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ON $b-$REPDIGITS WHICH ARE SUMS OF THREE NARAYANA NUMBERS WITH A CONSEQUENCE

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Abstract. *The sequence known as Narayana's cows sequence is defined by the third-order linear recurrence relation* $N_n = N_{n-1} + N_{n-3}$, *for* $n \geq 3$, *with the initial values given by* $N_0 = N_1 = N_2 = 1$. *In this paper, we explore the class of numbers known as b-repdigits that can be represented as the sum of three terms from this sequence. Specifically, we aim to identify all such numbers for bases in the range* $2 \leq b \leq 30$. *Our approach relies on precise lower bounds for linear forms in logarithms, along with an enhanced version of the Baker-Davenport reduction method applied to Diophantine equations.*

Keywords: Narayana's cows sequence, b-repdigits, diophantine equations, Mersenne primes, lower bounds, logarithms, Baker-Davenport reduction method, recurrence relations

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1. Introduction

In the 14th century, the Indian mathematician Narayana Pandit wrote the remarkable mathematical text *Ganita Kaumudi*, which includes a classical problem about the reproduction of cows. The scenario considers a cow that gives birth to one calf each year, and from their fourth year onward, these calves also start producing one calf annually. The question posed in this context is: how many calves will be born after 20 years? This

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situation can be modeled using a third-order linear recurrence relation defined as:

$$N_n = N_{n-1} + N_{n-3}, \quad \text{for } n \geq 3,$$

with the initial conditions:

$$N_0 = N_1 = N_2 = 1.$$

The first few terms of this sequence are:

$$1, 1, 1, 2, 3, 4, 6, 9, 13, 19, \ldots \quad (\text{sequence } A000930).$$

This sequence, commonly referred to as the Narayana's cows sequence, differs from the well-known Fibonacci sequence in that each term is obtained by adding the previous term to the term three steps earlier, creating a delayed recursion.

A positive integer is called a b-repdigit if it consists of repeated occurrences of a single digit when written in base b. In this paper, we examine the class of b-repdigits that can be expressed as the sum of three terms from the Narayana's cows sequence, specifically for bases in the range $2 \leq b \leq 30$. Such a repdigit has the form:

$$R_b(a, \ell) = \frac{a(b^\ell - 1)}{b - 1},$$

where a is a base-b digit satisfying $1 \leq a \leq b - 1$, and $\ell \geq 3$ represents the number of repeated digits.

The main goal of this paper is to identify all integer solutions to the Diophantine equation:

$$N_n + N_m + N_q = R_b(a, \ell),$$

under the conditions $0 \leq m \leq n \leq q$, $2 \leq b \leq 30$, $1 \leq a \leq b-1$, and $\ell \geq 3$. This equation equates the sum of three Narayana numbers to a b-repdigit.

Similar problems involving recurrence sequences have been widely studied in the mathematical literature. For instance, Luca [8] investigated repdigits in the Fibonacci and Lucas sequences, while Faye and Luca [7] showed that the Pell sequence does not contain any repdigits. Moreover, the study of repdigits in generalized Fibonacci sequences has been extended to sums involving two or three terms [4], [3] and even to combinations of four terms from various recursive sequences [10], [11]. Our approach builds upon the foundational work presented in [2].

This paper extends the investigation to the Narayana's cows sequence by systematically identifying all numbers that can be expressed as sums of three terms from this sequence. We focus on the non-trivial cases where $\ell \geq 3$ and explore the algebraic structure that underlies these Diophantine problems.

The results presented here provide a deeper understanding of the distribution of Narayana numbers across different numeral bases, offering new insights into the occurrence of repdigits within this specific recursive context.

More specifically, our aim is to find all integer solutions to the Diophantine equation

$$N_n + N_m + N_q = a\left(\frac{b^\ell - 1}{b - 1}\right), \tag{1}$$

where the parameters (n, m, q, ℓ, a, b) are integers satisfying the constraints $0 \leq q \leq m \leq n$, $2 \leq b \leq 30$, $1 \leq a \leq b-1$, and $\ell \geq 3$.

Our main result regarding the solutions of equation (1) is as follows:

Theorem 1. *The Diophantine equation* (1) *admits only finitely many non-trivial integer solutions* (n, m, q, ℓ, a, b) *under the constraints* $0 \leq q \leq m \leq n$, $2 \leq b \leq 30$, $1 \leq a \leq b-1$, *and* $\ell \geq 3$. *Furthermore, all such solutions with* $\ell \geq 3$ *are explicitly listed in Section 5.*

As immediate consequences of this result, we derive the following corollaries:

Corollary 1. *No number can be a b-repdigit in more than one base* $b \leq 30$ *while satisfying equation* (1).

Corollary 2. *The maximum possible value of* ℓ *for which equation* (1) *has a solution is* $\ell = 10$. *Specifically, we have:*

$$N_{30} + N_{18} + N_{10} = 59048 = 2\left(\frac{3^{10}-1}{3-1}\right) = [2222222222]_3.$$

Corollary 3. *The largest b-repdigit expressible as the sum of three Narayana numbers with* $b \leq 30$ *is*

$$591890 = [qqqqq]_{28} = N_{36} + N_{26} + N_{16}.$$

Corollary 4. *The only Mersenne prime that can be expressed as a sum of three Narayana numbers is*

$$\begin{aligned} 31 = [11111]_2 &= N_8 + N_7 + N_7 = N_9 + N_6 + N_6 = N_9 + N_7 + N_4 \\ &= N_{10} + N_3 + N_0 = N_{10} + N_3 + N_1 = N_{10} + N_3 + N_2. \end{aligned}$$

Additionally, no single Narayana number is a repdigit.

2. Preliminaries and Notations

In this section, we provide an overview of some fundamental properties of the Narayana's cows sequence, which is crucial for our subsequent analysis.

The Narayana's cows sequence $(N_n)_{n\geq 0}$ is governed by a characteristic polynomial that captures its recursive structure. This polynomial is given by:

$$f(x) = x^3 - x^2 - 1,$$

which is known to be irreducible over the rational field $\mathbb{Q}[x]$. This polynomial has a single positive real root, denoted by α, and a pair of complex conjugate roots, β and γ, such that $|\beta| = |\gamma| < 1$. The approximate numerical value of the dominant root is $\alpha \approx 1.46557$.

We summarize several key properties of this sequence in the following lemma.

Lemma 1. *For the Narayana's cows sequence* $(N_n)_{n\geq 0}$, *the following properties hold:*

1. *Growth Bounds:*

$$\alpha^{n-2} \leq N_n \leq \alpha^{n-1}, \text{ for all } n \geq 1.$$

2. Binet-like Representation:

$$N_n = a_1\alpha^n + a_2\beta^n + a_3\gamma^n, \text{ for all } n \geq 0,$$

where the coefficients are given by:

$$a_1 = \frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}, \quad a_2 = \frac{\beta}{(\beta-\alpha)(\beta-\gamma)}, \quad a_3 = \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}.$$

3. Alternative Form:

$$N_n = C_\alpha\alpha^{n+2} + C_\beta\beta^{n+2} + C_\gamma\gamma^{n+2},$$

where

$$C_x = \frac{1}{x^3+2},$$

which offers a more refined approximation of N_n for large n.

4. Bounds on the Dominant Root:

$$1.45 < \alpha < 1.5, \quad 5 < C_\alpha^{-1} < 5.15.$$

5. Small Remainder Term:

$$\zeta_n = C_\beta\beta^{n+2} + C_\gamma\gamma^{n+2},$$

satisfies

$$|\zeta_n| < \frac{1}{2}, \text{ for all } n \geq 1.$$

Proof. Part (a) can be established through mathematical induction, leveraging the definition of the Narayana sequence. The Binet-like expression in part (b) is a classical result in the study of linear recurrence sequences, as presented in [12]. The reformulation in part (c) follows directly from (b) by isolating the dominant term. Part (d) is a straightforward numerical approximation, and (e) is derived using the triangle inequality and the known magnitudes of β and γ. For brevity, we omit the detailed steps here. ◀

Definition. *The minimal primitive polynomial for C_α over the integers is given by:*

$$31x^3 - 31x^2 + 9x - 1,$$

with all its roots lying within the unit circle, reflecting its stability as a scaling factor.

3. Upper Bounds for the Number of Solutions

In this section, we aim to derive upper bounds for the possible values of n that appear in the solutions of equation (1). We assume that the tuple (n, m, q, ℓ, a, b) represents a valid set of positive integers satisfying this equation. Our first step is to establish a relationship between the parameters ℓ and n, which will be crucial in proving Theorem 1.

Lemma 2. *If $n \geq 4$ and equation (1) holds, then the parameters ℓ and n satisfy the inequality:*

$$(n-2)\frac{\log\alpha}{\log b} < \ell < 1.5n.$$

Proof. From Lemma 1 (a), we have the inequality

$$\alpha^{n-2} \leq N_n + N_m + N_q = a\left(\frac{b^\ell - 1}{b-1}\right) < b^\ell,$$

which leads to the bound

$$(n-2)\frac{\log\alpha}{\log b} < \ell.$$

To establish the upper bound, we note that

$$b^{\ell-1} < a\left(\frac{b^\ell - 1}{b-1}\right) = N_n + N_m + N_q \leq 3\alpha^{n-1}.$$

Given that $b \geq 2$ and $\alpha < 1.5$, we can further simplify this to obtain:

$$\ell < 1 + \frac{\log 3}{\log b} + (n-1)\frac{\log\alpha}{\log b} < 3 + (n-1)\frac{\log 1.5}{\log 2} < 1.5n,$$

which holds for all $n \geq 4$, thus completing the proof. ◀

Next, we proceed to find a more precise upper bound on the parameter n.

3. 1. Deriving an Upper Bound for n

Starting from Lemma 1 and equation (1), we can rewrite this relationship as:

$$C_\alpha\alpha^{n+2} - \frac{ab^\ell}{b-1} = -N_m - N_q - \frac{a}{b-1} - \zeta_n.$$

Taking the absolute value of this equation and dividing by $C_\alpha\alpha^{n+2}$, we obtain:

$$\begin{aligned}\left|1 - \frac{ab^\ell}{C_\alpha\alpha^{n+2}(b-1)}\right| &< \frac{N_m}{C_\alpha\alpha^{n+2}} + \frac{N_q}{C_\alpha\alpha^{n+2}} + \frac{3}{2C_\alpha\alpha^{n+2}}\\ &\leq \frac{\alpha^{m-1}}{C_\alpha\alpha^{n+2}} + \frac{\alpha^{q-1}}{C_\alpha\alpha^{n+2}} + \frac{3}{2C_\alpha\alpha^{n+2}}\\ &< \frac{5.15\alpha^{m-1}}{\alpha^{n+2}} + \frac{5.15\alpha^{q-1}}{\alpha^{n+2}} + \frac{7.725}{\alpha^{n+2}}.\end{aligned}$$

Given that $1.45 < \alpha$, this can be further simplified to:

$$\left|\alpha^{-(n+2)}b^\ell\frac{a}{C_\alpha(b-1)} - 1\right| < \frac{8}{\alpha^{n-m}}.$$

We now define the following notation:

$$\gamma_1 := \alpha, \quad \gamma_2 := b, \quad \gamma_3 := \frac{a}{C_\alpha(b-1)}, \quad b_1 := -(n+2), \quad b_2 := \ell, \quad b_3 := 1, \tag{2}$$

and introduce the quantity

$$\Lambda_1 := \gamma_1^{b_1}\gamma_2^{b_2}\gamma_3^{b_3} - 1.$$

From the above, we obtain the inequality:

$$|\Lambda_1| < \frac{8}{\alpha^{n-m}}. \tag{3}$$

To proceed, we need a lower bound for $|\Lambda_1|$, which we derive using the powerful result from Matveev [9], as restated in Theorem 9.4 of [5].

We now proceed to apply Matveev's result from [9] with $t = 3$ and the parameters defined in equation (2). We first note that the algebraic number field containing $\gamma_1, \gamma_2, \gamma_3$ is given by $\mathbb{K} = \mathbb{Q}(\alpha)$, which has degree

$$D = [\mathbb{K} : \mathbb{Q}] = \deg(x^3 - x^2 - 1) = 3.$$

The logarithmic heights of these parameters are as follows:

- for $\gamma_1 = \alpha$, we have

$$h(\gamma_1) = \frac{\log \alpha}{3},$$

- for $\gamma_2 = b$, the height is simply

$$h(\gamma_2) = \log b.$$

Thus, we can choose the initial bounds:

$$A_1 = \log \alpha, \quad A_2 = 3 \log b.$$

Next, we estimate the logarithmic height of γ_3 using the general properties of logarithmic heights. From Definition, we have

$$h\left(\frac{a}{C_\alpha(b-1)}\right) \leq h\left(\frac{a}{b-1}\right) + h(C_\alpha) = \log(b-1) + h(C_\alpha). \tag{4}$$

Given that the minimal primitive polynomial of C_α over $\mathbb{Z}$ is $31x^3 - 31x^2 + 10x - 1$, this height is approximately

$$h(C_\alpha) = \frac{\log 31}{3}.$$

Substituting this in equation (4), we obtain

$$h(\gamma_3) \leq \log(b-1) + \frac{\log 31}{3} < 2 \log b.$$

Hence, we can choose

$$A_3 = 6 \log b.$$

Additionally, by Lemma 2, we have

$$B = \max\{n+2, \ell, 1\} = n+2.$$

To apply Matveev's result from [9], we must verify that $\Lambda_1 \neq 0$. If $\Lambda_1 = 0$, then we would have the identity

$$C_\alpha \alpha^{n+2} = \frac{ab^\ell}{b-1}, \tag{5}$$

which implies

$$C_\beta \beta^{n+2} = \frac{ab^\ell}{b-1}$$

when the Galois automorphism σ is applied, such that $\sigma(\alpha) = \beta$. However, this leads to a contradiction because

$$\left|C_\beta \beta^{n+2}\right| < |C_\beta| = 0.407506\ldots < 1,$$

while the right-hand side of equation (5) is at least 1 for all $\ell \geq 2$. Thus, we conclude that $\Lambda_1 \neq 0$.

Applying Matveev's result from [9] now gives the lower bound

$$\Lambda_1 > \exp\left(-1.4 \times 30^6 \times 3^{4.5} \times 3^2 (1+\log 3)(1+\log(n+2))(\log\alpha)(3\log b)(6\log b)\right),$$

which must be smaller than the upper bound obtained in inequality (3). Taking the natural logarithm on both sides yields

$$(n-m)\log\alpha - \log 8 < 2.10^{13}(1+\log(n+2))\log^2 b.$$

Given that $1+\log(n+2) \leq 2\log n$ for $n \geq 5$, this simplifies to

$$(n-m)\log\alpha < 1.1 \times 10^{14} \log n \log^2 b. \tag{6}$$

Next, to derive an upper bound on n in terms of q, we return to equation (1) and rearrange it as

$$C_\alpha \alpha^{n+2} + C_\alpha \alpha^{m+2} - \frac{ab^\ell}{b-1} = -\zeta_n - \zeta_m - N_q - \frac{a}{b-1}.$$

This leads to the inequality

$$\left| C_\alpha \alpha^{n+2}(1+\alpha^{m-n}) - \frac{ab^\ell}{b-1} \right| < N_q + 2.$$

Dividing this by $C_\alpha \alpha^{n+2}(1+\alpha^{m-n})$ gives

$$\left| \alpha^{-(n+2)} b^\ell \frac{a}{(b-1)C_\alpha(1+\alpha^{m-n})} \right| < \frac{N_q}{C_\alpha \alpha^{n+2}(1+\alpha^{m-n})} + \frac{2}{C_\alpha \alpha^{n+2}(1+\alpha^{n})} <$$

$$< \frac{5.15\alpha^{q-1}}{C_\alpha \alpha^{n+2}(1+\alpha^{m-n})} + \frac{5}{\alpha^n} < \frac{2}{\alpha^{n-q}} \times \frac{1}{1+\alpha^{m-n}} + \frac{5}{\alpha^n} < \frac{1}{\alpha^{n-q}} + \frac{5}{\alpha^n},$$

which reduces to

$$\left| \alpha^{-(n+2)} b^\ell \frac{a}{(b-1)C_\alpha(1+\alpha^{m-n})} \right| < \frac{6}{\alpha^{n-q}}. \tag{7}$$

We now define the parameters:

$$\gamma_1 := \alpha, \quad \gamma_2 := b, \quad \gamma_3 := \frac{a}{(b-1)C_\alpha(1+\alpha^{m-n})}, \quad b_1 := -(n+2), \quad b_2 := \ell, \quad b_3 := 1.$$

and the corresponding expression

$$\Lambda_2 := \gamma_1^{b_1}\gamma_2^{b_2}\gamma_3^{b_3} - 1.$$

From equation (7), we have the bound

$$|\Lambda_2| < \frac{6}{\alpha^{n-q}}. \tag{8}$$

To derive a lower bound for this quantity using Matveev's theorem [9], we proceed as before, setting the relevant parameters. Here, we take the number field $\mathbb{K} = \mathbb{Q}(\alpha)$ with degree $D = 3$, and the height bounds as:

$$A_1 = \log\alpha, \quad A_2 = 3\log b, \quad B = n+2.$$

Next, we estimate the height of γ_3 in this context. Using the properties of logarithmic heights, we have:

$$h(\gamma_3) \le 3\log b + \frac{(n-m)\log\alpha}{3}.$$

Substituting the earlier bound from equation (6), we find:

$$h(\gamma_3) < 3\log b + \frac{1.1\times 10^{14}\log n\log^2 b}{3}.$$

Thus, we can set

$$A_3 = 1.2\times 10^{14}\log n\log^2 b.$$

With these parameters, applying Matveev's result provides a lower bound for $|\Lambda_2|$, which we then compare to the upper bound in equation (8) to conclude that

$$(n-q)\log\alpha - \log 6 < 7.5\times 10^{26}\log^2 n\log^3 b.$$

This further simplifies to:

$$(n-q)\log\alpha < 7.5\times 10^{26}\log^2 n\log^3 b.$$

Next, to find an upper bound for n in terms of b, we revisit equation (1) and rewrite it as:

$$C_\alpha\alpha^{n+2} + C_\alpha\alpha^{m+2} + C_\alpha\alpha^{q+2} - \frac{ab^\ell}{b-1} = -\zeta_n - \zeta_m - \zeta_q - \frac{a}{b-1}.$$

This leads to the inequality:

$$\left| C_\alpha\alpha^{n+2}(1+\alpha^{m-n}+\alpha^{q-n}) - \frac{ab^\ell}{b-1}\right| < \frac{5}{2}.$$

Dividing by $C_\alpha\alpha^{n+2}(1+\alpha^{m-n}+\alpha^{q-n})$ gives:

$$\left|\alpha^{-(n+2)}b^\ell\frac{a}{(b-1)C_\alpha(1+\alpha^{m-n}+\alpha^{q-n})} - 1\right| < \frac{2.1}{\alpha^n}. \tag{9}$$

For this third application of Matveev's result, we define:

$$\gamma_1 := \alpha, \quad \gamma_2 := b, \quad \gamma_3 := \frac{a}{(b-1)C_\alpha(1+\alpha^{m-n}+\alpha^{q-n})},$$
$$b_1 := -(n+2), \quad b_2 := \ell, \quad b_3 := 1,$$
$$\Lambda_3 = \gamma_1^{b_1}\gamma_2^{b_2}\gamma_3^{b_3} - 1.$$

From equation (9), we have the upper bound:

$$|\Lambda_3| < \frac{2.1}{\alpha^n}.$$

As before, we ensure that $\Lambda_3 \neq 0$ by verifying that this expression cannot vanish. Using the same parameters as in the previous steps:

$$D = 3, \quad A_1 = \log\alpha, \quad A_2 = 3\log b, \quad B = n+2.$$

The height of γ_3 in this context is estimated as:

$$\begin{aligned}
h(\gamma_3) &\leq h\left(\frac{a}{(b-1)C_\alpha}\right) + h(1+\alpha^{m-n}+\alpha^{q-n}) \\
&\leq 2\log b + h(\alpha^{m-n}+\alpha^{q-n}) + \log 2 \\
&\leq 2\log b + \frac{(n-m)\log\alpha}{3} + \frac{(n-q)\log\alpha}{3} + 2\log 2 \\
&< 4\log b + \frac{(n-m)\log\alpha}{3} + \frac{(n-q)\log\alpha}{3} \\
&< 4\log b + \frac{1}{3}\left(1.1\times 10^{14}\log n\log^2 b + 7.5\times 10^{26}\log^2 n\log^3 b\right),
\end{aligned}$$

which allows us to set

$$A_3 = 1.5\times 10^{27}\log^2 n\log^3 b.$$

Applying Matveev's result again, we obtain the final bound:

$$n\log\alpha - \log(2.1) < 9.3\times 10^{39}\log^3 n\log^4 b.$$

which reduces to

$$n < 2.5\times 10^{40}\log^3 n\log^4 b,$$

which can be rewritten as:

$$\frac{n}{\log^3 n} < 2.5\times 10^{40}\log^4 b. \tag{10}$$

Next, we apply an analytical argument, as developed by Sanchez and Luca in [13], to further refine this bound.

We now proceed with the final step of our analysis, using the bound from equation (10) to tighten our estimate for n. Applying the analytical lemma:

Lemma 3. *If x and T are real numbers with $T > 16^2$ and*

$$\frac{x}{\log^2 x} < T, \quad \textit{then} \quad x < 4T\log^2 T.$$

We substitute $T = 2.5 \times 10^{40} \log^4 b$ into the bound from equation (10), yielding:

$$\begin{aligned} n &< 4\left(2.5 \times 10^{40} \log^4 b\right)\left(\log(2.5 \times 10^{40} \log^4 b)\right)^2 \\ &< \left(10^{41} \log^4 b\right)(93.1 + 4\log b)^2 \\ &< 10^{41} \log^4 b\,(132.2 \log b)^2 \\ &< 1.75 \times 10^{45} \log^6 b. \end{aligned}$$

In the above estimates, we used the fact that $93.1 + 4\log b < 132.2 \log b$ for all $b \geq 2$. Combining this with Lemma 2 and equation (1) (which gives $\ell < 1.5n$ and $m \leq n$), we can summarize our findings as follows:

Theorem 2. *Let (n, m, q, ℓ, a, b) be a solution of equation (1) with $\ell \geq 3, b \geq 2$, and $1 \leq a \leq b - 1$. Then,*

$$\max(\ell, m) \leq n < 1.75 \times 10^{45} \log^6 b.$$

Remark. For a fixed base $b \geq 2$, equation (1) has only finitely many possible solutions.

Next, we focus on further reducing this bound using the reduction method developed by Dujella and Pethő [6], which generalizes a classical result of Baker and Davenport [1]. This approach will allow us to refine the upper limit on n considerably.

4. Reduction Lemma and Improved Bounds

We begin with a few elementary properties of the exponential function that will be instrumental in the reduction steps.

Lemma 4. *For any non-zero real number x, the following hold:*

1. *If $0 < x < 1$, then $0 < x < |e^x - 1|$.*
2. *If $x < 0$ and $|e^x - 1| < \frac{1}{2}$, then $|x| < 2|e^x - 1|$.*

We define the quantity

$$\Gamma_1 := \ell \log b - (n+2)\log\alpha + \log\left(\frac{a}{C_\alpha(b-1)}\right),$$

noting that $\Gamma_1 \neq 0$ since $e^{\Gamma_1} - 1 = \Lambda_1 \neq 0$.

Lemma 5. *If $m = 0$ and $n \geq 7$, then*

$$0 < |\Gamma_1| < \frac{16}{\alpha^n}.$$

Proof. If $m = 0$, then equation (3) can be rewritten as

$$|e^{\Gamma_1} - 1| < \frac{8}{\alpha^n}.$$

If $\Gamma_1 > 0$, we apply part (a) of Lemma 1 to obtain

$$|\Gamma_1| = \Gamma_1 < |e^{\Gamma_1} - 1| < \frac{8}{\alpha^n}.$$

If instead $\Gamma_1 < 0$, then the condition $|e^{\Gamma_1} - 1| < \frac{1}{2}$ holds for all $n \geq 7$, allowing us to use part (b) of Lemma 1 to get

$$|\Gamma_1| < 2|e^{\Gamma_1} - 1| < \frac{16}{\alpha^n}.$$

In both cases, we have the desired bound. ◀

Lemma 6. *If $m \geq 1$, then*

$$0 < \Gamma_1 < \frac{8}{\alpha^{n-m}}.$$

Proof. Starting from equation (3), we have

$$|e^{\Gamma_1} - 1| < \frac{8}{\alpha^{n-m}}.$$

Additionally, from equation (1) and Lemma 1, we know that

$$C_\alpha \alpha^{n+2} = N_n - \zeta_n < N_n + \frac{1}{2} < N_n + N_m + N_q < \frac{ab^\ell}{b-1},$$

which ensures that $\Gamma_1 > 0$. Thus, by part (a) of Lemma 1, we obtain

$$\Gamma_1 < |e^{\Gamma_1} - 1| < \frac{8}{\alpha^{n-m}}.$$

◀

To proceed, we restrict our attention to bases b in the range $2 \leq b \leq 30$ for computational feasibility. This choice, while somewhat arbitrary, covers a wide spectrum of practical cases without significantly changing the nature of our results.

The following lemma, due to Bravo, Gomez, and Luca [3], is a refinement of a classic result by Dujella and Pethő [6], itself a generalization of the original Baker-Davenport reduction [1]. We will use this result to further reduce our bounds on n.

Lemma 7. *Let A, B, γ, μ be positive real numbers and M a positive integer. If $\frac{p}{q}$ is a convergent of the continued fraction expansion of the irrational number γ such that $q > 6M$, define*

$$\epsilon := \|\mu q\| - M\|\gamma q\|,$$

where $\|\cdot\|$ denotes the distance to the nearest integer. If $\epsilon > 0$, then the inequality

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

has no positive integer solutions (u, v, w) under the conditions

$$u \leq M \quad \textit{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

4. 1. Bounding n: Step 1

Assume first that $m = 0$. By substituting the expression of Γ_1 from Lemma 5 into the corresponding inequality and dividing through by $\log\alpha$, we obtain:

$$0 < \left| \ell \left(\frac{\log b}{\log \alpha} \right) - n + \left(\frac{\log\left(\frac{a}{(b-1)C_\alpha}\right)}{\log\alpha} - 2 \right) \right| < 42\alpha^{-n}. \tag{11}$$

This inequality is now suitable for an application of Lemma 7 using the following parameters:

$$u := \ell, \quad v := n, \quad w := n, \quad \gamma := \frac{\log b}{\log\alpha}, \quad \mu := \frac{\log\left(\frac{a}{(b-1)C_\alpha}\right)}{\log\alpha} - 2, \quad A := 43, \quad B := \alpha.$$

Note that γ is irrational, since $\alpha > 1$ is a unit in the ring of integers $\mathcal{O}_{\mathbb{K}}$ of the number field $\mathbb{K}$, implying that α and b are multiplicatively independent.

We set the bound $M := M_b = 1.75 \times 10^{45} \log^6 b$. Applying Lemma 7 to inequality (11) for all values $b \in \{2, \dots, 30\}$ and $a \in \{1, \dots, b-1\}$, a computational check using *Mathematica* shows that if $m = 0$, then any solution $(n, 0, q, \ell, a, b)$ to equation (1) must satisfy $n \le 328$.

We now consider the case $m \ge 1$. In this situation, Lemma 6 yields:

$$0 < \ell \left(\frac{\log b}{\log \alpha} \right) - n + \left(\frac{\log\left(\frac{a}{(b-1)C_\alpha}\right)}{\log\alpha} - 2 \right) < 22\alpha^{-(n-m)}. \tag{12}$$

Once again, applying Lemma 7 to inequality (12) for each $b \in \{2, \dots, 30\}$ and $a \in \{1, \dots, b-1\}$, we deduce that for all solutions (n, m, q, ℓ, a, b) of (1) with $m \ge 1$, the difference $n - m$ must lie in the interval $[0, 328]$.

4. 2. Bounding n: Step 2

In this step, we aim to derive an upper bound on $n - q$ using the previous bound on $n - m$.

We define the quantity

$$\Gamma_2 = \ell \log b - (n+2)\log\alpha + \log\left(\frac{a}{(b-1)C_\alpha(1 + \alpha^{m-n})} \right),$$

and note that inequality (8) is equivalent to

$$\left| e^{\Gamma_2} - 1 \right| < \frac{6}{\alpha^n}. \tag{13}$$

Since $e^{\Gamma_2} - 1 = \Lambda_2 \neq 0$, we conclude that $\Gamma_2 \neq 0$. From inequality (13) and Lemma 2, it follows that

$$0 < |\Gamma_2| < \frac{12}{\alpha^n}.$$

Dividing this inequality by $\log\alpha$ and substituting the expression for Γ_2, we obtain

$$0 < \left|\ell\left(\frac{\log b}{\log\alpha}\right) - n + \left(\frac{\log\left(\frac{a}{(b-1)C_\alpha(1+\alpha^{m-n})}\right)}{\log\alpha} - 2\right)\right| < 32\alpha^{-(n-q)}. \qquad (14)$$

We now apply Lemma 7 again, with the following parameters:

$$u := \ell,\ v := n,\ w := n,\ \gamma := \frac{\log b}{\log\alpha},\ A := 33,\ B := \alpha,\ \mu := \frac{\log\left(\frac{a}{(b-1)C_\alpha(1+\alpha^{m-n})}\right)}{\log\alpha} - 2.$$

By setting $M := M_b = 1.75 \times 10^{45}\log^6 b$, we apply Lemma 7 to inequality (14) for all choices of $b \in \{2,\ldots,30\}$, $a \in \{1,\ldots,b-1\}$, and $n-m \in \{0,\ldots,328\}$. A computational analysis performed using *Mathematica* reveals that any solution (n,m,q,ℓ,a,b) to equation (1) satisfies $n-q \le 334$.

4. 3. Bounding n: Step 3

In this final step, we leverage the bounds on $n-m$ and $n-q$ obtained previously to derive an upper bound on the variable n itself.

To proceed, we define the quantity

$$\Gamma_3 = \ell\log b - (n+2)\log\alpha + \log\left(\frac{a}{(b-1)C_\alpha\left(1+\alpha^{m-n}+\alpha^{q-n}\right)}\right),$$

and observe that inequality (9) can be reformulated as:

$$\left|e^{\Gamma_3} - 1\right| < \frac{2.1}{\alpha^n}. \qquad (15)$$

Since $e^{\Gamma_3} - 1 = \Lambda_3 \neq 0$, we confirm that $\Gamma_3 \neq 0$. Then, applying Lemma 2 to inequality (15), we deduce the following bound:

$$0 < |\Gamma_3| < \frac{4.2}{\alpha^n}.$$

Dividing the inequality above by $\log\alpha$ and substituting the expression of Γ_3 yields:

$$0 < \left|\ell\left(\frac{\log b}{\log\alpha}\right) - n + \left(\frac{\log\left(\frac{a}{(b-1)C_\alpha(1+\alpha^{m-n}+\alpha^{q-n})}\right)}{\log\alpha} - 2\right)\right| < 11\alpha^{-n}. \qquad (16)$$

We apply Lemma 7 once again with the following choice of parameters:

$$u := \ell,\ v := n,\ w := n,\ \gamma := \frac{\log b}{\log\alpha},$$

$$A := 12,\ B := \alpha,\ \mu := \frac{\log\left(\frac{a}{(b-1)C_\alpha(1+\alpha^{m-n}+\alpha^{q-n})}\right)}{\log\alpha} - 2.$$

Table 1. *Solutions of equation 1 with $\ell \geq 3$*

Base	**Sum**	**Representation** (n, m, q)	**Base Representation**
2	15	(6, 6, 4), (7, 4, 4), (7, 5, 3), (8, 0, 0), (8, 1, 0), (8, 1, 1), (8, 2, 0), (8, 2, 1), (8, 2, 2)	$[1111]_2$
2	31	(8, 7, 7), (9, 6, 6), (9, 7, 4), (10, 3, 0), (10, 3, 1), (10, 3, 2)	$[11111]_2$
2	63	(11, 8, 7), (11, 9, 4), (12, 3, 0), (12, 3, 1), (12, 3, 2)	$[111111]_2$
2	255	(15, 12, 6)	$[11111111]_2$
3	40	(9, 9, 3), (10, 6, 6), (10, 7, 4)	$[1111]_3$
3	80	(12, 9, 0), (12, 9, 1), (12, 9, 2)	$[2222]_3$
3	121	(12, 12, 0), (12, 12, 1), (12, 12, 2)	$[11111]_3$
3	728	(18, 14, 5)	$[222222]_3$
3	3280	(21, 20, 14), (22, 17, 14)	$[11111111]_3$
3	59048	(30, 18, 10)	$[2222222222]_3$
4	85	(11, 11, 4), (12, 9, 6)	$[1111]_4$
4	170	(13, 11, 11), (14, 10, 8)	$[2222]_4$
4	255	(15, 12, 6)	$[3333]_4$
4	341	(16, 12, 5)	$[11111]_4$
5	468	(16, 15, 3), (17, 12, 3)	$[3333]_5$
5	624	(18, 10, 0), (18, 10, 1), (18, 10, 2)	$[4444]_5$
6	259	(14, 14, 0), (14, 14, 1), (14, 14, 2)	$[1111]_6$
6	1295	(20, 8, 5)	$[5555]_6$
6	1555	(19, 17, 16), (19, 18, 13), (20, 15, 13)	$[11111]_6$
6	3110	(22, 16, 13)	$[22222]_6$
6	7775	(24, 21, 6)	$[55555]_6$
7	2801	(22, 10, 10)	$[11111]_7$
8	28086	(28, 19, 8)	$[666666]_8$
9	3280	(21, 20, 14), (22, 17, 14)	$[4444]_9$
9	59048	(30, 18, 10)	$[88888]_9$
12	1885	(21, 6, 6), (21, 7, 4)	$[1111]_{12}$
12	9425	(25, 18, 15)	$[5555]_{12}$
12	18850	(27, 16, 8)	$[aaaa]_{12}$
13	309410	(33, 32, 4), (34, 29, 4)	$[aaaaa]_{13}$
14	5910	(24, 8, 0), (24, 8, 1), (24, 8, 2)	$[2222]_{14}$
14	11820	(24, 24, 10)	$[4444]_{14}$
16	8738	(25, 13, 7)	$[2222]_{16}$
18	6175	(24, 16, 3)	$[1111]_{18}$
23	12720	(26, 10, 10)	$[1111]_{23}$
23	165360	(32, 29, 5)	$[cccc]_{23}$
23	585122	(36, 24, 16)	$[22222]_{23}$
27	183960	(33, 11, 4)	$[9999]_{27}$
27	367920	(33, 33, 13)	$[iiiii]_{27}$
28	591890	(36, 26, 16)	$[qqqqq]_{28}$

By fixing $M := M_b = 1.75 \times 10^{45} \log^6 b$, we execute the application of Lemma 7 to inequality (16) over all combinations of $b \in \{2, \ldots, 30\}$, $a \in \{1, \ldots, b-1\}$, $n - m \in \{0, \ldots, 326\}$, and $n - q \in \{0, \ldots, 334\}$. A search performed with *Mathematica* confirms that for all such solutions (n, m, q, ℓ, a, b), the bound $n \leq 322$ holds.

5. Proof of Theorem 1

Based on the reduction steps above, we conclude that any solution (n, m, q, ℓ, a, b) of the Diophantine equation (1), under the constraints $0 \leq q \leq m \leq n$, $2 \leq b \leq 30$, $1 \leq a \leq b-1$, and $\ell \geq 2$, must satisfy $1 \leq n \leq 322$.

Within this finite range, all potential solutions are computed using *Mathematica*. We document and analyze the complete list of solutions to equation (1) for which $\ell \geq 4$.

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