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ON THE EXISTENCE OF SOLUTIONS FOR PERTURBED IMPULSIVE DIFFERENTIAL EQUATIONS WITH INITIAL TIME DIFFERENCES

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Abstract. *In this paper, we investigate the existence of solutions for a perturbed impulsive differential system, where the initial time differs from that of the corresponding unperturbed system. Our approach is based on the method of upper and lower solutions, which provides a framework for establishing the existence of solutions under suitable conditions.*

Keywords: existence, initial time difference, impulsive differential equation

Mathematics Subject Classification (2020): 34A12, 34A37, 34A34

1. Introduction

In many real-time systems, dynamic behaviors can undergo abrupt changes due to short-term disturbances whose effects are brief compared to the overall time scale of the system. These sudden changes, both in time and space, are often best described using impulses. Modeling such discontinuities is essential in a wide range of disciplines, including control theory, population dynamics, pharmacokinetics, epidemiology, and economics [3], [6]. Impulsive differential equations (IDEs) provide a powerful framework for representing these phenomena. They are widely used to model situations such as rapid switching in valves, pendulum motion influenced by sudden external forces, mechanical systems with vibrations, oscillators affected by intermittent impulses, satellite trajectory adjustments using short bursts of radial acceleration, and impulsive changes in ecological systems, including predator–prey interactions and population decreases due to sudden environmental effects [2].

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The concept of initial time difference (ITD), first introduced by Lakshmikantham and his colleagues, was mainly applied in their studies on stability analysis [4], [5] and [8]. In this study, we investigate the existence criteria for solutions of perturbed impulsive differential equations by comparing them with their unperturbed counterparts, taking into account a difference in initial time. To this end, we employ the method of upper and lower solutions along with comparison principles. As a starting point, we define a scalar impulsive differential equation and aim to transfer the existence properties of its solutions to the perturbed system relative to the unperturbed one.

This work offers two main contributions beyond the earlier study in [7]. First, we establish the existence of solutions to the perturbed system on the interval $t \geq \tau_0$, in relation to the unperturbed system defined on $t \geq t_0$. Second, we introduce a novel approach by applying a leftward time shift of the unperturbed system by $\mu = \tau_0 - t_0$, in contrast to the rightward shift used in prior work. While the rightward shift does not guarantee the existence of solutions, our leftward shift resolves this issue and provides a more robust framework for analyzing stability.

This paper is organized as follows: Section 2 introduces the necessary preliminaries. Section 3 presents the main existence result along with a uniqueness theorem adapted from the literature. Finally, Section 4 concludes with a summary of our contributions.

2. Preliminary

Consider the following unperturbed impulsive differential systems:

$$\begin{aligned} z'(t) &= F(t, z(t)), \quad t \neq t_j, \\ \Delta z(t_j) &= I_j(z(t_j^-)), \quad j = 1, 2, 3, \ldots, \\ z(t_0) &= z_0, \quad t_j \geq t_0; \end{aligned} \tag{1}$$

$$\begin{aligned} z'(t) &= F(t, z(t)), \quad t \neq \tau_j, \\ \Delta z(\tau_j) &= I_j(z(\tau_j^-)), \quad j = 1, 2, 3, \ldots, \\ z(\tau_0) &= w_0, \quad \tau_j \geq \tau_0, \end{aligned} \tag{2}$$

and perturbed system of (2)

$$\begin{aligned} w'(t) &= h(t, w(t)), \quad t \neq \tau_j, \\ \Delta w(\tau_j) &= I_j(w(\tau_j^-)), \quad j = 1, 2, 3, \ldots, \\ w(\tau_0) &= w_0, \quad \tau_j \geq \tau_0, \end{aligned} \tag{3}$$

where $h(t, w) = F(t - \mu, w) + R(t, w)$ with $R(t, w)$ is a perturbation function and

(1) $0 \leq t_0 < \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_j < \ldots$ and $\lim_{j\to\infty} \tau_j = \infty$ for $j = 1, 2, \ldots$;
(2) $\tau_0 > t_0$, $\mu = \tau_0 - t_0$;
(3) $t \in \mathbb{R}_+, z \in \Omega \subset \mathbb{R}^n$, Ω-open;
(4) $F, h : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$;
(5) $I_j : \Omega \to \mathbb{R}^n$ for $j = 1, 2, 3, \ldots$;

(6) $F(t,0) = 0$, $I_j(0) = 0$, for all $j = 1, 2, 3, \dots$;
(7) $R(t,w)$ is a continuous function and $\|R(t,w)\| \leq a(t)\|w\|$ for $(t,w) \in [0,\infty) \times S(\rho)$ where $a(t)$ is non-negative continuos function such that $\int_{\tau_0}^{\infty} a(s)ds < \infty$.

Let us consider the system (1) shifted by a time μ, where $\mu = \tau_0 - t_0$. We define the impulsive jump as $\Delta z(\tau_j) = z(\tau_j^+) - z(\tau_j)$, where $z(\tau_j) = z(\tau_j^-)$, and τ_j denotes the impulse points.

$$\begin{aligned} z^{*\prime}(t) &= F(t, z^*(t)), \quad t \neq \tau_j, \\ \Delta z^*(\tau_j) &= I_j(z^*(\tau_j^-)), \quad j = 1, 2, 3, \dots, \\ z^*(\tau_0) &= z_0, \quad \tau_j > \tau_0. \end{aligned}$$

The scalar IDE is described by

$$\begin{aligned} u'(t) &= f(t,u), \quad t \neq \tau_j, \\ \Delta u(\tau_j) &= J_j(u(\tau_j)), \quad j = 1, 2, 3, \dots, \\ u(\tau_0) &= u_0, \end{aligned} \tag{4}$$

where $u(t; \tau_0, u_0)$ is a solution of (4) in $[\tau_0, \infty)$. Also, $f(t,u)$ is nondecreasing function in u where

$$f(t,u) \geq \max_{\|w - z^*\| \in S(\rho)} \|h(t, w(t)) - F(t, z^*(t))\|,$$

$$J_j(u(\tau_j)) \geq \max_{\|w - z^*\| \in S(\rho)} \|I_j(w(\tau_j^-)) - I_j(z^*(t_j^-))\|$$

and u_0 can be specified in the scalar IDE (4).

Let $I = [t_0, b]$ and $\tilde{I} = [\tau_0, b + \eta]$, where $\tau_0 < b < \infty$. Suppose that $t_0 < \tau_0 < \tau_1 < \dots < \tau_p < \tau_{p+1} = b$ are given points.

Definition.

(i) Let $\beta \in C^{(1)}[\tilde{I}, \mathbb{R}]$ is said to be an upper solution of (4) if

$$\begin{aligned} \beta'(t) &\geqslant f(t, \beta(t)), \quad t \neq \tau_j, \ \tau_j > \tau_0, \\ \Delta\beta(\tau_j) &\geqslant I_j\left(\beta\left(\tau_j^-\right)\right), \quad j = 1, \dots, p, \\ \beta(\tau_0) &\geqslant \beta_0. \end{aligned}$$

(ii) Let $\alpha \in C^{(1)}[\tilde{I}, \mathbb{R}]$ is said to be a lower solution of (4) if

$$\begin{aligned} \alpha'(t) &\leqslant f(t, \alpha(t)), \quad t \neq \tau_j, \ \tau_j > \tau_0, \\ \Delta\alpha(\tau_j) &\leqslant I_j\left(\alpha\left(\tau_j^-\right)\right), \quad j = 1, \dots, p, \\ \alpha(\tau_0) &\leqslant \alpha_0. \end{aligned}$$

Theorem 1. *Let*

(i) The function $f : \tilde{I} \times \mathbb{R} \to \mathbb{R}$ is quasimonotone nondecreasing in u for each $\tilde{I}$, and satisfies a Lipschitz condition in u.

(ii) For each j, the functions $J_j : \mathbb{R} \to \mathbb{R}$ are nondecreasing in u and satisfy a Lipschitz condition.

(iii) α and β are lower and upper solutions of (4), respectively.

If $\alpha_0 \leq \beta_0$ holds, then this inequality holds for all $t \in \tilde{I}$, that is,

$$\alpha(t) \leq \beta(t).$$

Proof. This proof closely follows the structure of Theorem 2.6.1 in the book by Lakshmikantham et al. [3], with appropriate modifications to account for the initial time difference. ◀

3. Main Results

In this section, we present the main theorem about the existence of solutions for the impulsive differential equation we consider. We use the method of upper and lower solutions, which helps us show that there is at least one solution between these two functions. This method not only proves that a solution exists but also helps us understand how the solution behaves over the given time interval. Furthermore, we conclude this section with a uniqueness theorem, adapted from the existing literature [3] to suit our framework.

Theorem 2. *Assume the following conditions are satisfied:*

A_1. *the functions $\alpha(t)$ and $\beta(t)$ are lower and upper solutions of (4), respectively, such that $\alpha(t) \leq \beta(t)$ for all $t \in \tilde{I}$;*

A_2. *the function $f \in C(\tilde{I} \times \mathbb{R}, \mathbb{R})$ is nondecreasing in u for each fixed $t \in \mathbb{R}$, and satisfies*

$$\sup_{\alpha(t) \leqslant u \leqslant \beta(t)} |f(t,u)| \leqslant \lambda(t) \quad \text{almost everywhere on } \tilde{I},$$

where $\lambda \in L^1(\tilde{I})$;

A_3. *the functions $J_k : \mathbb{R} \to \mathbb{R}$ are continuous and nondecreasing for each $j = 1, \ldots, p$.*

Then, the system defined by (4) has a solution $u(t)$ such that $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in \tilde{I}$.

Proof. Consider the following initial value problem (IVP):

$$\begin{aligned} u'(t) =& G(t, u(t)), \quad t \neq \tau_j, \\ \Delta u(\tau_j) =& J_j\left(u\left(\tau_j^-\right)\right), \quad j = 1, \ldots, p, \\ u(\tau_0) =& u_0, \quad t_j > \tau_0, \end{aligned}$$

where $G : [\tau_0, \tau_1]$ is defined as follows:

$$G(t,u) = \begin{cases} f(t,\bar{u}) + \frac{\beta(t)-u}{1+u}, & \text{if} \quad u > \beta(t), \\ f(t,\bar{u}), & \text{if} \quad \alpha(t) \leq u \leq \beta(t), \\ f(t,\bar{u}) + \frac{\alpha(t)-u}{1+u}, & \text{if} \quad u < \alpha(t), \end{cases} \tag{5}$$

and

$$\bar{u} = \begin{cases} \beta(t), & \text{if} \quad u > \beta(t), \\ u, & \text{if} \quad \alpha(t) \leq u \leq \beta(t), \\ \alpha(t), & \text{if} \quad u < \alpha(t). \end{cases}$$

Since $\sup_{u \in \mathbb{R}} |G(t,u)| \leq \lambda(t)$ almost everywhere on $[\tau_0, \tau_1]$, it follows that (4) has a solution u_1 on $[\tau_0, \tau_1]$.

We aim to prove that

$$\alpha(t) \leq u_1(t) \leq \beta(t), \quad t \in [\tau_0, \tau_1].$$

To demonstrate that

$$u_1(t) \leq \beta(t), \quad t \in [\tau_0, \tau_1],$$

we proceed by contradiction. If this is false, then the function

$$m(t) = u_1(t) - \beta(t), \quad t \in [\tau_0, \tau_1],$$

reaches a positive maximum at some $t^* \in (\tau_0, \tau_1]$, such that

$$\begin{aligned} m(t^*) &> 0, \\ m'(t^*) &\geq 0. \end{aligned}$$

This implies that

$$\begin{aligned} u_1(t^*) &> \beta(t^*), \\ u_1'(t^*) &\geq \beta'(t^*). \end{aligned}$$

However, we have the inequality

$$0 \leq u_1'(t^*) - \beta'(t^*) \leq f\left(t^*, \bar{u}(t^*)\right) + \frac{\beta(t^*) - u_1(t^*)}{1 + |u_1(t^*)|} - f\left(t^*, \beta(t^*)\right) < 0.$$

Since $\bar{u}(t^*) = \beta(t^*)$ and $\bar{u}_1(t^*) \leq \beta(t^*)$, by the monotonicity of f, we obtain

$$f(t^*, \bar{u}(t^*)) \leq f(t^*, \beta(t^*)).$$

Similarly, we can show that

$$\alpha(t) \leq u_1(t), \quad t \in [\tau_0, \tau_1].$$

Given that

$$\alpha\left(\tau_1^-\right) \leq u_1\left(\tau_1^-\right) \leq \beta\left(\tau_1^-\right),$$

and since J_1 is nondecreasing, we have

$$J_1\left(\alpha\left(\tau_1^-\right)\right) \leq J_1\left(u_1\left(\tau_1^-\right)\right) \leq J_1\left(\beta\left(\tau_1^-\right)\right).$$

From this, using (3) and (4), we conclude that

$$\begin{aligned} \alpha\left(\tau_1\right) \leq \alpha\left(\tau_1^-\right) + J_1\left(\alpha\left(\tau_1^-\right)\right) \leq u_1\left(\tau_1^-\right) + J_1\left(\alpha_1\left(\tau_1^-\right)\right) \leq \\ \leq \beta\left(\tau_1^-\right) + J_1\left(\beta\left(\tau_1^-\right)\right) \leq \beta\left(\tau_1\right). \end{aligned}$$

By repeating this argument, we can show that the problem

$$\begin{cases} u'(t) = G(t, u(t)), \quad t \in [\tau_1, \tau_2], \\ u(\tau_1) = u_1(\tau_1^-) + J_1(u_1(\tau_1^-)), \end{cases}$$

where G is defined similarly as in (5), has a solution u_2 on $[\tau_1, \tau_2]$ such that

$$\alpha(t) \le u_2(t) \le \beta(t), \quad t \in [\tau_1, \tau_2].$$

Proceeding in this manner for $t \in [\tau_p, \tau_{p+1}]$, we consider the initial problem

$$\begin{cases} u'(t) = G(t, u(t)), \quad t \in [\tau_p, \tau_{p+1}], \\ u(\tau_p) = u_p(\tau_p^-) + J_p(u_p(\tau_p^-)). \end{cases}$$

Similarly, we can establish that this problem has a solution u_{p+1} such that

$$\alpha(t) \le u_{p+1}(t) \le \beta(t), \quad t \in [\tau_p, \tau_{p+1}].$$

Continuing with the proof, we define

$$u(t) = \begin{cases} u_1(t), & t \in [\tau_0, \tau_1), \\ u_2(t), & t \in [\tau_1, \tau_2), \\ \vdots & \vdots \\ u_{p+1}(t), & t \in [\tau_p, \tau_{p+1}]. \end{cases}$$

Therefore, $u(t)$ serves as a solution to the problem (4), satisfying $\alpha(t) \le u(t) \le \beta(t)$ for $t \in \tilde{I}$. ◄

As a result, we establish the existence of solutions to the perturbed impulsive differential equations relative to the unperturbed systems, taking into account the initial time difference. For further details, see [3], [7], and [9].

Remark 1. Suppose that the assumptions A_2 and A_3 of Theorem 2 and
$[(A_1{}^*)]$ $\alpha \in C^{(1)}(I, \mathbb{R})$ and $\beta \in C^{(1)}(\tilde{I}, \mathbb{R})$ such that $\alpha(t-\mu) \le \beta(t)$,
where $\mu = \tau_0 - t_0$ are satisfied, then the solution $u(t)$ satisfies

$$\alpha(t-\mu) \le u(t) \le \beta(t),$$

where $\mu = \tau_0 - t_0$.

Remark 2. Suppose that the assumptions A_2 and A_3 of Theorem 2 and
$[(A_1{}^{**})]$ $\alpha \in C^{(1)}(\tilde{I}, \mathbb{R})$ and $\beta \in C^{(1)}(\tilde{\tilde{I}}, \mathbb{R})$, where $\xi_0 \ge \tau_0$ such that
$\alpha(t) \le \beta(t+\sigma)$, where $\sigma = \xi_0 - \tau_0$ and $\tilde{\tilde{I}} = [\xi_0, b+T+\sigma]$ are satisfied, then the solution $u(t)$ satisfies

$$\alpha(t) \le u(t) \le \beta(t+\sigma).$$

Remark 3. Suppose that the assumptions A_2 and A_3 of Theorem 2 are satisfied and

[$(A_1{}^{***})$] $\alpha \in C^{(1)}(I,\mathbb{R})$ and $\beta \in C^{(1)}(\tilde{\tilde{I}},\mathbb{R})$, where $\xi_0 \geq \tau_0$ such that $\alpha(t-\mu) \leq u(t) \leq \beta(t+\sigma)$, where $\mu = \tau_0 - t_0$ and $\sigma = \xi_0 - \tau_0$ are satisfied, then the solution $u(t)$ satisfies

$$\alpha(t-\mu) \leq u(t) \leq \beta(t+\sigma).$$

The following theorem addresses the uniqueness of solutions. It is an adaptation of the result by Lakshmikantham [3] to the scalar IDE (4) under the ITD setting.

Theorem 3. *Suppose that* $f \in C[\tilde{R},\mathbb{R}]$, $g \in C[(\tau_0,\tau_0+h) \times [0,2l],\mathbb{R}_+]$ *and* (t,u), $(t,v) \in \tilde{R}$,

$$|f(t,u) - f(t,v)| \leq g(t,|u-v|),$$

where $\tilde{R} = [(t,u) : \tau_0 \leq t \leq \tau_0 + h$ *and* $|u - u_0| \leq k]$. *Suppose further that for any* $\tau_0 \leq t^* \leq \tau_0 + h$, *the scalar IDE (4) has a unique solution* $u(t) = 0$ *on* $[t^*,\tau_0+h]$. *Then the system (3) has atmost one solution with respect to the system (1).*

Proof. For the proof, we refer the reader to the book by Lakshmikantham et al. [3], with appropriate modifications made to account for the ITD. ◀

4. Conclusion

In this study, we examined the existence of solutions for perturbed impulsive differential equations with respect to their unperturbed one by using the concept of initial time difference. With the help of upper and lower solution methods and comparison results, we showed that applying a leftward time shift to the unperturbed system provides better conditions for existence compared to previous rightward shift approaches. This result improves the analysis of impulsive systems, which are important for understanding sudden changes in many real-world models. Additionally, we present a uniqueness theorem that has been modified from the literature to fit within the ITD framework.

As a continuation of this study, we intend to explore the use of monotone iterative techniques for IDEs with ITD, based on the authors' previous results on ITD stability [1] and current ongoing research.

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