for} \quad l,j = \overline{1,N} \quad \text{and} \quad j \neq l;$$

$$t_{N+l,l} = 0 \quad \text{for} \quad l = \overline{1,N};$$

$$t_{N+l,j} = 2(n_1(x(l)) - n_1(x(j))) \frac{\partial \Phi_k(x(l)\,,\,x(j))}{\partial x_2} mes\Omega_j \quad \text{for} \quad l,j = \overline{1,N} \quad \text{and} \quad j \neq l;$$

$$t_{N+l,N+l} = 0 \quad \text{for} \quad l = \overline{1,N};$$

$$t_{N+l,N+j} = 2\left((n_2(x(l)) - n_2(x(j))) \frac{\partial \Phi_k(x(l)\,,\,x(j))}{\partial x_2} - \frac{\partial \Phi_k(x(l)\,,\,x(j))}{\partial n(x(l))} \right) mes\Omega_j$$

$$\text{for} \quad l,j = \overline{1,N} \quad \text{and} \quad j \neq l;$$

$$t_{N+l,2N+l} = 0 \quad \text{for} \quad l = \overline{1,N};$$

$$t_{N+l,2N+j} = 2(n_3(x(l)) - n_3(x(j))) \frac{\partial \Phi_k(x(l)\,,\,x(j))}{\partial x_2} mes\Omega_j \quad \text{for} \quad l,j = \overline{1,N} \quad \text{and} \quad j \neq l;$$

$$t_{2N+l,l} = 0 \quad \text{for} \quad l = \overline{1,N};$$

$$t_{2N+l,j} = 2(n_1(x(l)) - n_1(x(j))) \frac{\partial \Phi_k(x(l)\,,\,x(j))}{\partial x_3} mes\Omega_j \quad \text{for} \quad l,j = \overline{1,N} \quad \text{and} \quad j \neq l;$$

$$t_{2N+l,N+l} = 0 \quad \text{for} \quad l = \overline{1,N};$$

$$t_{2N+l,N+j} = 2\,(n_2(x(l)) - n_2(x(j))) \frac{\partial \Phi_k(x(l)\,,\,x(j))}{\partial x_3} mes\Omega_j \quad \text{for} \quad l,j = \overline{1,N} \quad \text{and} \quad j \neq l;$$

$$t_{2N+l,2N+l} = 0 \quad \text{for} \quad l = \overline{1,N};$$

$$t_{2N+l,2N+j} = 2\left((n_3(x(l)) - n_3(x(j))) \frac{\partial \Phi_k(x(l)\,,\,x(j))}{\partial x_3} - \frac{\partial \Phi_k(x(l)\,,\,x(j))}{\partial n(x(l))} \right) mes\Omega_j$$

$$\text{for} \quad l,j = \overline{1,N} \quad \text{and} \quad j \neq l.$$

Proceeding in the same manner as in [4] and taking into account Lemma, it is not difficult to show that the expressions

$$\left(T_{mp}^{N}\varphi\right)(x(l))=\sum_{j=1}^{N}t_{(m-1)N+l,\,(p-1)N+j}\,\varphi(x(j)),\quad m,p=\overline{1,3},$$

evaluated at the reference points $x(l)$, $l=\overline{1,\,N}$, represent cubature formulas for the integrals $(T_{mp}\varphi)(x)$, $m,p=\overline{1,\,3}$, respectively, and moreover,

$$\max_{l=\overline{1,\,N}}\left|(T_{mp}\varphi)(x(l))-\left(T_{mp}^{N}\varphi\right)(x(l))\right|\le M\left(\,\|\varphi\|_{\infty}\,N^{-\frac{1}{2}}\ln N+\omega\left(\varphi,N^{-\frac{1}{2}}\right)\right),$$

where $\omega(\varphi,\delta)$ is the modulus of continuity of the function $\varphi\in C(\varOmega)$, i.e.

$$\omega(\varphi,\delta)=\max_{\substack{|x-y|\le\delta\\ x,y\in\varOmega}}|\varphi(x)-\varphi(y)|\;,\quad\delta>0.$$

Let C^{3N} be the space of vectors

$$z^{3N}=\left(z_{1}^{3N},z_{2}^{3N},...,z_{3N}^{3N}\right)^{\mathrm{T}},z_{l}^{3N}\in C,l=\overline{1,3N},$$

equipped with the norm $\left\|z^{3N}\right\|=\max\limits_{l=\overline{1,\,3N}}\left|z_{l}^{3N}\right|$, and let I^{3N} denote the identity operator in the space C^{3N}. Then, if we denote by z_{l}^{3N}, $l=\overline{1,\,N}$, the approximate values of $\mu_{1}(x(l))$; by z_{N+l}^{3N}, $l=\overline{1,\,N}$, the approximate values of $\mu_{2}(x(l))$; and by z_{2N+l}^{3N}, $l=\overline{1,\,N}$, the approximate values of $\mu_{3}(x(l))$, the systems of integral equations (3) and (4) are reduced to systems of algebraic equations with respect to $z^{3N}\in C^{3N}$, which we write in the form:

$$\left(I^{3N}-T^{3N}\right)z^{3N}=-2f^{3N}\tag{5}$$

and

$$\left(I^{3N}+T^{3N}\right)z^{3N}=2f^{3N},\tag{6}$$

respectively, where $f^{3N}=p^{3N}f^{\mathrm{T}}$ and $p^{3N}:\,C^{3}(\varOmega)\to C^{3N}$ is a bounded linear operator defined by

$$p^{3N}f^{\mathrm{T}}=\left(f_{1}(x(1)),...,f_{1}(x(N)),f_{2}(x(1)),...,f_{2}(x(N)),f_{3}(x(1)),...,f_{3}(x(N))\right)^{\mathrm{T}}.$$

Theorem 1. *Let $Im k>0$ and $f\in C^{3}(\varOmega)$. Then the systems of equations* (3) *and* (5) *have unique solutions $\mu_{*}^{\mathrm{T}}=(\mu_{1}^{*},\mu_{2}^{*},\mu_{3}^{*})^{\mathrm{T}}\in C^{3}(\varOmega)$ and $w^{3N}\in C^{3N}$, respectively, and the following estimate holds:*

$$\left\|\,w^{3N}-p^{3N}\mu_{*}^{\mathrm{T}}\right\|\le M\left(\omega\left(f,N^{-\frac{1}{2}}\right)+\|f\|_{3}\,N^{-\frac{1}{2}}\ln N\right).$$

Proof. To prove the theorem, we use G.M. Vainikko's theorem on the convergence of linear operator equations (see [11]). Let us verify the conditions of Theorem 4.2 from [11], using the notations, definitions, and propositions provided therein. In work [2, p. 124] it is proved that if $Imk > 0$, then the interior Maxwell boundary value problem admit at most one solution, i.e., if $Imk > 0$, then $Ker\ (I - T) = \{0\}$. It is clear that the operators $I^{3N} - T^{3N}$ are Fredholm operators of index zero. Taking into account the method of partitioning the surface Ω into "regular" elementary parts, it follows that for any vector function $\rho = (\rho_1, \rho_2, \rho_3)^{\mathrm{T}} \in C^3(\Omega)$, the identity

$$\lim_{N\to\infty} \left\| p^{3N}\rho \right\| = \lim_{N\to\infty} \max_{m=\overline{1,3}} \left\{ \max_{l=\overline{1,N}} \left| \rho_m\left(x\left(l\right)\right) \right| \right\} = \max_{m=\overline{1,3}} \left\{ \max_{x\in\Omega} \left| \rho_m\left(x\right) \right| \right\} = \|\rho\|_3$$

holds. Therefore, the operator system $P = \{ p^{3N} \}$ is linking the spaces $C^3(\Omega)$ and C^{3N}. Then $f^{3N} \overset{P}{\to} f^{\mathrm{T}}$, and by Definition 2.1 in [11], $I^{3N} - T^{3N} \overset{PP}{\to} I - T$. Since, by Definition 3.2 in [11], $I^{3N} \to I$ is stable, it remains, by Proposition 3.5 and Definition 3.3 in [11], to verify the compactness condition. According to Proposition 1.1 from [11], this is equivalent to the existence of a relatively compact sequence $\left\{T_{3N}\, z^{3N}\right\} \subset C^3(\Omega)$ such that

$$\left\| T^{3N} z^{3N} - p^{3N}\left(T_{3N}\, z^{3N}\right) \right\| \to 0 \ \text{ at } \ N \to \infty,$$

given that $\forall \left\{ z^{3N} \right\}$, $z^{3N} \in C^{3N}$ and $\left\| z^{3N} \right\| \le M$. As $\left\{T_{3N}\, z^{3N}\right\}$, we choose the sequence

$$\left(T_{3N}\, z^{3N}\right)(x) = \left(\sum_{j=1}^{3N} t_j^{(1)}(x)\ z_j^{3N},\ \sum_{j=1}^{3N} t_j^{(2)}(x)\ z_j^{3N},\ \sum_{j=1}^{3N} t_j^{(3)}(x)\ z_j^{3N} \right)^{\mathrm{T}},\ x \in \Omega,$$

where

$$t_j^{(1)}(x) = 2\int_{\Omega_j} \left(\left(n_1(x) - n_1(y)\right) \frac{\partial \Phi_k(x,\,y)}{\partial x_1} - \frac{\partial \Phi_k(x,\,y)}{\partial n(x)} \right) d\Omega_y \ \text{for}\ j = \overline{1,N},$$

$$t_j^{(1)}(x) = 2\int_{\Omega_{j-N}} \left(n_2(x) - n_2(y)\right) \frac{\partial \Phi_k(x,y)}{\partial x_1} d\Omega_y \ \text{ for } \ j = \overline{N,2N},$$

$$t_j^{(1)}(x) = 2\int_{\Omega_{j-2N}} \left(n_3(x) - n_3(y)\right) \frac{\partial \Phi_k(x,y)}{\partial x_1} d\Omega_y \ \text{for}\ j = \overline{2N,3N},$$

$$t_j^{(2)}(x) = 2\int_{\Omega_j} \left(n_1(x) - n_1(y)\right) \frac{\partial \Phi_k(x,\,y)}{\partial x_2} d\Omega_y \ \text{for}\ j = \overline{1,N},$$

$$t_j^{(2)}(x) = 2\int_{\Omega_{j-N}} \left(\left(n_2(x) - n_2(y)\right) \frac{\partial \Phi_k(x,\,y)}{\partial x_2} - \frac{\partial \Phi_k(x,\,y)}{\partial n(x)} \right) d\Omega_y \ \text{for}\ j = \overline{N,2N},$$

$$t_j^{(2)}(x) = 2\int_{\Omega_{j-2N}} \left(n_3(x) - n_3(y)\right) \frac{\partial \Phi_k(x,\,y)}{\partial x_2} d\Omega_y \ \text{for}\ j = \overline{2N,3N},$$

$$t_j^{(3)}(x) = 2\int_{\Omega_j} \left(n_1(x) - n_1(y)\right) \frac{\partial \Phi_k(x,\,y)}{\partial x_3} d\Omega_y \ \text{for}\ j = \overline{1,N},$$

$$t_j^{(3)}(x) = 2\int_{\Omega_{j-N}} (n_2(x) - n_2(y)) \frac{\partial \Phi_k(x, y)}{\partial x_3} d\Omega_y \text{ for } j = \overline{N, 2N},$$

$$t_j^{(3)}(x) = 2\int_{\Omega_{j-2N}} \left((n_3(x) - n_3(y)) \frac{\partial \Phi_k(x, y)}{\partial x_3} - \frac{\partial \Phi_k(x, y)}{\partial n(x)} \right) d\Omega_y \text{ for } j = \overline{2N, 3N}.$$

As can be seen, the expressions $t_j^{(m)}(x)$, $j = \overline{1, 3N}$, $m = \overline{1, 3}$, are weakly singular integrals. Therefore,

$$\left| \sum_{j=1}^{3N} t_j^{(m)}(x)\, z_j^{3N} \right| \le M \left\| z^{3N} \right\|, \forall x \in \Omega, m = \overline{1, 3}.$$

Moreover, following the approach in [8], it is not difficult to show that

$$\left| \sum_{j=1}^{3N} t_j^{(m)}(x')\, z_j^{3N} - \sum_{j=1}^{3N} t_j^{(m)}(x'')\, z_j^{3N} \right| \le$$

$$\le M \left\| z^{3N} \right\| |x' - x''| \left| \ln |x' - x''| \right|, \forall x', x'' \in \Omega, m = \overline{1, 3}.$$

Hence,

$$\left| \left(T_{3N}\, z^{3N} \right)(x) \right| \le M \left\| z^{3N} \right\|, \quad \forall x \in \Omega,$$

and

$$\left| \left(T_{3N}\, z^{3N} \right)(x') - \left(T_{3N}\, z^{3N} \right)(x'') \right| \le M \left\| z^{3N} \right\| |x' - x''| \left| \ln |x' - x''| \right|, \ \forall x', x'' \in \Omega.$$

Therefore, $\left\{ T_{3N}\, z^{3N} \right\} \subset C^3(\Omega)$, and taking into account condition $\left\| z^{3N} \right\| \le M$, we obtain the uniform boundedness and equicontinuity of the sequence $\left\{ T_{3N}\, z^{3N} \right\}$. Then, by the Arzela–Ascoli theorem, the sequence $\left\{ T_{3N}\, z^{3N} \right\}$ is relatively compact. In addition, by proceeding as in [4], one can show that

$$\left\| T^{3N}\, z^{3N} - p^{3N} \left(T_{3N}\, z^{3N} \right) \right\| \to 0 \text{ at } N \to \infty.$$

As a result, applying Theorem 4.2 from [11], we find that equations (3) and (5) have unique solutions $\mu_*^{\mathrm{T}} = (\mu_1^*, \mu_2^*, \mu_3^*)^{\mathrm{T}} \in C^3(\Omega)$ and $w^{3N} \in C^{3N}$, respectively, and

$$c_1\, \delta_N \le \left\| w^{3N} - p^{3N} \mu_*^{\mathrm{T}} \right\| \le c_2\, \delta_N,$$

where

$$c_1 = 1 / \sup_{N \ge N_0} \left\| I^{3N} - T^{3N} \right\| > 0, \; c_2 = \sup_{N \ge N_0} \left\| \left(I^{3N} - T^{3N} \right)^{-1} \right\| < +\infty,$$

$$\delta_N = \left\| \left(I^{3N} - T^{3N} \right) \left(p^{3N} \mu_*^{\mathrm{T}} \right) + 2 f^{3N} \right\|.$$

Since

$$-2 f^{3N} = -2 p^{3N} f^{\mathrm{T}} = p^{3N} \left(\mu_*^{\mathrm{T}} - T \mu_*^{\mathrm{T}} \right) = p^{3N} \mu_*^{\mathrm{T}} - p^{3N} \left(T \mu_*^{\mathrm{T}} \right),$$

then taking into account the error estimates of the cubature formulas for the integrals $(T_{mp}\varphi)(x)$, $m,p=\overline{1,\,3}$, we obtain

$$\delta_N = \left\|T^{3N}\left(p^{3N}\mu_*^{\mathrm{T}}\right) - p^{3N}\left(T\mu_*^{\mathrm{T}}\right)\right\| \le M\left(\left\|\mu_*^{\mathrm{T}}\right\|_3 N^{-\frac{1}{2}}\ln N + \omega\left(\mu_*^{\mathrm{T}}, N^{-\frac{1}{2}}\right)\right).$$

Moreover, following the same approach as in [8], one can show that

$$\omega\left(T\mu_*^{\mathrm{T}},\,\delta\right) \le M\left\|\,\mu_*^{\mathrm{T}}\right\|_3 \delta\,\left|\ln\delta\right|.$$

Then, taking into account the inequalities

$$\omega\left(\mu_*^{\mathrm{T}},\,N^{-\frac{1}{2}}\right) = \omega\left(-2f^{\mathrm{T}} + T\mu_*^{\mathrm{T}},\,N^{-\frac{1}{2}}\right) \le \omega\left(-2f^{\mathrm{T}},\,N^{-\frac{1}{2}}\right) + \omega\left(T\mu_*^{\mathrm{T}},\,N^{-\frac{1}{2}}\right) \le$$
$$\le M\left(\omega\left(f^{\mathrm{T}},\,N^{-\frac{1}{2}}\right) + \left\|\,\mu_*^{\mathrm{T}}\right\|_3 N^{-\frac{1}{2}}\ln N\right)$$

and

$$\left\|\,\mu_*^{\mathrm{T}}\,\right\|_3 = \left\|\,-2\,(I-T)^{-1}f^{\mathrm{T}}\,\right\|_3 \le 2\left\|\,(I-T)^{-1}\right\|\;\left\|\,f^{\mathrm{T}}\,\right\|_3,$$

we obtain

$$\delta_N \le M\left(\omega\left(f, N^{-\frac{1}{2}}\right) + \|f\|_3\,N^{-\frac{1}{2}}\ln N\right).$$

The theorem is proven. ◀

In a similar way, the following theorem can be proven.

Theorem 2. *Let $Im k > 0$ and $f \in C^3(\Omega)$. Then, the systems of equations* (4) *and* (6) *have unique solutions $\tilde{\mu}_*^{\mathrm{T}} = (\tilde{\mu}_1^*, \tilde{\mu}_2^*, \tilde{\mu}_3^*)^{\mathrm{T}} \in C^3(\Omega)$ and $\tilde{w}^{3N} \in C^{3N}$, respectively, and the following estimate holds:*

$$\left\|\,\tilde{w}^{3N} - p^{3N}\tilde{\mu}_*^{\mathrm{T}}\right\| \le M\left(\omega\left(f, N^{-\frac{1}{2}}\right) + \|f\|_3\,N^{-\frac{1}{2}}\ln N\right).$$

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