Baku Mathematical Journal
2025, Vol. 4, No 2, PP. 209-230
https://doi.org/10.32010/j.bmj.2025.15

MIXED PROBLEM FOR A NONLINEAR DIFFERENTIAL EQUATION OF THE SEVENTH ORDER WITH IDENTIFICATION SOURCE

T.K. YULDASHEV*, B.J. KADIRKULOV, M.M. BABAYEV

Received: 17.06.2025 / Revised: 17.07.2025 / Accepted: 29.07.2025

Abstract. *In this paper, a seventh order nonlinear partial differential equations with mixed and additional conditions and positive parameter is considered. The Fourier spectral method of separation of variables is applied. Countable systems of nonlinear functional-integral equations are derived. Theorem on the uniqueness and existence of the solution of inverse problem is proved for some values of parameter. The method of contraction mapping in Banach space is applied. The solution of the inverse problem is obtained in the form of Fourier series. Theorem on absolute and uniform convergence of Fourier series is proved.*

Keywords: inverse mixed problem, redefinition function, seventh order differential equation, unique solvability, real parameter

Mathematics Subject Classification (2020): 34B08, 34B10, 35G05

1. Formulation of the Problem Statement

Differential equations of parabolic and hyperbolic types are the base of the equations of mathematical physics. Along with equations of parabolic and hyperbolic types, pseudoparabolic and pseudohyperbolic differential equations are often studied.

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* Corresponding author

Tursun K. Yuldashev
Samarkand State University, Samarkand, Uzbekistan;
Osh State University, Osh, Kyrgyzstan
E-mail: tursun.k.yuldashev@gmail.com

Bakhtiyar J. Kadirkulov
Alfraganus University, Tashkent, Uzbekistan
E-mail: b.kadirkulov@afu.uz

Mahkambek M. Babayev
Tashkent State Transport University, Tashkent, Uzbekistan
E-mail: babayevm@mail.ru

Let us consider differential equations of the form

$$\left[\frac{\partial^k}{\partial t^k}+(-1)^m\frac{\partial^{k+2m}}{\partial t^k\partial x^{2m}}+(-1)^m\omega\frac{\partial^{2m}}{\partial x^{2m}}\right]U(t,x)=f(t,x),$$

where $k=1,2,3,\ \ m=1,2,3,...,n.$

This equation is sometimes called a Barenblatt–Zheltov–Kochina equation at $k=1$. And when $k=2$, it is often called a Boussinesq type differential equation. These types of equations have important applications in physics and mechanics. Many works have been devoted to the study of this equation for $k=1,2$ (see, for example, [1]-[9], [11]-[13], [15], [16],[18]-[20], [22]-[31], [33], [34], [36]). However, we have not yet encountered a single work on the study of an equation in the case of $k=3$.

When $k=3$ and $m=1$, the equation is studied in our works [32], [35] for Dirichlet conditions. In this paper we consider a case $k=3$ and $m=2$ with Samarskii–Ionkin type conditions and nonlinear right-hand side. We study the solvability of the inverse mixed problem for a seventh order nonlinear differential equation with identification source function and positive parameter. So, in the rectangular domain, $\Omega=\{0<t<T,\ 0<x<1\}$ we consider the following partial differential equation:

$$\left[\frac{\partial^3}{\partial t^3}+\frac{\partial^7}{\partial t^3\partial x^4}+\omega\frac{\partial^4}{\partial x^4}\right]V(t,x)=F\left(t,x,\int_0^T\int_0^1 G(s,y)V(s,y)dyds\right),\tag{1}$$

where $F(t,x,\cdot)\in C^{0,2}_{t,x}(\bar{\Omega}\times R)$, $0<G(t,x)\in C(\bar{\Omega})$, T is given positive number, ω is a positive parameter, $\bar{\Omega}=\left\{0\le t\le T,\ 0\le x\le 1\right\}$.

No great effort is required to compile applications of the equation (1) in the study of processes related to the oil and gas sector. In solving partial differential equation (1), we use the following Samarskii–Ionkin type conditions:

$$V(t,1)=0,\ \ V_{xx}(t,0)=0,\ \ V_x(t,0)=V_x(t,1),\ \ V_{xxx}(t,0)=V_{xxx}(t,1),\ \ 0\le t\le T,\tag{2}$$

and initial value condition

$$V(0,x)=\varphi_1(x),\quad V_t(0,x)=\varphi_2(x),\quad V_{tt}(0,x)=\varphi_3(x),\quad 0\le x\le 1,\tag{3}$$

where $\varphi_1(x)$ is redefinition function, $\varphi_2(x)$ and $\varphi_3(x)$ are given enough smooth functions on the segment $[0,1]$.

Direct Problem Statement. To find a function

$$V(t,x)\in C(\overline{\Omega})\cap C^{3,2}_{t,x}(\Omega),\tag{4}$$

which satisfies differential equation (1) and conditions (2) and (3).

Inverse Problem Statement. To find a pair of functions $\left\{V(t,x),\varphi_1(x)\right\}$, using conditions (2), (3) and following additional condition:

$$V(t_1,x)=\psi(x)\in C^{(4)}[0,1],\quad 0<t_1<T,\tag{5}$$

where $\psi(x)$ is given function.

2. Some Auxiliary Materials about Riesz Bases

From the work [10] we use some materials in our present studies. Let $\{b_n(x)\}$ and $\{c_n(x)\}$ be two complete systems of functions from $L_2[a,b]$. Let us denote by $(b,c)_0$ the scalar product of functions $b(x)$ and $c(x)$ in $L_2[a,b]$:

$$(b,c)_0 = (b,c)_{L_2[a,l]} = \int_a^l b(x)c(x)dx.$$

Definition 1. *Systems $\{b_n(x)\}$ and $\{c_n(x)\}$ form a biorthonormal system on some interval $[a,l]$, if*

$$\big(b_n, c_k\big)_0 = \int_a^l b_n c_k dx = \delta_{nk} = \begin{cases} 0, & n \neq k, \\ 1, & n = k, \end{cases}$$

in this case the system $\{c_n(x)\}$ is called biorthogonally adjoint to the system $\{b_n(x)\}$ on the interval $[a,l]$.

Definition 2. *A system of elements $\{b_n(x)\}$ is called minimal in $L_2[a,l]$, if none of its elements belongs to the closure of the linear span of other elements of this system.*

The minimality of the system ensures the existence of a biorthogonally conjugate system.

Definition 3. *The biorthogonal expansion of a function $f \in L_2[a,l]$ in a system $\{b_n(x)\}$ is called a series*

$$f(x) \sim \sum_{n=1}^{\infty} d_n b_n(x),$$

where $d_n = \big(f, c_n\big)_0$.

Definition 4. *A complete and minimal system of functions $\{b_n(x)\}$ is called Besselian, if for any $f \in L_2[a,l]$ the series of squares of the coefficients of its biorthogonal expansion in $\{b_n(x)\}$ converges:*

$$\sum_{n=1}^{\infty} \left|\big(f, c_n\big)_0\right|^2 < \infty,$$

where $\{c_n\}$ is biorthogonally conjugate system to $\{b_n(x)\}$.

Definition 5. *A complete and minimal system of functions $\{b_n(x)\}$ is called a Hilbert system, if for any sequence of numbers d_n, such that $\sum\limits_{k=1}^{\infty} d_n^2 < \infty$, there is one and only one function $f \in L_2[a,l]$, for which d_n are coefficients of its biorthogonal expansion on $\{b_n(x)\}$:*

$$d_n = \big(f, c_n\big)_0, \quad n = 1, 2, \ldots .$$

Definition 6. *A complete and minimal system is called a Riesz basis, if it is both Bessel and Hilbert basis.*

From the work [17] we use the following theorem.

Theorem 1. *The following statements are equivalent:*
1) the sequence of functions $\{c_j(x)\}_1^{\infty}$ *forms a basis, equivalent to the orthonormal one, in the space R;*
2) the sequence of functions $\{c_j(x)\}_1^{\infty}$ *will be an orthonormal basis of the space R with the appropriate replacement of the scalar product* (f,g) *by some new one* $(f,g)_1$*, topologically equivalent to the original one;*
3) the sequence of functions $\{c_j(x)\}_1^{\infty}$ *is complete in R and there exist constants,* $a_1,\ a_2\,(>0)$*, such that for any natural n and any complex numbers* $\gamma_1,\gamma_2,\dots,\gamma_n$ *there holds*

$$a_2\sum_{j=1}^{n}|\gamma_j|^2 \le \sum_{j=1}^{n}|\gamma_j c_j|^2 \le a_1\sum_{j=1}^{n}|\gamma_j|^2;$$

4) the sequence of functions $\{c_j(x)\}_1^{\infty}$ *is complete in R and its Gram matrix* $(c_j(x),c_k(x))_1^{\infty}$ *generates a bounded invertible operator in the space* ℓ_2*;*
5) the sequence of functions $\{c_j(x)\}_1^{\infty}$ *is complete in R, corresponds to complete biorthogonal sequence of functions* $\{\chi_j(x)\}_1^{\infty}$ *and for any* $f(x)\in R$ *is true that*

$$\sum_{j=1}^{n}|(f,c_j)|^2<\infty,\quad \sum_{j=1}^{n}|(f,\chi_j)|^2<\infty.$$

Lemma 1. *([21]) Let be* $f(x)\in L_2[0,1]$ *and* $a_k=\int_0^1 f(x)e^{-\lambda kx}dx,\ b_k=\int_0^1 f(x)e^{\lambda k(x-1)}dx$*, where* λ *is an arbitrary complex number with a positive real part* $(\lambda=\alpha+i\beta,\ \ \alpha>0)$*. Then the series* $\sum\limits_{k=1}^{\infty}|a_k|^2,\ \ \sum\limits_{k=1}^{\infty}|b_k|^2$ *are convergence.*

3. Eigenvalues and Eigenfunctions

First, we consider homogeneous differential equation

$$\frac{\partial^3 U(t,x)}{\partial t^3}+\frac{\partial^7 U(t,x)}{\partial t^3\,\partial x^4}+\frac{\partial^4 U(t,x)}{\partial x^4}=0. \tag{6}$$

We will look for a non-trivial particular solution of the equation in the form $U(t,x)=u(t)\cdot\vartheta(x)$. Substituting this product of functions, depending from different variables, into equation (6), we obtain

$$-\frac{u'''(t)}{u'''(t)+u(t)}=\frac{\vartheta^{(4)}(x)}{\vartheta(x)}.$$

Hence, equating second fraction into λ we obtain

$$\vartheta^{(4)}(x)-\lambda\vartheta(x)=0,\quad \lambda\ge 0. \tag{7}$$

Using conditions (3), from product of two functions we obtain conditions for the eigenvalues λ and eigenfunctions $\vartheta(x)$:

$$\vartheta(1)=0,\quad \vartheta''(0)=0,\quad \vartheta'(0)=\vartheta'(1),\quad \vartheta'''(0)=\vartheta'''(1). \tag{8}$$

Solving the spectral problem (7), (8), we derive the eigenvalues

$$\lambda_n=(2\pi n)^4,\quad n=0,1,2,\,\ldots\,. \tag{9}$$

Eigenfunctions, corresponding to the eigenvalues (9), have the forms

$$\vartheta_0(x)=2(1-x),\quad \vartheta_{1n}(x)=-2\sin 2\pi nx,\quad \vartheta_{2n}(x)=\frac{e^{2\pi nx}-e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\cos 2\pi nx. \tag{10}$$

The spectral problem (7), (8) is not self-adjoint and it is easy to see that the following problem will be adjoint to it:

$$\omega^{(4)}(x)-\lambda\omega(x)=0,\quad 0<x<1, \tag{11}$$

$$\omega(0)=\omega(1),\quad \omega'(1)=0,\quad \omega''(0)=\omega''(1),\quad \omega'''(0)=0. \tag{12}$$

Along with problem (7), (8), we also consider adjoint to it problem (11), (12). Solving this problem, it is not difficult to see that the eigenfunctions, corresponding to eigenvalues (9), have the form

$$\omega_0(x)=1,\quad \omega_{1n}(x)=\frac{e^{2\pi nx}+e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\sin 2\pi nx,\quad \omega_{2n}(x)=-2\cos 2\pi nx. \tag{13}$$

It should be noted that (10) and (13) are a non-orthogonal system of functions. Indeed, let us consider, for example, the system (10) and calculate it

$$\left(\vartheta_0(x)\,,\,\vartheta_n^{(1)}(x)\right)_0=-4\int_0^1(1-x)\sin 2\pi nx dx=-\frac{2}{\pi n}\neq 0.$$

Let us study the issues of the basis of systems (10) and (13) in $L_2[0,1]$.

Lemma 2. *Systems of functions* (10) *and* (13) *are biorthogonal systems in* $L_2[0,1]$*:*

$$\left(\vartheta_0,\omega_0\right)_0=1,\,\left(\vartheta_{ik},\omega_{jn}\right)_0=\begin{cases}1, & k=n,\ i=j\\ 0, & k\neq n,\ i\neq j\end{cases},\quad i,j=1,2,\quad n,k=1,2,\,\ldots\,.$$

Proof. We present the proof of Lemma 2 for the functions $\vartheta_{1n}(x)$ and $\omega_{1n}(x)$. According to Definition 1, we calculate the integral

$$\left(\vartheta_{1k},\omega_{1n}\right)_0=-2\int_0^1\sin 2\pi kx\left(\frac{e^{2\pi nx}+e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\sin 2\pi nx\right)dx=$$

$$-\frac{2}{e^{2\pi n}-1}\int_0^1\left(e^{2\pi nx}+e^{2\pi n(1-x)}\right)\sin 2\pi kx dx+2\int_0^1\sin 2\pi nx\sin 2\pi kx dx=I_{kn}+J_{kn}.$$

Simple calculations are showed that

$$I_{kn} = -\frac{2}{e^{2\pi n}-1}\int_0^1 (e^{2\pi nx}+e^{2\pi n(1-x)})\sin 2\pi kx dx = 0, \quad k,n \in N,$$

$$J_{kn} = 2\int_0^1 \sin 2\pi kx \cdot \sin 2\pi nx dx = \begin{cases} 1, & k=n \\ 0, & k\neq n \end{cases}, \quad k,n\in N.$$

Consequently, $(\vartheta_{1n},\omega_{1n})_0 = 1$ for $k=n$ and $\left(\vartheta_{1n},\omega_{1n}\right)_0 = 0$ for $k\neq n$. The Lemma 2 is proved. ◄

Lemma 3. *The systems of functions* (10) *and* (13) *are minimal in* $L_2[0,1]$.

The *proof* of Lemma 3 follows from the existence of a biorthonormal system, which was established in Lemma 2.

Theorem 2. *The system of functions* (10) *and* (13) *is complete in the space* $L_2[0,1]$.

Proof. First, we prove the completeness of (10). Assume the opposite, let the system of functions (10) be incomplete in $L_2[0,1]$. Then there exists a function $\psi(x)$ from $L_2(0,1)$, that is orthogonal to all functions of the system (10). We will expand the function $\psi(x)$ into a Fourier series

$$\psi(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos 2\pi nx + b_n \sin 2\pi nx),$$

which is convergence in $L_2[0,1]$. Since $\psi(x)$ is orthogonal to the system $\{-2\sin 2\pi nx\}_{n=1}^{\infty}$, the last expansion takes the form

$$\psi(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi nx. \tag{14}$$

By assumption, $\psi(x)$ is orthogonal to all functions of the form $\vartheta_0(x)$, $\vartheta_{2k}(x)$. Next, multiplying the series (14) successively by these functions and integrating over the interval $(0,1)$, we have

$$0 = 2\int_0^1 \psi(x)(1-x)dx = 2a_0\int_0^1(1-x)dx + 2\sum_{n=1}^{\infty} a_n\int_0^1 (1-x)\cos 2\pi nx dx = a_0,$$

$$0 = \int_0^1 \psi(x)\cdot\left(\frac{e^{2\pi kx}-e^{2\pi k(1-x)}}{e^{2\pi k}-1} - \cos 2\pi kx\right)dx =$$

$$= \sum_{n=1}^{\infty} a_n \int_0^1 \left(\frac{e^{2\pi kx}-e^{2\pi k(1-x)}}{e^{2\pi k}-1} - \cos 2\pi kx\right)\cos 2\pi nx dx = -\frac{1}{2}a_k, \quad k=1,2,3,\ldots.$$

Hence, follows that $a_k = 0, \quad k = 0,1,2,...$ Therefore, from (14) we obtain that $\psi(x) = 0$ on $[0,1]$, which contradicts the condition $\psi(x)\neq 0$. Thus, system (10) is complete in the space $L_2[0,1]$.

Now we will prove the completeness of the system (13). Let there $\psi(x)$ be a function from $L_2[0,1]$, different from zero, orthogonal to all functions of system (13). Since the function $\psi(x)$ is orthogonal to system $\{-2\cos 2\pi nx\}_{n=0}^{\infty}$, it can be represented in $L_2[0,1]$ as a sine series

$$\psi(x)=\sum_{n=1}^{\infty} b_n \sin 2\pi nx. \tag{15}$$

Next, multiplying the last series by $\omega_{1k}(x)$ and integrating over the segment $[0,1]$, taking into account the orthogonality of the functions $\psi(x)$ and $\omega_{1k}(x)$, we obtain

$$0=\int_0^1 \psi(x)\left(\frac{e^{2\pi kx}+e^{2\pi k(1-x)}}{e^{2\pi k}-1}-\sin 2\pi kx\right)dx=$$

$$=\sum_{n=0}^{\infty} b_n \int_0^1 \left(\frac{e^{2\pi kx}+e^{2\pi k(1-x)}}{e^{2\pi k}-1}-\sin 2\pi kx\right)\sin 2\pi nx dx=-\frac{1}{2}b_k, \quad k=1,2,\ldots,$$

i.,e. $b_k=0, \quad n=1,2,\ldots$ Then from (15) it follows that $\psi(x)=0$ on $[0,1]$, i.e. the system (13) is complete in $L_2[0,1]$. Theorem 2 is proved. ◀

Theorem 3. *The system of functions* (10) *and* (13) *forms the Riesz basis in* $L_2[0,1]$.

Proof. To prove the Riesz basis property of the systems (10) and (13), according to Theorem 1, it is sufficient to establish the completeness of these systems and the convergence for any $\psi(x)\in L_2[0,1]$ of the following series:

$$(\psi(x),2(1-x))_0^2+\sum_{n=1}^{\infty}(\psi(x),-2\sin 2\pi nx)_0^2+$$

$$+\sum_{n=1}^{\infty}\left(\psi(x),\frac{1}{e^{2\pi n}-1}(e^{2\pi nx}-e^{2\pi n(1-x)})-\cos 2\pi nx\right)_0^2, \tag{16}$$

$$(\psi(x),1)_0^2+\sum_{n=1}^{\infty}(\psi(x),-2\cos 2\pi nx)_0^2+$$

$$+\sum_{n=1}^{\infty}\left(\frac{1}{e^{2\pi n}-1}\left(e^{2\pi nx}+e^{2\pi n(1-x)}\right)-\sin 2\pi nx,\psi(x)\right)_0^2. \tag{17}$$

The completeness of the systems (10) and (13) follows from Lemma 2 and, therefore, we will show the convergence of series (16) and (17). Let us consider the series (11) and use the notation

$$I_1=4\left(\psi(x),\,(1-x)\right)_0^2, \quad I_2=4\sum_{n=1}^{\infty}\left(\psi(x),\,\sin 2\pi nx\right)_0^2,$$

$$I_3=\sum_{n=1}^{\infty}\left(\psi(x),\frac{e^{2\pi nx}-e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\cos 2\pi nx\right)_0^2.$$

Applying the Cauchy–Shwartz inequality, for I_1 we obtain

$$I_1 = 4\left(\int_0^1 (1-x)\psi(x)dx\right)^2 \leq 4\int_0^1 (1-x)^2 dx \int_0^1 \psi^2(x)dx = \frac{4}{3} \parallel \psi(x) \parallel^2{}_{L_2(0,1)} < \infty.$$

Further, we have

$$I_2 = 4\sum_{n=1}^{\infty} (\psi(x), \sin 2\pi nx)_0^2 = 2\sum_{n=1}^{\infty} \left(\psi(x), \sqrt{2}\sin 2\pi nx\right)^2 = 2\sum_{n=1}^{\infty} c_n^2,$$

where $c_n = \left(\psi(x), \sqrt{2}\sin 2\pi nx\right)$ are Fourier coefficients of a function $\psi(x)$ in an orthonormal system $\left\{\sqrt{2}\sin 2\pi nx\right\}$. Hence, applying Bessel's inequality, we obtain, that

$$I_2 = 2\sum_{n=1}^{\infty} c_n^2 \leq 2 \parallel \psi(x) \parallel_{L_2[0,1]}^2 < \infty.$$

Next, we consider I_3. Since

$$A = \left(\psi(x), \frac{e^{2\pi nx} - e^{2\pi n(1-x)}}{e^{2\pi n} - 1} - \cos 2\pi nx\right)_0^2 =$$

$$= \left((\psi(x), \frac{e^{2\pi nx} - e^{2\pi n(1-x)}}{e^{2\pi n} - 1}) - (\psi(x), \cos 2\pi nx)\right)_0^2,$$

applying the inequality $(a+b)^2 \leq 2(a^2+b^2)$, we obtain

$$A \leq 2\left(\psi(x), \frac{e^{2\pi nx} - e^{2\pi n(1-x)}}{e^{2\pi n} - 1}\right)_0^2 + 2\left(\psi(x), \cos 2\pi nx\right)_0^2 =$$

$$= 2\left(\left(\psi(x), \frac{e^{2\pi nx}}{e^{2\pi n} - 1}\right) - \left(\psi(x), \frac{e^{2\pi n(1-x)}}{e^{2\pi n} - 1}\right)\right)_0^2 + 2\left(\psi(x), \cos 2\pi nx\right)_0^2.$$

Applying the previous inequality again, we get that

$$A \leq 4\left(\psi(x), \frac{e^{2\pi nx}}{e^{2\pi n} - 1}\right)_0^2 + 4\left(\psi(x), \frac{e^{2\pi n(1-x)}}{e^{2\pi n} - 1}\right)_0^2 + 2\left(\psi(x), \cos 2\pi nx\right)_0^2.$$

Hence, we obtain

$$I_3 \leq 4\sum_{n=1}^{\infty}\left(\psi(x), \frac{e^{2\pi nx}}{e^{2\pi n} - 1}\right)_0^2 + 4\sum_{n=1}^{\infty}\left(\psi(x), \frac{e^{2\pi n(1-x)}}{e^{2\pi n} - 1}\right)_0^2 + 2\sum_{n=1}^{\infty}\left(\psi(x), \cos 2\pi nx\right)_0^2 =$$

$$= J_1 + J_2 + J_3.$$

For the J_3 we have

$$J_3 = 2\sum_{n=1}^{\infty}\left(\psi(x), \cos 2\pi nx\right)^2 = \sum_{n=1}^{\infty} a_n^2,$$

where $a_n = \big(\psi(x), \sqrt{2}\cos 2\pi n x\big)$ are Fourier coefficients for function $\psi(x)$ in orthonormal system $\big\{\sqrt{2}\cos 2\pi n x\big\}$. Then, applying Bessel's inequality, we obtain that

$$J_3 = \sum_{n=1}^{\infty} a_n^2 \le \| \, \psi(x) \, \|^2_{L_2[0,1]} < \infty.$$

Since

$$\left(\psi(x), \frac{e^{2\pi n x}}{e^{2\pi n}-1}\right)_0^2 = \left(\int_0^1 \psi(x) \cdot \frac{e^{2\pi n}}{e^{2\pi n}-1} e^{2\pi n(x-1)} dx\right)_0^2 =$$

$$= \left(\int_0^1 \psi(x) \left[1 + \frac{1}{e^{2\pi n}-1}\right] e^{2\pi n(x-1)} dx\right)^2 \le 4\left(\int_0^1 \psi(x) e^{2\pi n(x-1)} dx\right)^2,$$

for J_1 we derive

$$J_1 \le 16 \sum_{n=1}^{\infty} \left(\int_0^1 \psi(x) e^{2\pi n(x-1)} dx\right)^2 = 16 \sum_{n=1}^{\infty} b_n^2, \quad b_n = \int_0^1 \psi(x) e^{2\pi n(1-x)} dx.$$

Hence, taking into account Lemma 1, we obtain that J_1 is finite. Similarly, we obtain that J_2 is also finite. Thus, the series I_1 and I_2 converge. Therefore, the series (16) also converges. The convergence of the series (17) is proved similarly. Theorem 3 is proved. ◀

4. Construction of Solutions of the Direct Boundary Value Problem

Taking into account the formulas (10) and (13) we look for a solution to the problem (1)–(3) in the form of following Fourier series:

$$V(t,x) = v_0(t)\, b_0(x) + \sum_{n=1}^{\infty} \left(v_{1,n}(t)\, b_{1,n}(x) + v_{2,n}(t)\, b_{2,n}(x)\right), \tag{18}$$

where

$$b_0(x) = 2(1-x), \quad b_{1n}(x) = -2\sin 2\pi n x, \quad b_{2n}(x) = \frac{e^{2\pi n x} - e^{2\pi n(1-x)}}{e^{2\pi n}-1} - \cos 2\pi n x,$$

$$v_0(t) = \int_0^1 V(t,y) c_0(y) dy, \quad v_{1,n}(t) = \int_0^1 V(t,y) c_{1,n}(y) dy, \tag{19}$$

$$v_{2,n}(t) = \int_0^1 V(t,y) c_{2,n}(y) dy, \tag{20}$$

$$c_0(x) = 1, \quad c_{1n}(x) = \frac{e^{2\pi n x} + e^{2\pi n(1-x)}}{e^{2\pi n}-1} - \sin 2\pi n x, \quad c_{2n}(x) = -2\cos 2\pi n x.$$

Let the function $V(t,x)$ be a solution to the direct mixed problem (1)–(4). Then, substituting representation (18) into equation (1), we obtain

$$v_0'''(t)b_0(x)+\sum_{n=1}^{\infty}\left(v_{1,n}'''(t)\,b_{1,n}(x)+v_{2,n}'''(t)\,b_{2,n}(x)\right)+$$

$$+\sum_{n=1}^{\infty}\lambda_n\left[v_{1,n}'''(t)\,b_{1,n}(x)+v_{2,n}'''(t)b_{2,n}(x)\right]+\omega\sum_{n=1}^{\infty}\lambda_n\left[v_{1,n}(t)\,b_{1,n}(x)+v_{2,n}(t)b_{2,n}(x)\right]+$$

$$=F_0(t,\cdot)b_0(x)+\sum_{n=1}^{\infty}\left(F_{1,n}(t,\cdot)\,b_{1,n}(x)+F_{2,n}(t,\cdot)\,b_{2,n}(x)\right),$$

where

$$F_0(t,\cdot)=\int_0^1 F(t,\cdot)c_0(y)dy,\quad F_{\kappa,n}(t,\cdot)=\int_0^1 F(t,\cdot)c_{\kappa,n}(y)dy,\ \ \kappa=1,2. \tag{21}$$

Hence, by virtue of (19)–(21) into account, we obtain

$$v_0'''(t)=F_0(t,\cdot), \tag{22}$$

$$v_{\kappa,n}'''(t)+\mu_n(\omega)v_{\kappa,n}(t)=\frac{F_{\kappa,n}(t,\cdot)}{1+\lambda_n},\quad \kappa=1,2, \tag{23}$$

where

$$\mu_n(\omega)=\frac{\lambda_n\omega}{1+\lambda_n},\quad \lambda_n=(2n\pi)^4.$$

Taking into account the formulas (10) and (13), we consider the functions $\varphi_j(x)$, $j=1,2,3$ as in the case of (18):

$$\varphi_j(x)=\varphi_{j,0}\,b_0(x)+\sum_{n=1}^{\infty}\left(\varphi_{j,1,n}\,b_{1,n}(x)+\varphi_{j,2,n}\,b_{2,n}(x)\right), \tag{24}$$

where

$$\varphi_{j,0}=\int_0^1\varphi_j(y)c_0(y)dy,\quad \varphi_{j,\kappa,n}=\int_0^1\varphi_j(y)c_{\kappa,n}(y)dy,\ \ \kappa=1,2,\ \ j=1,2,3.$$

The differential equation (22) is scalar. The differential equation (23) is the countable system of differential equations. Taking (24) into account, from the conditions (2) we obtain

$$v_0^{(j-1)}(0)=\int_0^1\frac{\partial^{j-1}}{\partial t^{j-1}}[V(0,y)]c_0(y)dy=\int_0^1\varphi_j(x)c_0(y)dy=\varphi_{j,0},\ \ j=1,2,3, \tag{25}$$

$$v_{\kappa,n}^{(j-1)}(0)=\int_0^1\frac{\partial^{j-1}}{\partial t^{j-1}}[V(0,y)]c_{\kappa,n}(y)dy=\int_0^1\varphi_j(x)c_{\kappa,n}(y)dy=\varphi_{j,\kappa,n},\ j=1,2,3. \tag{26}$$

First, we integrate the equation (22) with conditions (25):

$$v_0(t)=J_0(t;v_0(t))\equiv P_0(t)+$$

$$+\int_0^t \frac{(t-s)^2}{2}\int_0^1 F\left(s,y,\int_0^T\int_0^1 G(\theta,z)v_0(\theta)b_0(z)dzd\theta\right)c_0(y)dyds, \tag{27}$$

where

$$P_0(t)=\varphi_{1,0}+\varphi_{2,0}t+\varphi_{3,0}t^2.$$

The presentation (27) is nonlinear integral equation and equivalent to the problem (22), (25).

Now we solve the countable system (23) of ordinary differential equations with initial value condition (26). From the works [32], [35] we have

$$v_{\kappa,n}(t,\omega)=P_{\kappa,n}(t,\omega)+\frac{1}{\sqrt[3]{\left(\lambda_n^2+\lambda_n^3\right)\omega}}\int_0^t Q_n(t,s,\omega)\times$$

$$\times\int_0^1 F\left(s,y,\int_0^T\int_0^1 G(\theta,z)\sum_{i=1}^{\infty}v_{\kappa,i}(\theta,\omega)b_{\kappa,i}(z)dzd\theta\right)c_{\kappa,n}(y)dyds, \tag{28}$$

where

$$P_{\kappa,n}(t,\omega)=\varphi_{1,\kappa,n}\sigma_{1,n}(t,\omega)+\frac{1}{\sqrt[3]{\mu_n(\omega)}}\varphi_{2,\kappa,n}\sigma_{2,n}(t,\omega)+\frac{1}{\sqrt[3]{\mu_n^2(\omega)}}\varphi_{3,\kappa,n}\sigma_{3,n}(t,\omega),$$

$$Q_n(t,s,\omega)=\frac{1}{3}\left[e^{-\sqrt[3]{\mu_n(\omega)}(t-s)}+2e^{\frac{\sqrt[3]{\mu_n(\omega)}}{2}(t-s)}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega)}(t-s)+\frac{\pi}{6}\right)\right],$$

$$\sigma_{1,n}(t,\omega)=\frac{1}{3}\left[e^{-\sqrt[3]{\mu_n(\omega)}t}+2e^{\frac{\sqrt[3]{\mu_n(\omega)}t}{2}}\cos\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega)}t\right],$$

$$\sigma_{2,n}(t,\omega)=\frac{1}{3}\left[e^{-\sqrt[3]{\mu_n(\omega)}t}-2e^{\frac{\sqrt[3]{\mu_n(\omega)}}{2}t}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega)}t+\frac{\pi}{6}\right)\right],$$

$$\sigma_{3,n}(t,\omega)=\frac{1}{3}\left[e^{-\sqrt[3]{\mu_n(\omega)}t}-2e^{\frac{\sqrt[3]{\mu_n(\omega)}}{2}t}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega)}t-\frac{\pi}{6}\right)\right].$$

The presentation (28) is countable system of nonlinear integral equations and equivalent to the problem (23), (26).

We note that if the functions $Q_n(t,s,\omega)$, $\sigma_{j,n}(t,\omega)$, $j=1,2,3$ become zero at some values of parameter ω, then for these values of parameter the functions in (28) will be trivial.

We obtain the following transcendental equation

$$\sin\left(\frac{\sqrt{3}}{2}y+\frac{\pi}{6}\right)=-\frac{1}{2}e^{\frac{-3}{2}y},\quad y=\sqrt[3]{\mu_n(\omega)}\,(t-s)>0,$$

for the function $Q_n(t,s,\omega)$ and

$$\cos\frac{\sqrt{3}}{2}y=-\frac{1}{2}e^{\frac{-3}{2}y},\qquad y=\sqrt[3]{\mu_n(\omega)}\,t>0,$$

for the function $\sigma_{1,n}(t,\omega)$, respectively, $\mu_n(\omega) = \frac{\lambda_n}{1+\lambda_n}\omega$. Functions $\sigma_{2,n}(t,\omega),\ \sigma_{3,n}(t,\omega)$ become zero at some values of parameter ω. We replace these equations by the following transcendental equations:

$$\sin\left(\frac{\sqrt{3}}{2}y + \frac{\pi}{6}\right) = \frac{1}{2}e^{\frac{-3y}{2}}, \quad y = \sqrt[3]{\mu_n(\omega)}\,t > 0,$$

$$\sin\left(\frac{\sqrt{3}}{2}y - \frac{\pi}{6}\right) = \frac{1}{2}e^{\frac{-3y}{2}}, \quad y = \sqrt[3]{\mu_n(\omega)}\,t > 0,$$

respectively.

The values of parameter ω, for which the functions above become zero, we denote by $\Lambda_j,\ j=1,2,3,4$, respectively. However, from the fact $\Lambda_1 \cap \Lambda_2 \cap \Lambda_3 \cap \Lambda_4 = \emptyset$ we deduce that the problem (1)–(3) is not trivial.

We will study every integral equation (27) and (28).

Smoothness condition 1. Let the functions $\varphi_j(x)\,(j=1,2,3)$ and $F(t,x,\cdot)$ satisfy the conditions

$$\varphi_j(x) \in C^{(5)}[0,1], \quad \varphi_j(1) = \varphi_j''(0) = \varphi_j^{(4)}(1) = 0, \quad \varphi_j'(0) = \varphi_j'(1), \quad \varphi_j'''(0) = \varphi_j'''(1),$$

$$F(t,x,\cdot) \in C^{0,1}_{t,x}(\bar{\Omega} \times R).$$

Then, we integrate by parts

$$\varphi_{j,0} = \int_0^1 \varphi_j(y)c_0(y)dy, \quad \varphi_{j,\kappa,n} = \int_0^1 \varphi_j(y)c_{\kappa,n}(y)dy, \quad \kappa=1,2, \quad j=1,2,3,$$

five times and (21) one times on the variable x, respectively, and obtain

$$\varphi_{j,1,n} = -\left(\frac{1}{2\pi}\right)^5 \frac{\varphi^{(5)}_{j,1,n}}{n^5}, \quad \varphi^{(5)}_{j,1,n} = \int_0^1 \frac{\partial^5\varphi_j(y)}{\partial y^5}\left(\frac{e^{2\pi ny} - e^{2\pi n(1-y)}}{e^{2\pi n}-1} + \cos 2\pi ny\right)dy,$$

$$\varphi_{j,2,n} = \left(\frac{1}{2\pi}\right)^5 \frac{\varphi^{(5)}_{j,2,n}}{n^5}, \quad \varphi^{(5)}_{j,2,n} = 2\int_0^1 \frac{\partial^5\varphi_j(y)}{\partial y^5}\sin 2\pi ny dy,$$

$$F_{1,n}(t,\cdot) = -\frac{1}{2\pi}\frac{F'_{1,n}(t,\cdot)}{n}, \quad F'_{1,n}(t,\cdot) = \int_0^1 \frac{\partial F(t,y,\cdot)}{\partial y}\left(\frac{e^{2\pi ny} - e^{2\pi n(1-y)}}{e^{2\pi n}-1} + \cos 2\pi ny\right)dy,$$

$$F_{2,n}(t,\cdot) = \frac{1}{2\pi}\frac{F'_{2,n}}{n}, \quad F'_{2,n}(t,\cdot) = 2\int_0^1 \frac{\partial F(t,y,\cdot)}{\partial y}\sin 2\pi ny dy.$$

It is easy to see that [14]

$$\left\|\boldsymbol{\varphi}^{(5)}_{j,\kappa}\right\|_{\ell_2} \le C_{1,\kappa}\left\|\frac{\partial^5\varphi_{j,\kappa}(x)}{\partial x^5}\right\|_{L_2[0,1]}, \quad j=1,2,3, \quad \|\mathbf{F}'_\kappa(t,\cdot)\|_{\ell_2} \le C_{2,\kappa}\left\|\frac{\partial F_\kappa(t,x,\cdot)}{\partial x}\right\|_{L_2[0,1]}.$$

5. Solvability of the Integral Equations (27), (28)

Theorem 4. *Let the conditions be fulfilled:*
1) $M_{F_0} = \left\| F\left(t, x, \int_0^T \int_0^1 G(\theta, z) v_0(\theta) b_0(z) dz d\theta\right) \right\|_{C(\bar{\Omega})} < \infty, \quad 0 < M_{F_0} = \text{const}$*;*
2) $|\, F(t, x, u_1) - F(t, x, u_2)\, | \leq L_{F_0}\, |\, u_1 - u_2\, |, \quad 0 < L_{F_0} = \text{const}$*;*
3) $M_{G_0} = \int_0^T \int_0^1 |\, G(t, x)\, |\, |\, b_0(x)\, |\, dx dt < \infty$*;*
4) $\rho_0 = \frac{T^3}{6} M_{F_0} L_{G_0} M_{G_0} < 1$.

Then the integral equation (27) *has a unique solution in the space* $C(\bar{\Omega})$*. This solution can be found by iteration process:*

$$\left[\begin{array}{l} v_0^0(t) = P_0(t) = \varphi_{1,0} + \varphi_{2,0} t + \varphi_{3,0} t^2, \\ v_0^{m+1}(t) = J_0(t; v_0^m(t)), \quad m = 0, 1, 2, \ldots . \end{array}\right.$$

Proof. According to the conditions of the theorem, we derive the following estimates:

$$\|\, v_0^0(t)\, \|_{C[0,T]} \leq |\, \varphi_{1,0}\, | + |\, \varphi_{2,0}\, | T + |\, \varphi_{3,0}\, | T^2;$$

$$\|\, v_0^1(t) - v_0^0(t)\, \|_{C[0,T]} \leq$$

$$\leq \max_{0 \leq t \leq T} \left| \int_0^t \frac{(t-s)^2}{2} \int_0^1 F\left(s, y, \int_0^T \int_0^1 G(\theta, z) v_0^0(\theta) b_0(z) dz d\theta\right) c_0(y) dy ds \right| \leq \frac{T^3}{6} M_{F_0};$$

$$\|\, v_0^{m+1}(t) - v_0^m(t)\, \|_{C[0,T]} \leq$$

$$\leq \max_{0 \leq t \leq T} \left| \int_0^t \frac{(t-s)^2}{2} L_{F_0} \int_0^T \int_0^1 |\, G(\theta, x)\, |\, b_0(x) dx d\theta\, \left|\, v_0^m(s) - v_0^{m-1}(s)\, \right| ds \right| \leq$$

$$\leq \rho_0 \cdot \left\|\, v_0^m(t) - v_0^{m-1}(t)\, \right\|_{C[0,T]},$$

where $\rho_0 = \frac{T^3}{6} M_{F_0} L_{G_0} M_{G_0}$.

From these estimates implies that the operator on the right-side of the equation (27) is constructing. Consequently, the nonlinear integral equation (27) has a unique solution in the space $C(\bar{\Omega})$. ◀

Theorem 5. *Let the smoothness conditions 1 be fulfilled and*
1) $\max_{0 \leq t \leq T} \|\, F_\kappa(t, x, \cdot)\, \|_{L_2[0,1]} \leq \delta_\kappa, \quad 0 < \delta_\kappa = \text{const} < \infty$*;*
2) $|\, F(t, x, u_1) - F(t, x, u_2)\, | \leq l(x)\, |\, u_1 - u_2\, |, \quad 0 < l(x) \in L_2[0, 1]$*;*
3) $\rho_\kappa = M_3 \|l_\kappa(x)\|_{L_2[0,1]} \int_0^T \|\, G_\kappa(t, x)\, \|_{L_2[0,1]} dt < 1$*, where* M_3 *determines from* (32) *below.*

Then for all values of the parameter ω *the equation* (28) *has a unique solution in the space* $B_2[0, T]$ *with norm*

$$\|\, \vec{v}_\kappa(t)\, \|_{B_2[0,T]} = \sqrt{\sum_{n=1}^{\infty} \left(\max_{t \in [0,T]} |\, v_{\kappa,n}(t)\, |\right)^2} < \infty.$$

Proof. We define the successive approximations for the (28) as:

$$\begin{cases} v^0_{\kappa,n}(t,\omega) = P_{\kappa,n}(t,\omega), \\ v^{m+1}_{\kappa,n}(t,\omega) = J(t; v^m_{\kappa,n}), \ \ m = 0, 1\,, 2\,, 3\,, \ldots . \end{cases} \tag{29}$$

We estimate the zero approximation. By virtue of smoothness conditions, applying the Cauchy–Shwartz inequality and Bessel inequality, from approximations (29) we have

$$\| \, \vec{v}^{\,0}_{\kappa}(t,\omega) \, \|_{B_2[0,T]} \le \sqrt{\sum_{n=1}^{\infty} \left[\max_{0\le t\le T} |\, v^0_{\kappa,n}(t,\omega) \,| \right]^2} \le \sum_{n=1}^{\infty} \max_{0\le t\le T} |\, P_{\kappa,n}(t,\omega) \,| \le$$

$$\le \max_{0\le t\le T} \sum_{n=1}^{\infty} \left| \varphi_{1,\kappa,n}\sigma_{1,n}(t,\omega) + \frac{1}{\sqrt[3]{\mu_n(\omega)}} \varphi_{2,\kappa,n}\sigma_{2,n}(t,\omega) + \frac{1}{\sqrt[3]{\mu^2_n(\omega)}} \varphi_{3,\kappa,n}\sigma_{3,n}(t,\omega) \right| \le$$

$$\le M_0 \left[\sum_{n=1}^{\infty} |\, \varphi_{1,\kappa,n}| + \sqrt[3]{\frac{e}{\omega}} \sum_{n=1}^{\infty} |\, \varphi_{2,\kappa,n} \,| + \sqrt[3]{\left(\frac{e}{\omega}\right)^2} \sum_{n=1}^{\infty} |\, \varphi_{3,\kappa,n} \,| \right] \le$$

$$\le C_0(\omega) M_0 \left(\frac{1}{2\pi}\right)^4 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{10}}} \left[\left\| \boldsymbol{\varphi}^{(5)}_{1,\kappa} \right\|_{\ell_2} + \left\| \boldsymbol{\varphi}^{(5)}_{2,\kappa} \right\|_{\ell_2} + \left\| \boldsymbol{\varphi}^{(5)}_{3,\kappa} \right\|_{\ell_2} \right] \le$$

$$\le M_{1,\kappa} \left[\left\| \frac{\partial^5 \varphi_{1,\kappa}(x)}{\partial\, x^5} \right\|_{L_2[0,1]} + \left\| \frac{\partial^5 \varphi_{2,\kappa}(x)}{\partial\, x^5} \right\|_{L_2[0,1]} + \left\| \frac{\partial^5 \varphi_{3,\kappa}(x)}{\partial\, x^5} \right\|_{L_2[0,1]} \right] < \infty, \tag{30}$$

where

$$\max_{j=1,2,3} \max_{0\le t\le T} |\, \sigma_{j,n}(t,\omega) \,| \le M_0 < \infty, \quad 0 < M_0 = \text{const} < \infty,$$

$$M_{1,\kappa} = C_0(\omega) C_{1,\kappa} M_0 \left(\frac{1}{2\pi}\right)^4 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{10}}}, \quad C_0(\omega) = \max\left\{ 1;\, \sqrt[3]{\frac{e}{\omega}};\, \sqrt[3]{\left(\frac{e}{\omega}\right)^2} \right\}.$$

Taking the conditions of the Theorem and estimate (29) into account, applying the Cauchy–Shwartz inequality and Bessel's inequality, for the first difference $v^1_{\kappa,n}(t) - v^0_{\kappa,n}(t)$ we obtain

$$\| \, \vec{v}^{\,1}_{\kappa}(t,\omega) - \vec{v}^{\,0}_{\kappa}(t,\omega) \, \|_{B_2[0,T]} \le \frac{1}{\sqrt[3]{\omega}} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \max_{0\le t\le T} \int_0^t |\, Q_n(t,s,\omega) \,| \times$$

$$\times \left| \int_0^1 F\left(s, y, \int_0^T \int_0^1 G(\theta,z) \sum_{r=1}^{\infty} v^0_{\kappa,r}(\theta,\omega) b_{\kappa,r}(z) dz d\theta \right) c_{\kappa,n}(y) dy \right| ds \le$$

$$\le \frac{M_2}{\sqrt[3]{\omega}} \left(\frac{1}{2\pi}\right)^5 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{10}}} \left\| \int_0^1 F\left(t, y, \cdot\right)\, c_{\kappa,n}(y) dy \right\|_{B_2[0,T]} \le$$

$$\le M_3 \max_{0\le t\le T} \|F_\kappa(t,x,\cdot)\|_{L_2[0,1]} < \infty, \tag{31}$$

where

$$M_2 = \max_{0\leq t\leq T} \int_0^t |\, Q_n(t,s,\omega)\,|\, ds, \quad M_3 = \frac{C_{2,\kappa} M_2}{\sqrt[3]{\omega}} \left(\frac{1}{2\pi}\right)^5 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{10}}}. \tag{32}$$

We take into account that the quantities

$$\int_0^1 l(y) c_{\kappa,n}(y)\, dy, \quad \int_0^1 |\, G(t,z)\,|\, c_{\kappa,r}(z) dz$$

are Fourier coefficients. So, we obtain

$$\|\, \vec{v}_\kappa^{m+1}(t,\omega) - \vec{v}_\kappa^{m}(t,\omega)\,\|_{B_2[0,T]} \leq$$

$$\frac{M_2}{\sqrt[3]{\omega}} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left| \int_0^1 l(y) \int_0^T \int_0^1 |G(t,z)| \sum_{r=1}^{\infty} \left| v_{\kappa,r}^m(t,\omega) - v_{\kappa,r}^{m-1}(t,\omega)\right| b_{\kappa,r}(z) dz dt c_{\kappa,n}(y) dy \right| \leq$$

$$\frac{M_2}{\sqrt[3]{\omega}} \left| \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \int_0^1 l(y) c_{\kappa,n}(y) dy \int_0^T \int_0^1 |G(t,z)| \sum_{r=1}^{\infty} \left| v_{\kappa,r}^m(t,\omega) - v_{\kappa,r}^{m-1}(t,\omega)\right| c_{\kappa,r}(z) dz dt \right| \leq$$

$$\frac{M_2}{\sqrt[3]{\omega}} \left| \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \int_0^1 l(y) c_{\kappa,n}(y) dy \right| \left| \int_0^T \sum_{r=1}^{\infty} \left| v_{\kappa,r}^m(t,\omega) - v_{\kappa,r}^{m-1}(t,\omega)\right| \int_0^1 |G(t,z)|\, b_{\kappa,r}(z) dz dt \right| \leq$$

$$\rho_\kappa \cdot \left\| \vec{v}_\kappa^{m}(t,\omega) - \vec{v}_\kappa^{m-1}(t,\omega) \right\|_{B_2[0,T]}, \tag{33}$$

where

$$\rho_\kappa = M_3 \, \|l_\kappa(x)\|_{L_2[0,1]} \int_0^T \|\, G_\kappa(t,x)\,\|_{L_2[0,1]}\, dt.$$

From estimates (30), (31), (33) it follows that the operator $J(t; v_{\kappa,n})$ on the right-hand side of (28) is contracting and there is a unique fixed point. So, the existence and uniqueness of the solution $\vec{v}_\kappa(t) \in B_2[0,T]$ to (28) are proved. The theorem is proved. ◀

6. Continuous Dependence of the Solution to the Problem (1)–(4) on Parameter ω

From the equations (27) and (28) we have to consider the Fourier series

$$V(t,x,\omega) = b_0(x) v_0(t) + \sum_{n=1}^{\infty} \sum_{\kappa=1}^{2} b_{n,\kappa}(x) v_{n,\kappa}(t) =$$

$$= b_0(x) \left[P_0(t) + \int_0^t \frac{(t-s)^2}{2} \int_0^1 F\left(s, y, \int_0^T \int_0^1 G(\theta,z) v_0(\theta) b_0(z) dz d\theta \right) c_0(y) dy ds \right] +$$

$$+\sum_{n=1}^{\infty}\sum_{\kappa=1}^{2} b_{\kappa,n}(x)\left[P_{\kappa,n}(t,\omega)+\frac{1}{\sqrt[3]{\left(\lambda_n^2+\lambda_n^3\right)\omega}}\int_0^t Q_n(t,s,\omega)\times\right.$$

$$\left.\times\int_0^1 F\left(s,y,\int_0^T\int_0^1 G(\theta,z)\sum_{i=1}^{\infty} v_{\kappa,i}(\theta,\omega)b_{\kappa,i}(z)dzd\theta\right) c_{\kappa,n}(y)dyds\right]. \qquad (34)$$

Theorem 6. *Let be fulfilled the conditions of the Theorem 5. Then for all values of the parameter ω the following estimate*

$$|\, V(t,x,\omega_1)-V(t,x,\omega_2)\,| \le L_s\,|\,\omega_1-\omega_2\,|\,, \quad 0<L_s=\text{const}$$

holds.

The proof of the theorem 6 is similar to the proof of the corresponding theorem from the works [32], [35].

7. Convergence of the Fourier Series

Theorem 7. *Let the conditions of the Theorem 5 be fulfilled. Then for all values of the parameter ω the series* (34) *converges absolute and uniform in the domain Ω.*

Proof. The *proof* of the theorem 7 is based on obtaining the estimates (30) and (31) for the Fourier series (34). Indeed, we have

$$|\, V(t,x,\omega)\,| \le \sum_{n=1}^{\infty}|\, b_{\kappa,n}(x)\,|\left[|\, P_{\kappa,n}(t,\omega)\,|+\frac{1}{\sqrt[3]{\left(\lambda_n^2+\lambda_n^3\right)\omega}}\left|\int_0^t Q_n(t,s,\omega)\times\right.\right.$$

$$\left.\left.\times\int_0^1 F\left(s,y,\int_0^T\int_0^1 G(\theta,z)\sum_{i=1}^{\infty} v_{\kappa,i}(\theta,\omega)b_{\kappa,i}(z)dzd\theta\right) c_{\kappa,n}(y)dyds\right|\right] \le$$

$$\le \bar{b}_{0,\kappa}M_{1,\kappa}\left[\left\|\frac{\partial^5\varphi_{1,\kappa}(x)}{\partial x^5}\right\|_{L_2[0,1]}+\left\|\frac{\partial^5\varphi_{2,\kappa}(x)}{\partial x^5}\right\|_{L_2[0,1]}+\left\|\frac{\partial^5\varphi_{3,\kappa}(x)}{\partial x^5}\right\|_{L_2[0,1]}\right]+$$

$$+\bar{b}_{0,\kappa}M_3\max_{0\le t\le T}\|F_\kappa(t,x,\cdot)\|_{L_2[0,1]}<\infty,$$

where

$$\bar{b}_{0,\kappa}=\max_{n=1,2,\ldots}\max_{0\le x\le 1}|\, b_{\kappa,n}(x)\,|\,, \quad \kappa=1,2.$$

◄

8. Inverse Problem

Inverse problem we solve only for such values $\omega \notin \Lambda_1$ of the parameter, for which $\sigma_{1,n}(t_1,\omega) \neq 0$ holds. Putting the series (34) into the additional condition (5), determine the Fourier coefficients of redefinition function

$$\varphi_{1,0} = \psi_0(\omega) - \varphi_{2,0}t_1 - \varphi_{3,0}t_1^2 -$$

$$-\int_0^{t_1} \frac{(t_1-s)^2}{2} \int_0^1 F\left(s, y, V_0(\theta,z) \int_0^T \int_0^1 G(\theta,z)V_0(\theta,z)dzd\theta\right) c_0(y)dyds, \qquad (35)$$

$$\varphi_{1,\kappa,n} = \frac{1}{\sigma_{1,n}(t_1,\omega)}\psi_{\kappa,n}(\omega) - \frac{1}{\sqrt[3]{\mu_n(\omega)}} \frac{\sigma_{2,n}(t_1,\omega)}{\sigma_{1,n}(t_1,\omega)}\varphi_{2,\kappa,n} -$$

$$-\frac{1}{\sqrt[3]{\mu_n^2(\omega)}} \frac{\sigma_{3,n}(t_1,\omega)}{\sigma_{1,n}(t_1,\omega)}\varphi_{3,\kappa,n} - \frac{1}{\sqrt[3]{(\lambda_n^2+\lambda_n^3)\omega}} \int_0^{t_1} \frac{Q_n(t_1,s,\omega)}{\sigma_{1,n}(t_1,\omega)} \times$$

$$\times \int_0^1 F\left(s, y, \int_0^T \int_0^1 G(\theta,z)V_\kappa(\theta,z,\omega)dzd\theta\right) c_{\kappa,n}(y)dyds. \qquad (36)$$

From the representations (35) and (36) we obtain redefinition function as a Fourier series

$$\varphi_1(x) = b_0(x)\Big[\psi_0(\omega) - \varphi_{2,0}t_1 - \varphi_{3,0}t_1^2 -$$

$$-\int_0^{t_1} \frac{(t_1-s)^2}{2} \int_0^1 F\left(s, y, \int_0^T \int_0^1 G(\theta,z)V_0(\theta,z)dzd\theta\right) c_0(y)dyds\Bigg] +$$

$$+\sum_{n=1}^{\infty}\sum_{\kappa=1}^{2} b_{\kappa,n}(x)\left[\frac{1}{\sigma_{1,n}(t_1,\omega)}\psi_{\kappa,n}(\omega) - \frac{1}{\sqrt[3]{\mu_n(\omega)}} \frac{\sigma_{2,n}(t_1,\omega)}{\sigma_{1,n}(t_1,\omega)}\varphi_{2,\kappa,n} -\right.$$

$$-\frac{1}{\sqrt[3]{\mu_n^2(\omega)}} \frac{\sigma_{3,n}(t_1,\omega)}{\sigma_{1,n}(t_1,\omega)}\varphi_{3,\kappa,n} - \frac{1}{\sqrt[3]{\left(\lambda_n^2+\lambda_n^3\right)\omega}} \int_0^{t_1} \frac{Q_n(t_1,s,\omega)}{\sigma_{1,n}(t_1,\omega)} \times$$

$$\left.\times \int_0^1 F\left(s, y, \int_0^T \int_0^1 G(\theta,z)V_\kappa(\theta,z,\omega)dzd\theta\right) c_{\kappa,n}(y)dyds\right]. \qquad (37)$$

The redefinition function $\varphi_1(x)$ is formal determined by the presentation (37). Now we must prove convergence of the series in (37). However, first substituting the equations (35) and (36) into presentation (34), obtain the formal solution of the inverse problem (1)–(5):

$$V(t,x,\omega) = b_0(x)\Big[\psi_0(\omega) + \varphi_{2,0}(t-t_1) + \varphi_{3,0}(t^2-t_1^2) +$$

$$+\int_{t_1}^{t} \frac{(t-s)^2}{2} \int_0^1 F\left(s, y, \int_0^T \int_0^1 G(\theta,z)V_0(\theta,z)dzd\theta\right) c_0(y)dyds\Bigg] +$$

$$+\sum_{n=1}^{\infty}\sum_{\kappa=1}^{2} b_{\kappa,n}(x)\left[\chi_{1,n}(t,\omega)\psi_{\kappa,n}(\omega)+\frac{1}{\sqrt[3]{\mu_n(\omega)}}\chi_{2,n}(t,\omega)\varphi_{2,\kappa,n}+\right.$$

$$+\frac{1}{\sqrt[3]{\mu_n^2(\omega)}}\chi_{3,n}(t,\omega)\varphi_{3,\kappa,n}+\frac{1}{\sqrt[3]{\left(\lambda_n^2+\lambda_n^3\right)\omega}}\int_0^t H_n(t,s,\omega)\times$$

$$\left.\times\int_0^1 F\left(s,y,\int_0^T\int_0^1 G(\theta,z)V_\kappa(\theta,z,\omega)dzd\theta\right)c_{\kappa,n}(y)dyds\right], \tag{38}$$

where

$$\chi_{1,n}(t,\omega)=\frac{\sigma_{1,n}(t,\omega)}{\sigma_{1,n}(t_1,\omega)},\quad \chi_{2,n}(t,\omega)=\sigma_{2,n}(t,\omega)-\frac{\sigma_{2,n}(t_1,\omega)}{\sigma_{1,n}(t_1,\omega)}\sigma_{1,n}(t,\omega),$$

$$\chi_{3,n}(t,\omega)=\sigma_{3,n}(t,\omega)-\frac{\sigma_{3,n}(t_1,\omega)}{\sigma_{1,n}(t_1,\omega)}\sigma_{1,n}(t,\omega),$$

$$H_n(t,s,\omega)=\begin{cases}Q_n(t,s,\omega)-\frac{\sigma_{1,n}(t,\omega)}{\sigma_{1,n}(t_1,\omega)}Q_n(t_1,s,\omega), & 0\le s\le t_1,\\ Q_n(t,s,\omega), \ t_1\le s\le t.\end{cases}$$

To find the function $V(t,x,\omega)$ from (38), we solve the following equations

$$v_0(t)=\psi_0(\omega)+\varphi_{2,0}(t-t_1)+\varphi_{3,0}\left(t^2-t_1^2\right)+$$

$$+\int_{t_1}^t\frac{(t-s)^2}{2}\int_0^1 F\left(s,y,\int_0^T\int_0^1 G(\theta,z)v_0(\theta)b_0(z)dzd\theta\right)c_0(y)dyds, \tag{39}$$

$$v_{\kappa,n}(t)=\sum_{n=1}^{\infty}\left[\chi_{1,n}(t,\omega)\psi_{\kappa,n}(\omega)+\frac{1}{\sqrt[3]{\mu_n(\omega)}}\chi_{2,n}(t,\omega)\varphi_{2,\kappa,n}+\right.$$

$$+\frac{1}{\sqrt[3]{\mu_n^2(\omega)}}\chi_{3,n}(t,\omega)\varphi_{3,\kappa,n}+\frac{1}{\sqrt[3]{\left(\lambda_n^2+\lambda_n^3\right)\omega}}\int_0^t H_n(t,s,\omega)\times$$

$$\left.\times\int_0^1 F\left(s,y,\int_0^T\int_0^1 G(\theta,z)\sum_{i=1}^{\infty}v_{\kappa,i}(\theta,\omega)b_{\kappa,i}(z)dzd\theta\right)c_{\kappa,n}(y)dyds\right]. \tag{40}$$

For the solvability of the equations (39) and (40) hold the Theorems 4 and 5, if the following smoothness condition 2 is fulfilled.

Smoothness condition 2. Let the function $\psi(x)$ satisfies the conditions

$$\psi(x)\in C^{(5)}[0,1],\quad \psi(1)=\psi''(0)=\psi^{(4)}(1)=0,\quad \psi'(0)=\psi'(1),\quad \psi'''(0)=\psi'''(1).$$

Then, we integrate by parts

$$\psi_0=\int_0^1\psi_0(y)c_0(y)dy,\quad \psi_{\kappa,n}=\int_0^1\psi_\kappa(y)c_{\kappa,n}(y)dy,\ \ \kappa=1,2,$$

five times on the variable x, and obtain

$$\psi_{1,n} = -\left(\frac{1}{2\pi}\right)^5 \frac{\psi^{(5)}_{1,n}}{n^5}, \quad \psi^{(5)}_{1,n} = \int_0^1 \frac{\partial^5 \psi(y)}{\partial y^5} \left(\frac{e^{2\pi n y} - e^{2\pi n(1-y)}}{e^{2\pi n} - 1} + \cos 2\pi n y \right) dy,$$

$$\psi_{2,n} = \left(\frac{1}{2\pi}\right)^5 \frac{\psi^{(5)}_{2,n}}{n^5}, \quad \psi^{(5)}_{2,n} = 2\int_0^1 \frac{\partial^5 \psi(y)}{\partial y^5} \sin 2\pi n y \, dy,$$

$$\left\| \boldsymbol{\psi}^{(5)}_{\kappa} \right\|_{\ell_2} \leq C_{3,\kappa} \left\| \frac{\partial^5 \psi_\kappa(x)}{\partial x^5} \right\|_{L_2[0,1]}.$$

Theorem 8. *Let the smoothness conditions 2 and the conditions of the Theorems 4 and 5 be fulfilled. Then for values of the parameter ω from the set Λ_1 the series* (38) *converges absolute and uniform in the domain Ω. Moreover, the solution of the inverse mixed problem* (1)–(3), (5) *belongs to the class of functions* (4).

Proof. As a case of proof of the Theorem 7, we have

$$|\, V_\kappa(t,x,\omega) \,| \leq \sum_{n=1}^{\infty} |\, b_{\kappa,n}(x) \,| \left[\left(\frac{1}{2\pi}\right)^5 \max_{0\leq t\leq T} |\, \chi_{1,n}(t,\omega) \,| \frac{\left| \psi^{(5)}_{\kappa,n} \right|}{n^5} + \right.$$

$$+ \frac{1}{\sqrt[3]{\mu_n(\omega)}} \left(\frac{1}{2\pi}\right)^5 \max_{0\leq t\leq T} |\, \chi_{2,n}(t,\omega) \,| \frac{\left| \varphi^{(5)}_{2,\kappa,n} \right|}{n^5} +$$

$$+ \frac{1}{\sqrt[3]{\mu_n^2(\omega)}} \left(\frac{1}{2\pi}\right)^5 \max_{0\leq t\leq T} |\, \chi_{3,n}(t,\omega) \,| \frac{\left| \varphi^{(5)}_{3,\kappa,n} \right|}{n^5} + \frac{1}{\sqrt[3]{(\lambda_n^2 + \lambda_n^3)\omega}} \max_{0\leq t\leq T} \int_0^t |\, H_n(t,s,\omega) \,| \times$$

$$\left. \times \left| \int_0^1 F\left(s, y, \int_0^T \int_0^1 G(\theta,z) V_\kappa(\theta,z,\omega) dz d\theta \right) c_{\kappa,n}(y) dy \right| ds \right].$$

Hence, using the Cauchy–Schwarz inequality, we derive

$$|\, V_\kappa(t,x,\omega) \,| \leq \chi_0 C_0(\omega) \left(\frac{1}{2\pi}\right)^5 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{10}} \max_{0\leq x\leq 1} |\, b_{\kappa,n}(x) \,|} \left[\left\| \boldsymbol{\psi}^{(5)}_{\kappa} \right\|_{\ell_2} + \left\| \boldsymbol{\varphi}^{(5)}_{2,\kappa} \right\|_{\ell_2} + \right.$$

$$\left. + \left\| \boldsymbol{\varphi}^{(5)}_{3,\kappa} \right\|_{\ell_2} \right] + \max_{0\leq x\leq 1} |\, b_{\kappa,n}(x) \,| \left(\frac{1}{2\pi}\right)^4 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^8} \max_{0\leq t\leq T} \int_0^t |\, H_n(t,s,\omega) \,| ds} \times$$

$$\times \left\| \int_0^1 F\left(t, y, \int_0^T \int_0^1 G(\theta,z) V_\kappa(\theta,z,\omega) dz d\theta \right) c_{\kappa,n}(y)\, dy \right\|_{B_2[0,T]}, \tag{41}$$

where

$$\chi_0 = \max_{n=1,2,\ldots} \left\{ \max_{0\leq t\leq T} |\, \chi_{1,n}(t,\omega) \,|\, ; \max_{0\leq t\leq T} |\, \chi_{2,n}(t,\omega) \,|\, ; \max_{0\leq t\leq T} |\, \chi_{3,n}(t,\omega) \,| \right\}.$$

Using the Bessel inequality, from the estimate (41) we obtain the necessary estimate on convergence of series (38)

$$|\, V_\kappa(t,x,\omega) \,| \leq \chi_0 \bar{b}_{0,\kappa} C_0(\omega) \left(\frac{1}{2\pi}\right)^5 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{10}}} \times$$

$$\times \left[\left\| \frac{\partial^5 \psi_\kappa(x)}{\partial x^5} \right\|_{L_2[0,1]} + \left\| \frac{\partial^5 \varphi_{2,\kappa}(x)}{\partial x^5} \right\|_{L_2[0,1]} + \left\| \frac{\partial^5 \varphi_{3,\kappa}(x)}{\partial x^5} \right\|_{L_2[0,1]} \right] +$$

$$+ \bar{b}_{0,\kappa} M_1 \left(\frac{1}{2\pi}\right)^4 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^8} \max_{0\leq t\leq T} \| F_\kappa(t,x,\cdot) \|_{L_2[0,1]}} < \infty, \tag{42}$$

where $\bar{b}_{0,\kappa} = \max\limits_{n=1,2,\ldots} \max\limits_{0\leq x\leq 1} |\, b_{\kappa,n}(x) \,|, \quad \kappa = 1,2.$

From the estimate (42) implies the convergence of the series (38). The proof of belongness of the solution to inverse mixed problem (1)–(3), (5) to the class of functions (4) is similar.

Substituting the solutions of the equations (39) and (40) into series (37), we determined redefinition functions $\varphi_1(x)$. The proof of convergence of the series (37) is similar to the proof of the Theorem 8. ◀

9. Conclusion

It is considered a seventh order nonlinear partial differential equations (1) with mixed conditions (2) and (3) and with a positive parameter ω. The Fourier spectral method of separation of variables is applied. The eigenvalues and eigenfunctions of the spectral (adjoint spectral) problems (7), (8) ((11), (12)) are calculated (see, (9), (10) and (13)). Systems of nonlinear functional-integral equations (27), (28) and (39), (40) are derived. Theorems on the uniqueness and existence of the solution of mixed problem (1)–(3) is proved for all values of parameter. The method of contraction mapping is applied for solving the systems (27) and (28) in the Banach spaces $C[0,T]$ and $B_2[0,T]$, respectively. The solution of the inverse mixed problem (1)–(3), (5) is obtained in the form of Fourier series (38). Analogously, the redefinition function is determined as a Fourier series (37). Theorem 5 on absolute and uniform convergence of Fourier series (34) of direct mixed problem (1)–(3) is proved for all values of the parameter ω. Theorem 8 on absolute and uniform convergence of Fourier series (38) of inverse mixed problem (1)–(3), (5) is proved for values of the parameter ω from the set Λ_1.

We hope that this work can serve as a basis for further development of the theory of partial differential equations of mathematical physics.

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