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ON ONE PROBLEM OF FEEDBACK CONTROL OF A HEATING SYSTEM

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Abstract. *We consider a problem of controlling a heating device intended for heating a coolant supplying heat to a closed system is considered. Feedback is used to control the process, in which information about the process state is continuously or discretely received from individual points of the device, where temperature sensors are installed. The mathematical model of the controlled process in both cases is described by a point-loaded first-order hyperbolic equation. Formulas for the gradient of the objective functional of the problem are obtained, allowing the use of first-order numerical optimization methods for solving problems. Numerical experiments are carried out using the example of solving several test problems.*

Keywords: optimal control, feedback, loaded differential equation

Mathematics Subject Classification (2020): 49N35, 49M05

1. Introduction

The article proposes an approach to constructing a feedback control system for an object with distributed parameters. The object under consideration is a heat supply system fed by liquid heated in a heat exchanger, which is a steam jacket [23]. Temperature sensors are installed at some points of the heat exchanger, based on the readings of which heat is supplied to the heat exchanger. The heat exchange process in the heat exchanger is described by a first-order hyperbolic transfer equation [23]. The boundary conditions include a delay argument due to the time it takes for the heated liquid to pass through the heat supply system.

It should be noted that interest in recent years has grown in problems of optimal control of objects with distributed parameters described by various types of partial differential equations with various types of initial boundary conditions [5], [6], [14], [15], [18], [20], [22], [23], [25]. Of particular complexity are problems of control (regulation) with feedback. While these problems have been studied quite well for objects with lumped parameters [7], [24], problems of control of objects with distributed parameters, on the contrary, have not been studied sufficiently [4]–[8], [10], [11], [15], [20], [24], [25]. Firstly, this is due to the complexity of the practical implementation of control systems for objects described by partial differential equations distributed in space and time [19]. This complexity is due to the impossibility of continuous or even discrete-time operational receipt of information about the state of the entire object (at all its points). Secondly, there are mathematical and computational problems, since the solution of initial boundary problems for partial differential equations

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requires, to a certain extent, a lot of time, which often makes it impossible to build control systems for objects with distributed parameters in real time.

The approach to the synthesis of optimal control of the heat supply process proposed in the article is based on the use of information about the state of the process at a finite number of measurement points. In this case, the problem involves optimizing both the position of the measurement points themselves and their number. Formulas are obtained for the gradient of the target functional for the optimized parameters of feedback control, which are used in the numerical solution of the problem using iterative first-order optimization methods. These formulas allow us to derive the necessary optimality conditions similar to the Pontryagin maximum principle.

2. Problem Statement

The process of heating the coolant in the furnace of the heated device (heat exchanger) of the heating system can be described by the transfer equation (see [23], [26]):

$$\frac{\partial u(x,t)}{\partial t} + a\frac{\partial u(x,t)}{\partial x} = \alpha\left[\vartheta(t) - u(x,t)\right], \ (x,t) \in \Omega = (0,l) \times (0,T], \tag{1}$$

where $u = u(x,t)$ is the temperature of the heat-carrying agent at the point x of the heat exchanger at the point of time t; l -the length of the heating tube, in which the heat-carrying agent is heated; a the velocity of the heat-carrying agent in the heat supply system, the value of which is constant for all points of the heat supply system, i.e. the motion is assumed to be steady (stationary); α the given value of the heat transfer coefficient between the furnace and the heat-carrying agent in the heating apparatus; $\vartheta(t)$ the temperature inside the furnace, by means of which the heating process is controlled, subject to the technological limit:

$$\underline{\vartheta} \leq \vartheta(t) \leq \bar{\vartheta}. \tag{2}$$

Let L be the linear length of the whole heat supply system, and L far exceeds l, i.e. $L >> l$. Then the heat-carrying agent heated in the furnace needs the time $T^d = L/a$ in order to return to the beginning of the furnace, i.e.

$$u(0,t) = (1-\gamma)\, u(l, t - T^d), \ t > 0\,, \tag{3}$$

where γ is the constant value that determines the heat lost during the motion in the heating system, which, in essence, depends considerably on the temperature of the external environment. On the basis of practical considerations, we have the obvious condition:

$$0 \leq \gamma \leq 1. \tag{4}$$

Denote by Γ the set of all possible values of γ, determining the amount of lost heat, satisfying (3) and (4). It is assumed that a density function $\rho_\Gamma(\gamma)$ is given on this set satisfying the condition:

$$\rho_\Gamma(\gamma) \geq 0\,, \ \gamma \in \Gamma\,, \int_\Gamma \rho_\Gamma(y) dy = 1,$$

$$\rho_\Gamma(\gamma) \in [0,1], \ \gamma \in \Gamma.$$

Let the inital history be given by:

$$u(x,t) = const, \ x \in [0,l], \ -T^d < t < 0. \tag{5}$$

The problem of controlling the of heating of the heat-carrying agent consists in maintaining the furnace temperature at such a level that provides a certain temperature $u(l,t) = V =$

const, $t \in (0,T]$, of the heat-carrying agent at the exit of the furnace under all possible admissible values of the heat lost by the heat-carrying agent when it moves in the heat supply system, determined by the values $\gamma \in \Gamma$.

Let sensors be installed at M arbitrary points $\xi_i \in [0,l],\ i = 1,2,...,M$, of the heating apparatus, at which temperature measurements are taken continuously:

$$u_i(t) = u(\xi_i, t), \quad t \in [0,T],$$

or at discrete points of time

$$u_{ij} = u(\xi_i, t_j), \quad t_j \in [0,T],\ j = 1,2,...,m.$$

To construct a heating control system with a continuous feedback, consider the following variant of the temperature control system:

$$\vartheta(t) = \frac{1}{l} \sum_{i=1}^{M} \lambda_i \tilde{k}_i [u(\xi_i, t) - z_i]\,, \tag{6}$$

where $\tilde{k}_i$ is the amplification coefficient; z_i the effective temperature at the point ξ_i, at which we need to control the amount of deviation from this value; $\lambda_i = const$ the weighting coefficient, determining the importance of taking a measurement at the point $\xi_i,\ i = 1,2,...,M$,

$$\lambda \in \Lambda = \left\{ \lambda \in \mathbb{R}^M :\ 0 \le \lambda_i \le 1,\ \ i = 1,2,...,M,\ \sum_{i=1}^{M} \lambda_i = 1 \right\}.$$

We introduce complex parameters:

$$k_i = \frac{\lambda_i \tilde{k}_i}{l}, i = 1,\ 2\ ,\ ...,M.$$

In this case the formula for the temperature in the furnace (6) takes the form:

$$\vartheta(t) = \vartheta(t; y) = \sum_{i=1}^{M} k_i [u(\xi_i, t) - z_i]\,. \tag{7}$$

Here $y = (\xi, k, z)^* \in \mathbb{R}^{3M}$ is the vector of parameters of the feedback that determines the current control value (furnace temperature) depending on the measured temperature values at the heat exchanger measurement points; "*" is the transposition sign.

Substituting (6) into (1), we obtain:

$$\frac{\partial u(x,t)}{\partial t} + a\frac{\partial u(x,t)}{\partial x} = \alpha \left[\sum_{i=1}^{M} k_i [u(\xi_i, t) - z_i] - u(x,t) \right],$$

$$(x,t) \in \Omega = (0,l) \times (0,T]\,. \tag{8}$$

The minimized criterion of the control quality is given by the following form:

$$J(y;\gamma) = \int_{\Gamma} I(y;\gamma) \rho_{\Gamma}(\gamma)\, d\gamma, \tag{9}$$

$$I(y;\gamma) = \int_0^T \left[u(l,t;y,\gamma) - V\right]^2 dt + \sigma \left\| y - \hat{y} \right\|_{\mathbb{R}^{3M}}^2 , \tag{10}$$

where $u(x,t;y,\gamma)$ is the solution of initial boundary-value problem (1), (3), (5) under specified feasible values of feedback parameters y and heat loss parameters $\gamma \in \Gamma$;

$\hat{y} = (\hat{k}, \hat{z}, \hat{\xi}) \in \mathbb{R}^{3M}$ and σ, a small positive quantity are the regularization parameters. Thus, initial feedback control problem the reduces to a parametric optimal control problem. The peculiarity of the problem is that it follows a point wise loaded differential equation (8) (due to the presence of the spatial variable of a value of the unknown function $u(x,t)$ at given points $\xi_i, i = 1, 2, ..., M$ in the equation, under boundary conditions with the lagging argument (3) [1]–[3], [5], [9], [12], [13], [16], [21].

Taking into account transformations (6), the optimizable feedback control parameters y can be constrained based on technical and process considerations:

$$0 \leq \xi_i \leq l, \quad \underline{k}_i \leq k_i \leq \bar{k}_i, \quad \underline{z_i} \leq z_i \leq \bar{z}_i, \quad i = 1,\, 2,\, ...,\, M. \tag{11}$$

Here, $\underline{k}_i$, $\bar{k}_i$, $\underline{z}_i$, $\bar{z}_i$, $i = 1, 2, ..., M$, are given values. The values $\underline{k}_i$, $\bar{k}_i$, $i = 1, 2, ..., M$, are derived from formula (7) taking into account constraints (2) the a priori information on possible and permissible values of steam and coolant temperature. The values $\underline{z}_i$, $\bar{z}_i$, $i = 1, 2, ..., M$, are mainly determined by the desired value of coolant temperature V at the heater outlet.

3. Derivation of the Formulas for the Gradient of the Functional

For numerical solution to the obtained parametric optimal control problem for a loaded system with distributed parameters, we propose to use first-order methods, for example, the gradient projection method (see [27]). To construct a minimizing sequence $y^\nu,\ \nu = 0, 1, ...$, an iterative process is constructed:

$$y^{\nu+1} = P_{(11)}[y^\nu - \mu_\nu grad\, J(y^\nu)]\ ,\ \nu = 0, 1, ...\ . \tag{12}$$

Here $P_{(11)}(y)$ is the projection operator of a $3M$ - dimensional point $y = (\xi, k, z)^*$ on the set defined by the constraints (11); $\mu_\nu > 0$ the step in the direction of the projected anti-gradient. The initial approximation y^0 can be arbitrary, satisfying, in particular, the conditions (11). Considering the simplicity of the structure of the admissible set of optimizable parameters defined by the constraints (11), the projection operator has a constructive character and is easy to implement.

The criterion for stopping the iterative process (14) can be the fulfillment of one of the following inequalities:

$$J(y^\nu) - J(y^{\nu+1}) \leq \epsilon_1, \quad \left\| y^\nu - y^{\nu+1} \right\| \leq \epsilon_2, \tag{13}$$

where ϵ_1 and ϵ_2 is a given positive numbers.

To build the procedure (12), we obtain formulas for the components of the gradient of the functional (9), (10) with respect to the optimizable parameters:

$$grad\, J(y) = \left(\frac{\partial J(\xi,k,z)}{\partial \xi}, \frac{\partial J(\xi,k,z)}{\partial k}, \frac{\partial J(\xi,k,z)}{\partial z} \right)^* .$$

To this end, we use the well-known technology of obtaining formulas for an increment of the functional obtained at the expense of the increment of the optimizable arguments of the functional (see [27]). In this case, the linear part of the increment of the functional with respect to each of the arguments will be the desired component of the gradient of the functional with respect to the corresponding argument.

Before to obtaining formulas for the gradient components of the functional, we note the following. Taking into account that the parameter $\gamma \in \Gamma$, determining the amount of lost heat, does not depend on the process of heating the heat-carrying agent in the heat exchanger, from (9),(10) it follows that:

$$grad\, J(y) = grad \int\limits_{\Gamma} I(y;\gamma)\rho_{\Gamma}(\gamma)d\gamma = \int\limits_{\Gamma} grad\, I(y;\gamma)\rho_{\Gamma}(y)d\gamma dy. \tag{14}$$

Therefore, we obtain the formula $grad\, I(y;\gamma)$ for any one arbitrarily given parameter $\gamma \in \Gamma$.

Let $u(x,t;y,\gamma)$ be the solution to the loaded initial- and boundary-value problem (8), (3)–(5) for an arbitrary chosen vector of the optimizable parameters $y = (\xi, k, z)^*$ and for a given value of the parameter $\gamma \in \Gamma$. For brevity, where this does not cause ambiguity, the parameters y, γ will be omitted from the solution $u(x,t;y,\gamma)$.

Let the parameters $y = (\xi, k, z)^*$ have obtained some admissible increments $\Delta y = (\Delta\xi, \Delta k, \Delta z)^*$, and $\tilde{u}(x,t) = \tilde{u}(x,t;\tilde{y}) = u(x,t) + \Delta u(x,t)$ be the solution to the problem (8), (3)–(5), that corresponds to the incremented vector of arguments $\tilde{y} = y + \Delta y$.

Substituting the function $\tilde{u}(x,t)$ into the conditions (8), (3)–(5), we obtain the following initial- and boundary-value problem accurate within the terms of the first order of smallness with respect to the increment $\Delta u(x,t)$ of the phase variable:

$$\Delta u_t(x,t) + a\Delta u_x(x,t) = \alpha \sum_{i=1}^{M} [k_i \Delta u(\xi_i,t) + k_i u_x(\xi_i,t)\Delta\xi_i +$$

$$+ \left(u(\xi_i,t) - z_i\right)\Delta k_i - k_i \Delta z_i] - \alpha\Delta u(x,t)\,, \;\; (x,t) \in \Omega\,, \tag{15}$$

$$\Delta u(x,0) = 0, \;\; x \in [0,l], \tag{16}$$

$$\Delta u(0,t) = \begin{cases} 0, & t \le T^d, \\ (1-\gamma)\Delta u(l, t - T^d), & t \ge T^d. \end{cases} \tag{17}$$

In obtaining formula (15) we used the relation:

$$u(\xi_i + \Delta\xi_i, t) = u(\xi_i,t) + u_x(\xi_i,t)\Delta\xi_i + o(|\Delta\xi_i|).$$

For the increment of the functional (10), it is not difficult to obtain directly the representation:

$$\Delta I(y;\gamma) = I(\tilde{y};\gamma) - I(y;\gamma) = I(y + \Delta y;\gamma) - I(y;\gamma) =$$

$$= 2\int_0^T \left[u(l,t\,;y,\gamma) - V\right]\Delta u(l,t)dt + 2\sigma \sum_{i=1}^{3M} (y_i - y_i^0)\Delta y_i\,,$$

$$\sum_{i=1}^{3M} (y_i - y_i^0)\Delta y_i = \sum_{i=1}^{3M} \left[(\xi_i - \xi_i^0)\Delta\xi_i + (k_i - k_i^0)\Delta k_i + (z_i - z_i^0)\Delta z_i\right].$$

Let $\psi(x,t) = \psi(x,t;y,\gamma)$ be yet an arbitrary function continuous everywhere on Ω, except at the points $x = \xi_i$, $i = 1,2,...,M$, differentiable with respect to x for $x \in (\xi_i, \xi_{i+1})$, $i = 0,1,...,M, \xi_0 = 0,\; \xi_{M+1} = l$, differentiable with respect to t for $t \in (0,T)$. The presence of the arguments y and γ in the function $\psi(x,t;y,\gamma)$ indicate that it can vary when the feedback parameter vector y and the parameter γ change. Where it is possible, we will omit the parameters y and γ in the function $\psi(x,t;y,\gamma)$. We multiply equation (15) by $\psi(x,t)$ and integrate it over the rectangle Ω. Taking into account the assumed assumptions and conditions (16), (17), we have:

$$\int\limits_0^T \int\limits_0^l \psi(x,t)\Delta u_t(x,t)dxdt + a\sum_{i=0}^{M} \int\limits_{\xi_i}^{\xi_{i+1}} \int\limits_0^T \psi(x,t)\Delta u_x(x,t)dtdx -$$

$$-\alpha\int\limits_0^T\int\limits_0^l \psi(x,t)\sum_{i=1}^{M}\left[k_i\Delta u(\xi_i,t)+k_iu_x(\xi_i,t)\Delta\xi_i+\left(u(\xi_i,t)-z_i\right)\Delta k_i-k_i\Delta z_i\right]dx\,dt+$$

$$+\alpha\int\limits_0^T\int\limits_0^l \psi(x,t)\Delta u(x,t)dx\,dt\ =0\,. \tag{18}$$

Using integration by parts for the first and second terms of (18) separately, and taking (16), (17) into account, we obtain:

$$\int\limits_0^T\int\limits_0^l \psi(x,t)\Delta u_t(x,t)dxdt=$$

$$=\int\limits_0^l \psi(x,T)\Delta u(x,T)dx-\int\limits_0^T\int\limits_0^l \psi_t(x,t)\Delta u(x,t)dxdt, \tag{19}$$

$$a\sum_{i=0}^{M}\int\limits_{\xi_i}^{\xi_{i+1}}\int\limits_0^T \psi(x,t)\Delta u_x(x,t)dtdx=a\int\limits_0^T\left[\psi(l,t)\Delta u(l,t)-\psi(0,t)\Delta u(0,t)\right]dt+$$

$$+a\sum_{i=1}^{M}\int\limits_0^T\left[\psi(\xi_i^-,t)-\psi(\xi_i^+,t)\right]\ \Delta u(\xi_i,t)\,dt-a\int\limits_0^T\int\limits_0^l \psi_x(x,t)\Delta u(x,t)dxdt=$$

$$=a\int\limits_0^T \psi(l,t)\Delta u(l,t)dt-a(1-\gamma)\int\limits_{T^d}^T \psi(0,t)\Delta u(l,t-T^d)dt+$$

$$+a\sum_{i=1}^{M}\int\limits_0^T\left[\psi(\xi_i^-,t)-\psi(\xi_i^+,t)\right]\ \Delta u(\xi_i,t)\,dt-a\int\limits_0^T\int\limits_0^l \psi_x(x,t)\Delta u(x,t)dxdt=$$

$$=a\int\limits_0^T \psi(l,t)\Delta u(l,t)dt-a(1-\gamma)\int\limits_0^{T-T^d} \psi(0,t+T^d)\Delta u(l,t)dt+$$

$$+a\sum_{i=1}^{M}\int\limits_0^T\left[\psi(\xi_i^-,t)-\psi(\xi_i^+,t)\right]\ \Delta u(\xi_i,t)\,dt-a\int\limits_0^T\int\limits_0^l \psi_x(x,t)\Delta u(x,t)dxdt. \tag{20}$$

Here we have used the notation

$$\psi(\xi_i^-,t)=\psi(\xi_i-0,t),\ \ \psi(\xi_i^+,t)=\psi(\xi_i+0,t).$$

Taking (18)–(20) into account, we obtain for the increment of the functional:

$$\Delta I=\int\limits_{T-T^d}^T\left[a\psi(l,t)+2(u(l,t)-V)\right]\Delta u(l,t)dt+\int\limits_0^l \psi(x,T)\Delta u(x,T)dx+$$

$$+\int\limits_0^{T-T^d}\left[a\psi(l,t)+a(1-\gamma)\psi(0,t+T^d)+2(u(l,t)-V)\right]\Delta u(l,t)dt+$$

$$+\int_0^T\int_0^l[-\psi_t(x,t)-a\psi_x(x,t)+\alpha\psi(x,t)]\Delta u(x,t)dxdt+$$

$$+a\sum_{i=1}^{M}\int_0^T\left[\psi(\xi_i^-,t)-\psi(\xi_i^+,t)-\frac{\alpha}{a}k_i\int_0^l\psi(x,t)dx\right]\ \Delta u(\xi_i,t)\,dt-$$

$$-\alpha\int_0^T\int_0^l\psi(x,t)\sum_{i=1}^{M}\left[k_iu_x(\xi_i,t)\Delta\xi_i+\left(u(\xi_i,t)-z_i\right)\Delta k_i-k_i\Delta z_i\right]dx\,dt+$$

$$+2\sigma\sum_{i=1}^{M}\left[(\xi_i-\xi_i^0)\Delta\xi_i+(k_i-k_i^0)\Delta k_i+(z_i-z_i^0)\Delta z_i\right].$$

Since the function $\psi(x,t)$ is arbitrary, we require that it be almost everywhere a solution of the following adjoint initial- and boundary-value problem:

$$\psi_t(x,t)+a\psi_x(x,t)=\alpha\psi(x,t),\ (x,t)\in\Omega, \tag{21}$$

$$\psi(x,T)=0,\quad x\in[0,l], \tag{22}$$

$$\psi(l,t)=-\frac{2}{a}(u(l,t)-V),\ t\in(T-T^d,T], \tag{23}$$

$$\psi(l,t)=-\frac{\alpha}{a}(1-\gamma)\psi(0,t+T^d)-\frac{2}{a}(u(l,t)-V),\ t\in(0,T-T^d], \tag{24}$$

and at the points $\xi_i,\ i=1,2,...,M$ for $t\in[0,T]$, it satisfy the condition:

$$\psi(\xi_i^-,t)=\psi(\xi_i^+,t)+\frac{\alpha}{a}k_i\int_0^l\psi(x,t)dx,\ \ i=1,2,...M\,. \tag{25}$$

Taking into account that the components of the gradient of the functional are determined by the linear part of the increment of the functional under the increments of the corresponding arguments, we obtain:

$$grad_{\xi_i}I=-\alpha k_i\int_0^T\left(\int_0^l\psi(x,t)dx\right)u_x(\xi_i,t)dt+2\sigma(\xi_i-\xi_i^0),\ \ i=1,2,....,M,$$

$$grad_{k_i}I=-\alpha\int_0^T\left(u(\xi_i,t)-z_i\right)\left(\int_0^l\psi(x,t)dx\right)dt+2\sigma(k_i-k_i^0),\ \ i=1,2,....,M,$$

$$grad_{z_i}I=\alpha k_i\int_0^l\psi(x,t)dx+2\sigma(z_i-z_i^0),\ \ i=1,2,....,M.$$

Thus, we can consider the following theorem to be proved.

Theorem. *For the optimality of the vector of parameters $y^*\in\mathbb{R}^{3M}$ in the problem (8), (3)-(5), (9)-(10), it is necessary and sufficient that*

$$(gradJ(y^*),\,y^*-y)\le 0$$

for all admissible control parameters $y \in \mathbb{R}^{3M}$ *satisfying conditions (11). The components of the gradient vector* $gradJ(y)$ *are defined by the formulas:*

$$grad_{\xi_i} J(y) = \int\limits_{\Gamma} \left\{ -\alpha k_i \int\limits_0^T \left(\int\limits_0^l \psi(x,t;y,\gamma)dx \right) u_x(\xi_i,t;y,\gamma)dt \,+\, 2\sigma(\xi_i - \xi_i^0) \right\} \rho_\Gamma(\gamma)d\gamma,$$

$$grad_{k_i} J(y) = \int\limits_{\Gamma} \left\{ -\alpha \int\limits_0^T (u(\xi_i,t;y,\gamma) - z_i) \left(\int\limits_0^l \psi(x,t;y,\gamma)dx \right) dt + \, 2\sigma(k_i - k_i^0) \right\} \rho_\Gamma(\gamma)d\gamma,$$

$$grad_{z_i} J(y) = \int\limits_{\Gamma} \left\{ \alpha k_i \int\limits_0^l \psi(x,t;y,\gamma)dx + 2\sigma(z_i - z_i^0) \right\} \rho_\Gamma(\gamma)d\gamma.$$

where $i = 1,2,....,M$, $u(x,t;y,\gamma)$; $\psi(x,t;y,\gamma)$ *are the solutions to the direct and adjoint boundary-value problems (8), (3)–(5) and (21)–(25), respectively.*

In numerical solution of the initial problem of optimizing the parameters to be synthesized, each iteration of procedure (12) involves solving direct (8), (3)-(5) and adjoint (21)–(25) boundary-value problems with the specifics described above. Numerical solution of loaded boundary-value problems can be obtained using methods of meshes or lines. Their application to solution of similar problems was studied, for example, in [1], [3]. Lagging under boundary conditions can be taken into account using the "step method" [17].

4. Results of the Numerical Experiments

In this section, we present the results of the solution of the following model problem. The process is described by the boundary-value problem (1)–(5). It is required to design an optimal control (regulation) system for the coolant heating process, first, with two feedback points, i.e., $M = 2$. Thus, it is required to determine $\xi = (\xi_1,\ \xi_2)$, that is, locations of two temperature sensors, as well as feedback parameters $k, z \in \mathbb{R}^2$. Hence, the total number of parameters to be synthesized is six.

The problem is considered solved at the following values of parameters comprising its statement: $l = 1$; $a = 1$; $\alpha = 0,1$; $T^d = 0.2$, $T = 5$, $V = 70$, $\Gamma = [0;\, 0.2]$, $\underline{\vartheta} = 55$, $\bar{\vartheta} = 75$, $\bar{k}_1 = \bar{k}_2 = 8$, $\underline{k}_1 = \underline{k}_2 = 1$, $\bar{z}_1 = \bar{z}_2 = 75$, $\underline{z}_1 = \underline{z}_2 = 57$. In calculations, the density function $\rho_\Gamma(\gamma)$ has been taken uniformly distributed on [0; 0.2], while approximation of the integral over Γ was performed using the of method rectangle with the step 0.05. It should be noted that values $\bar{k}_1$ and $\bar{k}_2$ chosen using the results of trial calculations performed, which required the process constraint (2) to be true for given $\underline{\vartheta}$ and $\bar{\vartheta}$.

The numerical experiments have been carried for different initial values of parameters $(y^0)^j = (k_1^0,\, k_2^0, z_1^0,\, z_2^0, \xi_1^0,\, \xi_2^0)^j$, $j = 1,2,...,5$, used in iterative optimization procedure (12).

Table 1. Initial values of the parameters $(y^0)^j$, $j = 1,2,...,5$, **to be optimized and the corresponding values of the functional**

j	Values of the parameters to be optimized						Value of the functional
	$(k_1^0)^j$	$(k_2^0)^j$	$(z_1^0)^j$	$(z_2^0)^j$	$(\xi_1^0)^j$	$(\xi_2^0)^j$	$J(y^0)^j$
1	4	6	61	63	0,1	0,8	363.210004
2	3	5	65	60	0,2	0,9	357.150011
3	1	8	62	63	0,4	0,8	257.310003
4	5	2	63	66	0,5	0,7	165.150016
5	6	4	66	62	0,2	0,7	205.190007

Table 1 gives these values, as well as the corresponding values of the functional at these points.

Table 2 gives the values of parameters $(y^{(*)})^j = (k_1^{(*)}, k_2^{(*)}, z_1^{(*)}, z_2^{(*)}, \xi_1^{(*)}, \xi_2^{(*)})^j$ and the functional $J(y^*)^j$ obtained using the gradient projection method, (12), (13) at $\delta_1 = 0.005$ and $\delta_2 = 0.001$ starting from the initial points $(y^0)^j$, $j = 1, 2, ..., 5$, specified in Table 1.

Table 2. Values of parameters and the functional obtained at the sixth iterations of process (12) for different initial values $(y^0)^j$, $j = 1, 2, ..., 5$

j	Values of the parameters to be optimized						Value of the functional
	$(k_1^*)^j$	$(k_2^*)^j$	$(z_1^*)^j$	$(z_2^*)^j$	$(\xi_1^*)^j$	$(\xi_2^*)^j$	$J(y^*)^j$
1	5.9956	3.9952	66.9945	68.9949	0.2994	0.5994	0.3422
2	5.9977	3.9983	66.9978	68.9954	0.3000	0.6000	0.3259
3	5.9962	3.9988	66.9951	68.9948	0.2971	0.5971	0.3538
4	5.9978	3.9971	66.9991	68.9975	0.3000	0.6000	0.3145
5	5.9991	3.9961	66.9964	68.9973	0.3000	0.6000	0.3062

Numerical experiments have been performed, in which exact values of the process states observed at sensor points $u(\xi_1, t)$ and $u(\xi_2, t)$ have been perturbed with random noise as follows:

$$u(\xi_i, t) = u(\xi_i, t)\left(1 + \chi(2\theta_i - 1)\right), \quad i = 1, 2,$$

where θ_i is a random value uniformly distributed on the interval [0, 1], and χ is the noise level.

Table 3 gives the obtained values of the functional and relative deviations between the obtained and the desired temperature at the unit outlet for noise levels equal to 0% (no noise), 1%, 3%, and 5%, which correspond to values $\chi = 0$ (no noise), 0.01, 0.03, and 0.05.

As can be seen from Table 3, feedback control of the coolant heating process in the furnace of the heated apparatus is quite resistant to measurement errors.

Table 3. Values of the functional and relative deviations between the obtained and the desired temperature at the unit outlet for different noise levels in measurement

Noise level χ	Relative deviation $\max_{t\in[0,5]} \lvert u(l,t) - V\rvert \Big/ \lvert V\rvert$	Value of the functional $J^*(y)$
0.00	0.021941	0.3023
0.01	0.033052	0.3543
0.03	0.038311	0.3762
0.05	0.064574	0.3916

5. Conslusion

Automatic feedback control systems for technical objects and technological processes with distributed parameters have become widespread due to the significantly increased capabilities of measuring and computing equipment. The paper studies the problem of controlling a heating device for heating a coolant that supplies heat to a closed heat supply system. The specificity of the problem under study, described by a first-order hyperbolic equation, lies in the presence of a time-delayed argument in its boundary conditions. The mathematical model of the controlled process is reduced to a point-loaded hyperbolic equation, and the problem under consideration is reduced to a parametric optimal control problem. To use first-order optimization methods for the numerical solution of the problem of optimizing the location of sensors and parameters of feedback control actions, formulas for the gradient of the target functional are obtained.

The statement of the problem and the approach used in the paper to obtaining calculation formulas for its numerical solution can be generalized to cases of feedback control of many other processes described by other types of partial differential equations.

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