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CONVEXITY OF OUT-CHEBYSHEV SETS

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Abstract. *A set is an out-Chebyshev set if any point lying at a positive distance from this set has a unique best approximant in this set. We extend the classical Berdyshev–Brøndsted–Brown theorem on convexity of Chebyshev sets in spaces of dimension* ≤ 4 *to the case of out-Chebyshev sets.*

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1. Introduction

It is well known that many classical aggregates of approximation (exponential sums, generalized rational fractions, splines of degree m of defect k with n free knots, products $\{vw \mid v \in V,\, w \in W\}$) are in general nonclosed sets (see, for example, [9, Chap. VI], [4, § 11.4, § 7.12]). This suggests that we study approximative and geometrical properties of sets which are not necessarily closed. Correspondingly, this leads to the concept of out-Chebyshev sets — unlike the classical definition of a Chebyshev set, here a best approximant from a set M is required to exist and be unique only for points from the metrical exterior $\operatorname{out}(M)$ of M, i.e., for points which lie at a positive distance from M. For out-Chebyshev sets in finite-dimensional spaces, we consider the problem of characterization of spaces in which any such a set is preconvex, i.e., its closure is convex. We also obtain balayage type results for this problem.

Below, $X = (X, \|\cdot\|)$ is a finite-dimensional real asymmetric normed space (in particular, X is a finite-dimensional normed linear space [11]);

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$B(x,r) = \{y \in X \mid \|y-x\| \le r\}$ is the closed ball with center x and radius r;
$S(x,r) = \{y \in X \mid \|y-x\| = r\}$ is the sphere;
$B := B(0,1)$ is the unit ball, $S = S(0,1)$ is the unit sphere.

For $\varnothing \ne M \subset X$, the *distance* from a point $x \in X$ to a set $M \subset X$ is defined by $\rho(x,M) := \inf_{y\in M} \|y-x\|$. The set of *nearest points* (best approximants) for a point $x \in X$ in a set M is defined by

$$P_M x := \big\{y \in M \mid \rho(x,M) = \|y-x\|\big\}.$$

The *metrical exterior* $\operatorname{out}(M)$ of a set M is the set of all points lying at a positive distance from M, i.e.,

$$\operatorname{out}(M) := \{x \in X \mid \rho(x,M) > 0\}.$$

Definition 1. A set $\varnothing \ne M \subset X$ is a Chebyshev set with respect to K if $P_M x$ is a singleton for any $x \in K$; M is a *Chebyshev set* if $K = X$; if $K = \operatorname{out}(M)$, then M is an out-*Chebyshev set*. (A Chebyshev set is closed *qua* an existence set; an out-Chebyshev set is not necessarily closed.)

In the main Theorem, we extend the classical Berdyshev–Brøndsted–Brown theorem (equivalence b) $\Leftrightarrow$ c) in Theorem) on convexity of Chebyshev sets in spaces of dimension ≤ 4 to the case of not necessarily closed Chebyshev sets (out-Chebyshev sets).

The study of geometry of Chebyshev sets in finite-dimensional normed linear spaces was pioneered by L. N. H. Bunt, H. Mann, and T. Motzkin (see, for example, [7, § 1.1]). In particular, it was shown that in the finite-dimensional Euclidean space $\mathbb{R}^n$ the class of Chebyshev sets coincides with the c;ass of closed convex sets. Brøndsted [10] and Berdyshev [8] independently characterized the three-dimensional (asymmetric) normed spaces in which any Chebyshev set is convex. (The answer in the two-dimensional setting was known much earlier.) A. L. Brown characterized the four-dimensional normed spaces, where each Chebyshev set is convex; later Alimov [1] extended Brown's result to the asymmetric case. For an account of these results and relevant references, see [7], [1], [12] and [6].

In many problems of geometric approximation theory, it suffices to consider not the entire unit sphere, but rather the part of the unit sphere consisting of acting points (for a given set M) — these being the (acting) points of the unit sphere such that M can be touched by an "analog" of such a point on some homothetic copy of the unit ball whose interior has no common points with M. In this regard, in Theorem we also obtain so-called balayage results for the problem of convexity of out-Chebyshev sets. In balayage theorems, which date back to the Fermat's Rule from calculus and the Chebyshev equioscillation theorem from approximation theory, one gets rid of unnecessary points in the domain of a given functional without changing its optimal value. For more results on this topic, see [1], [2], and [3].

Definition 2. Let $M \subset X$ be a nonempty set. A point $s \in S$ is an M-*acting point* (see [1]) of the unit sphere S if

$$s \in (P_M x - x)/\rho \quad \text{for some} \quad x \notin M, \quad \text{where} \quad \rho = \rho(x, P_M x)$$

here, "M" refers to the set under consideration M.

Definition 3. A point s of the unit sphere S is a *smooth point* of S (or of the unit ball B) if there is only one support hyperplane to B at s (equivalently, if the norm is Gâteaux differentiable at s). A point s is an *exposed* point of the ball B if there is a support hyperplane H to the ball B at the point s such that $H \cap B = \{s\}$.

Definition 4. The *projection boundary* $\operatorname{pb} M$ of a set M in an asymmetric normed space $(X, \|\cdot\|)$ is defined by (see [5])

$$\operatorname{pb}(M) := \{y \in \overline{M} \mid y \in P_{\overline{M}}x \ \text{ for some point } \ x \in \operatorname{out}(M)\}.$$

A set $M \neq \varnothing$ is *projection closed* if it contains its projection boundary (in other words, the projection boundary of the set consists of its touchable points [5]).

Example. Let $\mathscr{R}_{n,m}$, $m \geq 1$, be the set of classical rational functions in $L^p[a,b]$, $1 \leq p < \infty$ (see [4, Chap. 11]). The set

$$M := \mathscr{R}_{n,m} \setminus \mathscr{R}_{n-1,m-1}$$

gives a classical example of a projection closed set which is not closed. This property follows from one well-known result of J. Batter (see, for example, [4, § 11.3]), who proved that, for the rational L^p-approximation, a best approximant has always maximal degree of either numerator or denominator, i.e., if u^* is a nearest point of L^p-approximation from $\mathscr{R}_{n,m}$, $1 < p < \infty$, for an $f \notin \mathscr{R}_{n,m}$, $n \geq 1$, then $u^* \notin \mathscr{R}_{n-1,m-1}$. Note that according to Ch. Dunham this nondegeneracy property does not hold in L^1 (see, for example, [4, § 11.3]). In the space $C[a,b]$, the Chebyshev equioscillation theorem implies that any element from (the Chebyshev sun) $\mathscr{R}_{n,m}$ is a touchable point of $\mathscr{R}_{n,m}$.

We also note that in $C[a,b]$, a nonempty set $M \subset \mathscr{R}_{n,m}$ is projection closed if and only if it is closed.

The set $\varPi = \{x \in \ell^2 \mid |x^{(n)}| \leq 1/2^{n-1}\}$ (the Hilbert parallelotope) is also an example of a convex compact set for which the projection boundary is different from the boundary of this set.

Definition 5. A set M is a *uniqueness set* if any point x has at most one nearest point in M (0 lies in the boundary of M, but 0 does not lie in the projection boundary of M).

For the following result, see [5].

Theorem A. *Let X be a finite-dimensional space, $\varnothing \neq M \subset X$ be a projection closed uniqueness set. Then*

$$\operatorname{pb} M = \operatorname{bd} \overline{M},$$

i.e., the projection boundary of M coincides with the boundary of its closure $\overline{M}$.

Corollary. *If M is an* out-*Chebyshev set in a finite-dimensional space, then M is projection closed.*

Definition 6. A set is *preconvex* if its closure is convex.

The main result, which extends the classical Berdyshev–Brøndsted–Brown theorem on convexity of Chebyshev sets in spaces of dimension ≤ 4 to the case of out-Chebyshev sets, is as follows.

Theorem. *Let X be a normed* (*or asymmetric normed*) *linear space,* $\dim X \leq 4$. *Then the following assertions are equivalent*:

a) *each* out-*Chebyshev set M in X is preconvex*;

b) *each Chebyshev set M in X is convex*;

c) *each exposed point of the ball B is a smooth point of B*;

d) *for each Chebyshev set M in X, each exposed M-acting point of the ball B is a smooth point of the ball B*;

e) *for each* out-*Chebyshev set M in X, each exposed $\overline{M}$-acting point of the ball B is a smooth point of the ball B*;

f) *for each* out-*Chebyshev set M in X, each exposed M-acting point of the ball B is a smooth point of the ball B.*

Remark 1. In relation to Theorem, we recall that if M is an out-Chebyshev set in a finite-dimensional space, then M is projection closed (see Corollary and, further, the projection boundary of the set M coincides with the boundary of its closure $\overline{M}$ (see Theorem A.).

Remark 2. Assertion a) of Theorem is also true for arbitrary finite-dimensional spaces satisfying the well-known A. L. Brown's condition for the system of faces of the unit ball (this condition holds automatically for the spaces of dimension ≤ 4). For further details, see [7, § 1.1] and [1, § 3].

Remark 3. For the two-dimensional spaces, assertion c) of Theorem assumes the form: "any point of the unit sphere B is a smooth point"; and assertion d) takes the form: "for each Chebyshev set in X any M-acting point of the ball B is a smooth point of B", etc.

For a proof of Theorem, we need the following lemma.

Lemma. *If M is an* out-*Chebyshev set in a finite-dimensional asymmetric space, then $\overline{M}$ is a Chebyshev set.*

Proof. Assume on the contrary that some point $x \in \operatorname{out} M$ $(= \operatorname{out} \overline{M})$ has two nearest points y and y' in the set $\overline{M}$; $P_M x = \{y\}$. Given a point $x' \in (x, y')$, we set $r' := \rho(x', M) = \|x' - y'\|$. Since $x' \in \operatorname{out} M$, we have $r' > 0$. Let $P_M x' = \{v'\}$. If a point x' is sufficiently close to y', then $y \notin B(x', r')$. For $\alpha \in [0, 1]$, we set $x'_\alpha := \alpha x + (1 - \alpha)x'$. We have $\rho(x, M) = \|x - y'\|$, and hence $\|x'_\alpha - y'\| = \|x'_\alpha - v'\|$, whence, making $\alpha \to 1$, we find that

$$\|x - y'\| = \|x - v'\| = \|x - y\|.$$

Hence $y, v' \in P_M x$. This, however, contradicts the fact that M is an out-Chebyshev set. Lemma is proved. ◄

Let us now prove Theorem. Equivalence b) $\Leftrightarrow$ c) is well known. In the three-dimensional case, this result is due to V. I. Berdyshev [8] and A. Brøndsted [10]; in the four-dimensional case, the corresponding characterization is due to A. L. Brown in the symmetric case and A. R. Alimov in the asymmetric case; see [7].

Equivalence b) $\Leftrightarrow$ d) was proved by the author of the present note in [1].

Implication a) $\Rightarrow$ b) is clear, because any Chebyshev set is an out-Chebyshev set.

Let us verify implication c) $\Rightarrow$ a). Let M be an out-Chebyshev set in a space X, where each exposed point of the unit ball is a smooth point of this ball. By Lemma, $\overline{M}$ is a Chebyshev set. By implication c) $\Rightarrow$ b) the set $\overline{M}$ is convex. Hence M is preconvex.

Assume that, for each out-Chebyshev set M in X, each exposed M-acting point of the ball B is a smooth point of the ball B. Any Chebyshev set is an out-Chebyshev set, and hence, by the assumption, for each Chebyshev set N in X, each exposed N-acting point of the ball B is a smooth point of the ball B. This proves implication f) $\Rightarrow$ d) ($\Leftrightarrow$ a) $\Leftrightarrow$ b) $\Leftrightarrow$ c)).

By Corollary, in a finite-dimensional space any out-Chebyshev set is necessarily projection closed. Hence, by Remark 1, for each out-Chebyshev set M in X, the set of M-acting points of the ball B coincides with the set of $\overline{M}$-acting points of the ball B. This proves equivalence f) $\Leftrightarrow$ e). Implication d) $\Rightarrow$ e) is clear, because any Chebyshev set is an out-Chebyshev set. This proves Theorem. ◄

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