Baku Mathematical Journal
2025, Vol. 4, No 2, PP. 137-152
https://doi.org/10.32010/j.bmj.2025.10

# ON THE USAGES OF GREEN'S FUNCTION ON TIME SCALES

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Received: 10.11.2024 / Revised: 14.05.2025 / Accepted: 17.06.2025

**Abstract.** *We deal with Green's functions (Green's-f) for 2-nd order dynamic boundary value problems (BV-P's) on $\mathbb{T}$. Green's function is established on $\mathbb{T}$ for different situations to solve constructed problems. These results show how important Green's-f is for mathematical physics on $\mathbb{T}$.*

**Keywords**: time scales, Green's function, conjugate problems, eigenvalue problems

**Mathematics Subject Classification (2020):** 34N05, 34B27

## 1. Introduction

Green's-f are used in mathematics to specify a method used to solve inhomogeneous differential equations under desired boundary conditions and the calculated function associated with this method. It was first used by the mathematician George Green. In 1877 Carl Neumann (1832–1925) used the idea of Green's-f in his work on the Laplace equation in the plane. With the success of Green's-f in solving Laplace's equation, solutions began to be obtained by using Green's-f in other equations. For the heat equation, Hobson (1856–1933) defined Green's-f in one, two, and three dimensions, and Appell (1855–1930) realized that there was a formula similar to Green's formula for one-dimensional heat equation. However, the task of introducing the modern theory of its function for the heat equation fell to Sommerfeld (1868–1951). The greatest guiding development in the field of Green's-f was Kirchhoff (1824-1887) with his work in the three-dimensional wave

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equation. The application of Green's-f to differential equations involving BV-P's began with the work of Burkhardt (1861–1914). Burkhardt derived the Green's-f using the results of Picard's theory of ODE's. Later, Bocher (1867–1918) published these results to expanded to $n$ order BV-P's [14].

In mathematical physics, Green's-f creates a highly effective technique to solve IV-P's or BV-P's [1], [2], [5], [10], [14], [23], [25], [27], [30]. Green's-f are a tool to solve hard ODE's and PDE's which may be unsolvable by other ways. To get a solution for an IV-P or BV-P, we need a mapping whose kernel is a Green's-f. One of the remarkable features of Green's-f is their ability to express solutions in a highly general form. They are widely utilized in mathematical treatment of problems arising in mathematical physics, natural sciences, engineering, and biomedical modeling. Owing to this versatility, Green's-f hold a central role in applied mathematics. A Green's-f is impulse response of a nonhomogeneous linear differential operator defined on a domain with specified initial or boundary conditions.

Green's-f play a central role in the solution of physical and mathematical systems. Mathematically, especially in solving boundary value problems, Green's-f provide solutions that coincide with the fixed points of associated integral operators defined by differential equations. Due to this property, Green's-f are a critical tool for examining the existence and accuracy of solutions under boundary conditions [11], [26]. Physically, Green's-f are used to model the effects of external forces or sources on a system. For example, in processes like electromagnetic fields, heat propagation, or diffusion, Green's-f help in representing the impact of a specific source within a system, thereby enabling the resolution of physical processes. Therefore, Green's-f occupy an important place both in the depth of mathematical theory and in physical applications [22]. There are numerous studies in literature about Green's-f in usual case.

This important function was defined on $\mathbb{T}$ in the following years and its properties were examined in detail. Among the studies addressing spectral properties of second-order boundary value problems on measure chains, Chyan et al. [13] proved the existence of the smallest positive eigenvalue by applying the theory of positive operators on cones in Banach spaces along with the sign characteristics of the corresponding Green's-f in 1998. Let's give a summary of the history for Green's-f on $\mathbb{T}$. Erbe and Peterson [16] considered a self adjoint equation and examined Green's-f associated with this equation on $\mathbb{T}$ in 1999. In 2000, they [17] established the existence of positive solutions for a nonlinear boundary value problem and obtained significant results concerning Green's-f. In 2001, Bohner and Peterson [7] redefined Green's-f on $\mathbb{T}$. In 2003, they established Green's-f for BV-P's and applied to different problems [8]. The sign features of Green's-f are presented and existence results for positive solutions of non-linear BV-P's are obtained by Atici and Guseinov [3] on $\mathbb{T}$ in 2002. Hoffacker [21] viewed Green's-f for $n-$ th order focal BV-P's on $\mathbb{T}$ in 2003. In 2003, Eloe [15] studied on sign features of Green's-f for disconjugate dynamic equations. In 2004, Cabada [9] developed the expression of Green function associated with higher order periodic boundary value problems on time scales and determined the conditions for the existence of extreme solutions of this function. In 2005, Guseinov [18] considered self-adjoint BV-P's with symmetric Green's-f. Lin and Du [24] deal with the existence of multiple positive solutions for 2-nd order dynamic equations in 2008. Xie *et. al* [28] considered Green's-f and positive solutions of an $n-$th order singular three-point BV-P on $\mathbb{T}$ in 2009. Cetin and Topal [12] solved existence results for solutions of integral BV-P's on $\mathbb{T}$ with Green's-f in 2013. Bendouma and Hammoudi [6] construct existence of solutions for conformable fractional dynamic equations with non-linear functional boundary conditions with fractional Green's-f in 2019. Before addressing problems we will be dealing with on $\mathbb{T}$, let's give a brief about time scale theory.

Time scale calculus was introduced by Hilger [19] in his Ph.D. thesis in 1988 [4], [20]. It is a unification of standard calculus of functions with domain $\mathbb{R}$ and discrete calculus defined on $\mathbb{Z}$. More rigorously, $\mathbb{T}$ is an arbitrary, non-empty, closed subset of $\mathbb{R}$. Our study will give a new viewpoint for mathematicians to solve BV-P's on $\mathbb{T}$ with Green's-f. Before expressing the main results, it needs to be reminded some basic notions about $\mathbb{T}$. It is firstly compulsory to mention on $\Delta$ calculus of $\mathbb{T}$. Let $a = \inf \mathbb{T}$ and $b = \sup \mathbb{T}$. Since $\mathbb{T}$ is not necessarily connected, forward and backward-jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \ \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

respectively such that $a < t < b$, $t < \sup \mathbb{T}$, $\inf \phi = \sup \mathbb{T}$, $\sup \phi = \inf \mathbb{T}$ where $\phi$ is empty set and $t \in \mathbb{T}$. It can be written $\sigma(b) = b$ and $\rho(a) = a$ provided that $\mathbb{T}$ is bounded. The related forward-step function $\mu$ is

$$\mu : \mathbb{T}^{\kappa} \to \mathbb{R}^{+}, \quad \mu(t) = \sigma(t) - t.$$

Moreover, $t \in \mathbb{T}$ is left dense, left scattered, right dense, right scattered, isolated and dense iff $\rho(t) = t$, $\rho(t) < t, \sigma(t) = t, \sigma(t) > t$, $\rho(t) < t < \sigma(t)$ and $\rho(t) = t = \sigma(t)$, respectively. We also should remind the set $\mathbb{T}^{\kappa}$ along with $\mathbb{T}$ to define $\Delta$-derivative of a function (see [7], [8]). $\mathbb{T}^{\kappa} = \mathbb{T} \backslash \{b\}$ if $\mathbb{T}$ is bounded above where $b$ is left-scattered; otherwise $\mathbb{T}^{\kappa} = \mathbb{T}$. $f : \mathbb{T} \to \mathbb{R}$ is *rd*-continuous at $t \in \mathbb{T}$ if there is some $\delta > 0$ such that $|f(t) - f(s)| < \varepsilon$ for all $s \in [t, t + \delta)$ and $\varepsilon > 0$. $C_{rd}(\mathbb{T})$ indicates the set of all *rd*-continuous functions on $\mathbb{T}$. One can define $f^{\Delta}(t)$ to be value for $t \in \mathbb{T}^{\kappa}$, if one exists, there is a neighborhood $U$ of $t$ such that for all $s \in U$ and $\varepsilon > 0$

$$\left|[f^{\sigma}(t) - f(s)] - f^{\Delta}(t)\,(\sigma(t) - s)\right| < \varepsilon\,|\sigma(t) - s|\,.$$

Here, $f$ is $\Delta$-differentiable on $\mathbb{T}^{\kappa}$ if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$. Let $f \in C_{rd}(\mathbb{T})$. Then, there exists a function $F$ such that $F^{\Delta}(t) = f(t)$, and $\Delta$ integral is constructed by

$$\int\limits_a^b f(t)\Delta t = F(b) - F(a).$$

Spectral theory on time scales has been extensively studied in the literature [29]. This study also aims to contribute to this literature. The rest of study is arranged as follows: In Section 2, we give some fundamental properties of Green's-f in two points and conjugate problems on $\mathbb{T}$. Using some techniques, we get solutions of some eigenvalue problems on $\mathbb{T}$ in Section 3. In Section 4, we give a conclusion to sum up our study.

## 2. Green's-f for Two Points and Conjugate Problems on $\mathbb{T}$

In this section, it will be expressed how Green's-f is set up on $\mathbb{T}$ for varied cases. The infrastructure will be created for the solution of different IV-P's and BV-P's that we will discuss in the last section. Let's first recall some Green's-f forms studied by Bohner and Peterson on $\mathbb{T}$.

Let the second-order dynamic equation [16]

$$Lx(t) = (p(t)x^{\Delta}(t))^{\Delta} + q(t)x^{\sigma}(t),$$

be considered, where $L : \mathbb{D}_1 \to C_{rd}$, $p, q \in C_{rd}$ and $p(t) \neq 0$ for all $t \in \mathbb{T}$. Define $\mathbb{D}_1$ as the set of all functions $x : \mathbb{T} \to \mathbb{R}$ such that delta derivative $x^{\Delta} : \mathbb{T}^{\kappa} \to \mathbb{R}$ is continuous and the composition $(px^{\Delta})^{\Delta} : \mathbb{T}^{\kappa^2} \to \mathbb{R}$ is rd-continuous. $x \in \mathbb{D}_1$ is said to be a solution of equation $Lx = 0$ if the equality $Lx(t) = 0$ holds for $\forall t \in \mathbb{T}^{\kappa^2}$.

**Theorem 1.** *[16] Suppose that $L : \mathbb{D}_1 \to C_{rd}$,*

$$\begin{aligned} Lx(t) &= 0, \\ \alpha x(a) - \beta x^{\Delta}(a) &= 0, \\ \gamma x(\sigma(b)) + \delta x^{\Delta}(\sigma(b)) &= 0, \end{aligned} \tag{1}$$

*has only trivial solution where $\alpha, \beta, \gamma$ and $\delta$ be constants where $\alpha^2 + \beta^2 > 0$ and $\gamma^2 + \delta^2 > 0$. Then, below non-homogeneous BV-P*

$$\begin{aligned} Lx(t) &= h(t), \\ \alpha x(a) - \beta x^{\Delta}(a) &= A, \\ \gamma x(\sigma(b)) + \delta x^{\Delta}(\sigma(b)) &= B, \end{aligned} \tag{2}$$

*has a unique solution where $A, B$ are given constants and $h \in C_{rd}$. Now, let's express how a Green's-f is constructed for a two-point BV-P on $\mathbb{T}$.*

**Theorem 2.** *[16] Assume that BV-P (1) has only trivial solution. For each fixed $s \in [a, b]$, let $u(\cdot, s)$ be unique solution of below BV-P*

$$\begin{aligned} Lu(\cdot, s) &= 0, \\ \alpha u(a, s) - \beta u^{\Delta}(a, s) &= 0, \\ \gamma u\left(\sigma^2(b), s\right) + \delta u^{\Delta}(\sigma(b), s) &= -\gamma x\left(\sigma^2(b), s\right) - \delta x^{\Delta}(\sigma(b), s), \end{aligned}$$

*where $L : \mathbb{D}_1 \to C_{rd}$, $\alpha^2 + \beta^2 > 0$ and $\gamma^2 + \delta^2 > 0$. Furthermore, consider the below equation*

$$Lx(t) = \left(px^{\Delta}\right)^{\Delta}(t) + q(t)x^{\sigma}(t) = 0, \tag{3}$$

*where $x(t)$ is Cauchy function. Then, Green's-f for the problem (1) is defined by*

$$G : \left[a, \sigma^2(b)\right] \times [a, b] \to \mathbb{R},\ G(t, s) = \begin{cases} u(t, s), & \text{if} \quad t \leq s, \\ v(t, s), & \text{if} \quad t \geq \sigma(s)\ , \end{cases}$$

*where* $v(t,s) = u(t,s) + x(t,s)$ *for* $t \in \left[a, \sigma^2(b)\right]$, $s \in [a,b]$. *Then, for each fixed* $s \in [a,b]$, $v(\cdot, s)$ *is a solution of* (3) *and satisfies second boundary condition in* (1). *If* $h \in C_{rd}$,

$$x(t) = \int_a^{\sigma(b)} G(t,s)h(s)\Delta s, \tag{4}$$

*is solution of nonhomogeneous* $BV$-$P$ (2) *with* $A = B = 0$.

**Lemma 1.** *[7] If* $BV$-$P$ (1) *has only trivial solution, then Green's-f for problem* (1) *satisfies the jump condition*

$$G^{\Delta}\left(s^{+}, s\right) - G^{\Delta}\left(s^{-}, s\right) = \frac{1}{p(s)},$$

*for* $s \in \left(a, \sigma^2(b)\right)$ *such that* $\sigma(s) = s$ *and* $p \in C_{rd}$.

Now let's remind Green's-f for below conjugate BV-P.

**Theorem 3.** *[7] Suppose that below* $BV$-$P$

$$\begin{gathered} Lx(t) = 0, \\ x(a) = 0, \\ x\left(\sigma^2(b)\right) = 0, \end{gathered} \tag{5}$$

*has only trivial solution where* $x(t)$ *is Cauchy function. For each fixed* $s \in \mathbb{T}$, *let* $u(.,s)$ *be unique solution of* $BV$-$P$

$$\begin{aligned} Lu(\cdot,s) =& 0, \\ u(a,s) =& 0, \\ u(\sigma^2(b),s) =& -x(\sigma^2(b),s). \end{aligned} \tag{6}$$

*Then,*

$$G(t,s) = \begin{cases} u(t,s), & if \quad t \le s, \\ v(t,s), & if \quad t \ge \sigma(s) \end{cases}$$

*is Green's-f for* (6) *where* $v(t,s) = u(t,s) + x(t,s)$. *Moreover, for each fixed* $s \in [a,b]$, $v(\cdot,s)$ *is a solution of* (5) *and* $v(\sigma^2(b),s) = 0$. *Green's-f for* $BV$-$P$ (6) *with* $q = 0$ *is given by*

$$G(t,s) = \begin{cases} -\dfrac{\int\limits_a^t \frac{1}{p(\tau)}\Delta\tau \int\limits_{\sigma(s)}^{\sigma^2(b)} \frac{1}{p(\tau)}\Delta\tau}{\int\limits_a^{\sigma^2(b)} \frac{1}{p(\tau)}\Delta\tau}, & if \quad t \le s, \\ -\dfrac{\int\limits_a^{\sigma(s)} \frac{1}{p(\tau)}\Delta\tau \int\limits_t^{\sigma^2(b)} \frac{1}{p(\tau)}\Delta\tau}{\int\limits_a^{\sigma^2(b)} \frac{1}{p(\tau)}\Delta\tau}, & if \quad t \ge \sigma(s). \end{cases}$$

*Now, let us express the Green's-f $G(t,s)$ for some common time scales. For $\mathbb{T}=\mathbb{R}$, we obtain*

$$G(t,s)=\begin{cases} -\dfrac{\int_a^t \frac{1}{p(\tau)}d\tau \int_s^{\sigma^2(b)} \frac{1}{p(\tau)}d\tau}{\int_a^{\sigma^2(b)} \frac{1}{p(\tau)}d\tau}, & if \quad t\le s,\\ -\dfrac{\int_a^s \frac{1}{p(\tau)}d\tau \int_t^{\sigma^2(b)} \frac{1}{p(\tau)}d\tau}{\int_a^{\sigma^2(b)} \frac{1}{p(\tau)}d\tau}, & if \quad t\ge s. \end{cases}$$

*If $\mathbb{T}=\mathbb{Z}$, Green's-f for BV-P is*

$$G(t,s)=\begin{cases} -\dfrac{\sum_{\tau=a}^{t-1} \frac{1}{p(\tau)} \sum_{\tau=\sigma(s)}^{\sigma^2(b)-1} \frac{1}{p(\tau)}}{\sum_{\tau=a}^{\sigma^2(b)-1} \frac{1}{p(\tau)}}, & if \quad t\le s,\\ -\dfrac{\sum_{\tau=a}^{\sigma(s)-1} \frac{1}{p(\tau)} \sum_{\tau=t}^{\sigma^2(b)-1} \frac{1}{p(\tau)}}{\sum_{\tau=a}^{\sigma^2(b)-1} \frac{1}{p(\tau)}}, & if \quad t\ge s+1. \end{cases}$$

*Also for $\mathbb{T}=2^{\mathbb{N}_0}$,*

$$G(t,s)=\begin{cases} -\dfrac{\sum_{\tau\in[a,t)} \tau\frac{1}{p(\tau)} \sum_{\tau\in[2s,\sigma^2(b))} \tau\frac{1}{p(\tau)}}{\sum_{\tau\in[a,\sigma^2(b))} \tau\frac{1}{p(\tau)}}, & if \quad t\le s,\\ -\dfrac{\sum_{\tau\in[a,2s)} \tau\frac{1}{p(\tau)} \sum_{\tau\in[t,\sigma^2(b))} \tau\frac{1}{p(\tau)}}{\sum_{\tau\in[a,\sigma^2(b))} \tau\frac{1}{p(\tau)}}, & if \quad t\ge s+1. \end{cases}$$

*With the same logic, Green's-f for below BV-P [7]*

$$x^{\Delta\Delta}(t)=0,$$

$$x(a)=0,$$
$$x\left(\sigma^2(b)\right)=0$$

*is*

$$G(t,s)=\begin{cases} -\frac{(t-a)\left(\sigma^2(b)-\sigma(s)\right)}{\sigma^2(b)-a}, & if \quad t\le s,\\ -\frac{(\sigma(s)-a)\left(\sigma^2(b)-t\right)}{\sigma^2(b)-a}, & if \quad t\ge \sigma(s). \end{cases}$$

Following examples show that Green's-f has different forms when time scale changes.

**Example 1.** Let's solve following BV-P with the help of Green's-f.

$$x^{\Delta\Delta}=1, \tag{7}$$

$$x(2)=0=x(64). \tag{8}$$

When problem (7), (8) is considered on $\mathbb{T}=\mathbb{Z}$, corresponding Green's-f and the solution take the following form. Green's-f corresponding to this problem is given by

$$G(t,s)=\begin{cases}\dfrac{(t-2)(s-63)}{62}, & \text{if } t\leq s,\\ \dfrac{(s-1)(t-64)}{62}, & \text{if } t\geq s+1.\end{cases}$$

Using the Green's-f representation, solution can be written as

$$x(t)=\int_2^{64} G(t,s)1\,\Delta s=$$

$$=\int_2^t \frac{(s-1)(t-64)}{62}\Delta s+\int_t^{64}\frac{(t-2)(s-63)}{62}\Delta s=$$

$$=\frac{t-64}{62}\sum_{s=2}^{t-1}(s-1)+\frac{t-2}{62}\sum_{s=t}^{63}(s-63)=\frac{(t-64)(t-2)}{2}.$$

Let us now compute Green's-f and corresponding solution of problem (7), (8) for $\mathbb{T}=2^{\mathbb{N}_0}$ and $\mathbb{T}=\mathbb{R}$.

Green's-f for $\mathbb{T}=2^{\mathbb{N}_0}$ is

$$G(t,s)=\begin{cases}\frac{(2-t)(64-2s)}{62}, & \text{if } \quad t\leq s,\\ \\ \frac{(2s-2)(t-64)}{62}, & \text{if } \quad t\geq\sigma(s),\end{cases}$$

and the solution is given by

$$x(t)=\int_2^{64}G(t,s)h(s)\,\Delta s=\frac{(t-64)}{62}\sum_{s\in[2,t)} s\,(2s-2)+\frac{(2-t)}{62}\sum_{s\in[t,64)} s\,(64-2s)\,.$$

Also, we can construct Green's-f for (7), (8) as

$$G(t,s)=\begin{cases}\frac{(t-2)(s-64)}{62}, & \text{if } \quad t\leq s,\\ \\ \frac{(s-2)(t-64)}{62}, & \text{if } \quad s\leq t,\end{cases}$$

for $\mathbb{T}=\mathbb{R}$. Therefore, solution of the problem (7), (8) is

$$x(t)=\int_2^t\frac{(s-2)(t-64)}{62}\,ds+\int_t^{64}\frac{(t-2)(s-64)}{62}\,ds=\frac{(t-64)(t-2)}{2}.$$

**Example 2.** Consider

$$(3^t x^{\Delta})^{\Delta}=2^t, \tag{9}$$

$$x(2)=0=x(16), \tag{10}$$

for $\mathbb{T}=\mathbb{Z}$. Let us firstly construct Green's-f for (9), (10) as

$$G(t,s)=\begin{cases}-\frac{\left(1-\frac{1}{3^{t-2}}\right)\left(\frac{1}{2\cdot 3^{s+1}}\left(1-\frac{1}{3^{15-s}}\right)\right)}{1-\frac{1}{3^{14}}}, & \text{if} \quad t\le s,\\ -\frac{\left(1-\frac{1}{3^{s-1}}\right)\left(\frac{1}{2\cdot 3^{t}}\left(1-\frac{1}{3^{16-t}}\right)\right)}{1-\frac{1}{3^{14}}}, & \text{if} \quad t\ge \sigma(s).\end{cases}$$

With the help of this function on $\mathbb{T}=\mathbb{Z}$, solution of given problem (9), (10) is obtained as follows:

$$x(t)=-\frac{\left(\frac{1}{2\cdot 3^{t}}\left(1-\frac{1}{3^{16-t}}\right)\right)}{1-\frac{1}{3^{14}}}\sum_{s=2}^{t-1}2s\left(1-\frac{1}{3^{s-1}}\right)+-\frac{\left(1-\frac{1}{3^{t-2}}\right)}{1-\frac{1}{3^{14}}}\sum_{s=t}^{15}2s\left(\frac{1}{2\cdot 3^{s+1}}\left(1-\frac{1}{3^{15-s}}\right)\right).$$

For $\mathbb{T}=2^{\mathbb{N}_0}$, Green's-f of problem (9), (10) is obtained in the form of

$$G(t,s)=\begin{cases}-\frac{\sum\limits_{\tau\in[2,t)}\frac{\tau}{3^\tau}\sum\limits_{\tau\in[2s,16)}\frac{\tau}{3^\tau}}{\sum\limits_{\tau\in[2,16)}\frac{\tau}{3^\tau}}, & \text{if} \quad t\le s,\\ -\frac{\sum\limits_{\tau\in[2,2s)}\frac{\tau}{3^\tau}\sum\limits_{\tau\in[t,16)}\frac{\tau}{3^\tau}}{\sum\limits_{\tau\in[2,16)}\frac{\tau}{3^\tau}}, & \text{if} \quad t\ge 2s.\end{cases}$$

Accordingly, the solution $x(t)$ is derived as follows

$$x(t)=-\frac{\sum\limits_{\tau\in[t,16)}\frac{\tau}{3^\tau}}{\sum\limits_{\tau\in[2,16)}\frac{\tau}{3^\tau}}\sum_{s\in[2,t)}2s^2\left[\sum_{\tau\in[2,2s)}\frac{\tau}{3^\tau}\right]-\frac{\sum\limits_{\tau\in[2,t)}\frac{\tau}{3^\tau}}{\sum\limits_{\tau\in[2,16)}\frac{\tau}{3^\tau}}\sum_{s\in[t,16)}2s^2\left[\sum_{\tau\in[2s,16)}\frac{\tau}{3^\tau}\right].$$

Since the solution is obtained in a closed form, it can be computed from equation (4) when a specific value is assigned to the parameter $t$.

For $\mathbb{T}=\mathbb{R}$, the Green's-f is given by

$$G(t,s)=\begin{cases}-\frac{1}{\ln 3}\frac{\left(\frac{1}{9}-\frac{1}{3^t}\right)\left(\frac{1}{3^s}-\frac{1}{3^{16}}\right)}{\left(\frac{1}{9}-\frac{1}{3^{16}}\right)}, & \text{if} \quad t\le s,\\ -\frac{1}{\ln 3}\frac{\left(\frac{1}{9}-\frac{1}{3^s}\right)\left(\frac{1}{3^t}-\frac{1}{3^{16}}\right)}{\left(\frac{1}{9}-\frac{1}{3^{16}}\right)}, & \text{if} \quad t\ge s.\end{cases}$$

From this, solution function is obtained using formula (4):

$$\begin{aligned}x(t)&=-\frac{1}{\ln 3}\frac{\left(\frac{1}{3^t}-\frac{1}{3^{16}}\right)}{\left(\frac{1}{9}-\frac{1}{3^{16}}\right)}\int_2^t 2s\left(\frac{1}{9}-\frac{1}{3^s}\right)ds-\frac{1}{\ln 3}\frac{\left(\frac{1}{9}-\frac{1}{3^t}\right)}{\left(\frac{1}{9}-\frac{1}{3^{16}}\right)}\int_t^{16}2s\left(\frac{1}{3^s}-\frac{1}{3^{16}}\right)ds=\\
&=-\frac{1}{\ln 3}\frac{\left(\frac{1}{3^t}-\frac{1}{3^{16}}\right)}{\left(\frac{1}{9}-\frac{1}{3^{16}}\right)}\left(\frac{t^2-4}{9}+\frac{2t\cdot 3^{-t}}{\ln 3}+\frac{2\cdot 3^{-t}}{(\ln 3)^2}-\left(\frac{4\cdot 3^{-2}}{\ln 3}+\frac{2\cdot 3^{-2}}{(\ln 3)^2}\right)\right)-\\
&-\frac{1}{\ln 3}\frac{\left(\frac{1}{9}-\frac{1}{3^t}\right)}{\left(\frac{1}{9}-\frac{1}{3^{16}}\right)}\left(2\left[\left(\frac{-16\cdot 3^{-16}}{\ln 3}-\frac{3^{-16}}{(\ln 3)^2}\right)-\left(\frac{-t\cdot 3^{-t}}{\ln 3}-\frac{3^{-t}}{(\ln 3)^2}\right)-\frac{1}{3^{16}}\cdot\frac{256-t^2}{2}\right]\right).\end{aligned}$$

Now, let us consider the structure of Green's-f for a different type of problem on time scales.

**Theorem 4.** *[8] Consider*

$$u^{\Delta\Delta}(t) + f(u(\sigma(t))) = 0, \quad t \in [0,1], \tag{11}$$

*subject to the boundary conditions*

$$\alpha y(0) - \beta y^{\Delta}(0) = 0, \quad \gamma y(\sigma(1)) + \delta y^{\Delta}(\sigma(1)) = 0, \tag{12}$$

*where* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous and* $d = \gamma\beta + \alpha\delta + \alpha\gamma(\sigma(1)) > 0$*. To construct corresponding Green's function* $G(t,s)$*, consider the associated homogeneous problem*

$$-y^{\Delta\Delta}(t) = 0,$$

*with the same boundary conditions*

$$\alpha y(0) - \beta y^{\Delta}(0) = 0, \quad \gamma y(\sigma(1)) + \delta y^{\Delta}(\sigma(1)) = 0.$$

*Green's function is given by*

$$G(t,s) = \begin{cases} \frac{1}{d}(\alpha t + \beta)(\gamma(\sigma(1) - \sigma(s)) + \delta), & \text{if } t \le s, \\ \\ \frac{1}{d}(\alpha\sigma(s) + \beta)(\gamma(\sigma(1) - t) + \delta), & \text{if } \sigma(s) \le t, \end{cases}$$

*where* $d = \gamma\beta + \alpha\delta + \alpha\gamma(\sigma(1)) > 0$*.*

*The function* $u(t)$ *satisfies equations* (11) *and* (12) *if and only if it can be represented by the following integral equation:*

$$u(t) = \int_0^{\sigma(1)} G(t,s) f(u(\sigma(s)))\, \Delta s, \quad t \in [0, \sigma^2(1)].$$

**Example 3.** For the problem (11), (12), we set the Green's function by choosing the constants as $\alpha = \beta = \gamma = \delta = 1$ for $\mathbb{T} = \mathbb{R}$, and proceed to solve the problem accordingly. Since $\sigma(s) = s$, it yields $d = 3 > 0$ and $\gamma(\sigma(1) - \sigma(s)) = 1$. So, Green's-f for the given problem (11), (12) is

$$G(t,s) = \begin{cases} \frac{1}{3}(t+1)(2-s), \text{ if } & t \le s, \\ \\ \frac{1}{3}(s+1)(2-t), \text{ if } & s \le t. \end{cases}$$

On the other hand, we get

$$u(t) = \frac{2-t}{3} \int_0^t (s+1) f(u(s)) ds + \frac{t+1}{3} \int_t^{\sigma(1)} (2-s) f(u(s)) ds.$$

If $\mathbb{T}$ =$\mathbb{Z}$, we get $d = 4$, $\sigma(s) = s + 1$ and

$$G(t,s)=\begin{cases}\frac{1}{4}(t+1)(2-s), \text{ if } & t\le s,\\ \\ \frac{1}{4}(s+2)(3-t), \text{ if } & s+1\le t.\end{cases}$$

If the similar process is repeated, we get as follows

$$u(t)=\frac{3-t}{4}\sum_{s=0}^{t-1}(s+2)f(u(s+1))+\frac{t+1}{4}\sum_{s=t}^{\sigma(1)-1}(2-s)f(u(s+1)).$$

Finally, if $\mathbb{T}=2^{\mathbb{N}_0}$, we get $d=4$, $\sigma(s)=2s$ and

$$G(t,s)=\begin{cases}\frac{1}{4}(t+1)(3-2s), \text{ if } & t\le s,\\ \\ \frac{1}{4}(2s+1)(3-t), \text{ if } & 2s\le t.\end{cases}$$

In this way, er get

$$u(t)=\frac{3-t}{4}\sum_{s\in[0,t)}2s(2s+1)f(u(2s))+\frac{t+1}{4}\sum_{s\in[t,\sigma(1))}2s(3-2s)f(u(2s)).$$

As demonstrated in this example, the Green's-f can be effectively utilized to obtain solutions under various parameter choices and across different time scales.

Now let's express the Green's-f for nonlinear type problems. As known, nonlinear problems are very difficult to solve in classical case. However, Green's-f method is an effective method for such problems where many methods do not work.

## 3. Some Results for Non-Linear Type BV-P's on $\mathbb{T}$

In this section, let's explain how to construct the Green's-f for some special nonlinear type eigenvalue problems on $\mathbb{T}$. Some interesting problems will be tried to be solved with the help of Green's-f, taking into account the theorems and lemmas stated in the previous section.

**Theorem 5.** *[8] Consider*

$$y(t)=y^{\Delta\Delta}(t)+\lambda_1 p(t)y^{\sigma}(t)=0, \tag{13}$$
$$y(t)=y^{\Delta\Delta}(t)+\lambda_2 q(t)y^{\sigma}(t)=0,$$

*where $t\in[0,1]$ with conditions*

$$y(0)=0=y(\sigma^2(1)), \tag{14}$$

*for $0<p(t)\le q(t)$ and $p,q\in C_{rd}$. Now, consider the linear operators $L_1, L_2:\mathcal{B}\to\mathcal{B}$ by*

$$L_1x(t)=\int_0^{\sigma(1)}G(t,s)p(s)x(\sigma(s))\Delta s, \tag{15}$$

$$L_2x(t) = \int_0^{\sigma(1)} G(t,s)q(s)x(\sigma(s))\Delta s,$$

*where* $G(t,s)$ *is Green's-f for* $-x^{\Delta\Delta} = 0$ *which satisfy* (14) *and* $\mathcal{B}$ *be a Banach space defined by*

$$\mathcal{B} = \left\{x : \left[0,\sigma^2(1)\right] \to \mathbb{R};\quad x^{\Delta} \text{ exists and bounded on } [0,\sigma(1)] \text{ and } x \text{ satisfies } (14)\right\}.$$

*That is,*

$$G(t,s) = \begin{cases} \frac{t\left(\sigma^2(1)-\sigma(s)\right)}{\sigma^2(1)}, & \text{if} \quad 0 \le t \le s \le \sigma(1), \\ \frac{\sigma(s)\left(\sigma^2(1)-t\right)}{\sigma^2(1)}, & \text{if} \quad 0 \le \sigma(s) \le t \le \sigma^2(1), \end{cases}$$

*on* $\left[0,\sigma^2(1)\right] \times [0,\sigma(1)]$. *Here,* $G(t,s) > 0$ *on* $\left(0,\sigma^2(1)\right) \times (0,\sigma(1))$.

**Lemma 2.** *[8] Let* $\lambda_1$ *be an eigenvalue of* (13), (14) *and let* $u(t)$ *be corresponding eigenfunction. Then, we get*

$$u(t) = \lambda_1 \int_0^{\sigma(1)} G(t,s)p(s)u(\sigma(s))\Delta s.$$

*That is,* $\frac{1}{\lambda_1}u(t) = L_1u(t)$. *It means that eigenvalues of* (13), (14) *are reciprocals of the eigenvalues of* (15) *and conversely.*

**Example 4.** Consider BV-P as

$$y^{\Delta\Delta}(t) + \lambda_1 p(t)y^{\sigma}(t) = 0,\ t \in [0,1],$$

$$y(0) = 0 = y(\sigma^2(1)).$$

Let us get Green's-f and solution for given problem. If the methods in previous section are used, we get

$$G(t,s) = \begin{cases} \frac{t\left(\sigma^2(1)-\sigma(s)\right)}{\sigma^2(1)}, & \text{if} \quad 0 \le t \le s \le \sigma(1), \\ \\ \frac{\sigma(s)\left(\sigma^2(1)-t\right)}{\sigma^2(1)}, & \text{if} \quad 0 \le \sigma(s) \le t \le \sigma^2(1), \end{cases}$$

and

$$u(t) = \lambda_1 \int_0^{\sigma(1)} G(t,s)p(s)u(\sigma(s))\Delta s.$$

In some special cases as $\mathbb{T} = \mathbb{R}$ and $p(s) = 1$, we get

$$G(t,s) = \begin{cases} t(1-s), & \text{if} \quad 0 \le t \le s \le \sigma(1), \\ s(1-t), & \text{if} \quad 0 \le \sigma(s) \le t \le \sigma^2(1), \end{cases}$$

and

$$u(t)=\lambda_1(1-t)\int_0^t su(s)ds+\lambda_1 t\int_t^1 (1-s)u(s)ds.$$

For $\mathbb{T}=\mathbb{Z}$, we obtain

$$G(t,s)=\begin{cases}\frac{t(4-s)}{4}, & \text{if} \quad 0\le t\le s\le \sigma(1),\\ \\ \frac{(s+1)(4-t)}{4}, & \text{if} \quad 0\le \sigma(s)\le t\le \sigma^2(1),\end{cases}$$

and

$$u(t)=\lambda_1\frac{(4-t)}{4}\sum_{s=0}^{t-1}(s+1)u(s+1)ds+\lambda_1\frac{t}{4}\sum_{s=t}^{\sigma(1)-1}(4-s)u(s+1)ds.$$

Finally, for $\mathbb{T}=2^{\mathbb{N}_0}$, the following computations can be carried out:

$$G(t,s)=\begin{cases}\frac{t\left(4-4s^2\right)}{4}, & \text{if} \quad 0\le t\le s\le \sigma(1),\\ \frac{2s(4-t)}{4}, & \text{if} \quad 0\le \sigma(s)\le t\le \sigma^2(1),\end{cases}$$

and

$$u(t)=\lambda_1\frac{(4-t)}{4}\sum_{s=0}^{t-1}2su(2s)ds+\lambda_1\frac{t}{4}\sum_{s=t}^{\sigma(1)-1}(4-4s^2)u(2s)ds.$$

**Theorem 6.** *[8],[7] Consider the non-linear equation*

$$Lx(t)=f\left(t,x^{\sigma}(t)\right)=-\left[r(t)x^{\Delta}(t)\right]^{\Delta}.$$

*Let's deal with generating a Green's-f for problems involving these types of equations. Suppose that below BV-P*

$$Mx(t)=-\left[r(t)x^{\Delta}(t)\right]^{\Delta}+q(t)x\left(\sigma(t)\right)=0,$$

$$\alpha x(a)-\beta x^{\Delta}(a)=0,$$

$$\gamma x\left(\sigma(b)\right)+\delta x^{\Delta}\left(\sigma(b)\right)=0,$$

*has only trivial solution. Here, $M:\mathbb{D}_2\to C_{rd}$ be an operator where $\mathbb{D}_2$ denotes the set of all functions $x:\mathbb{T}\to\mathbb{R}$ for which the delta derivative $x^{\Delta}(t)$ is continuous on $\mathbb{T}$ and the expression $\left(r(t)x^{\Delta}(t)\right)^{\Delta}$ is right-dense continuous on $\mathbb{T}$. For each fixed $s\in[a,b]$, assume that $u(t,s)$ is unique solution of BV-P*

$$Mu(t,s)=0,$$

$$\alpha u(a,s)-\beta u^{\Delta}(a,s)=0,$$

$$\gamma u\left(\sigma(b),s\right)+\delta u^{\Delta}\left(\sigma(b),s\right)=-\gamma x\left(\sigma(b),s\right)-\delta x^{\Delta}\left(\sigma(b),s\right),$$

*where $x(t,s)$ is a Cauchy function for $Mx = 0$. Then,*

$$G(t,s) = \begin{cases} u(t,s), \textit{if} \quad t \leq s, \\ u(t,s) + x(t,s), \ \textit{if}\ \sigma(s) \leq t, \end{cases}$$

*is Green's-f for below BV-P:*

$$Lx(t) = 0,$$

$$\alpha x(a,s) - \beta x^{\Delta}(a,s) = 0,$$

$$\gamma x(\sigma(b),s) + \delta x^{\Delta}(\sigma(b),s) = 0.$$

**Lemma 3.** *[8] The number*

$$d = \frac{\gamma\beta}{r(a)} + \frac{\alpha\delta}{r(\sigma(b))} + \alpha\gamma \int_a^{\sigma(b)} \frac{1}{r(\tau)} \Delta\tau,$$

*is different from zero iff BV-P*

$$-\left[r(t)x^{\Delta}(t)\right]^{\Delta} = 0, \tag{16}$$

$$\alpha x(a,s) - \beta x^{\Delta}(a,s) = 0, \tag{17}$$

$$\gamma x(\sigma(b),s) + \delta x^{\Delta}(\sigma(b),s) = 0, \tag{18}$$

*has only trivial solution. If $d \neq 0$, Green's-f for (16)-(18) is given by*

$$G(t,s) = \begin{cases} \frac{1}{d} u(t) v(\sigma(s)), \textit{if} \quad t \leq s, \\ \\ \frac{1}{d} u(\sigma(s)) v(t), \textit{if} \quad \sigma(s) \leq t, \end{cases}$$

*where*

$$u(t) = \alpha \int_a^t \frac{1}{r(\tau)} \Delta\tau + \frac{\beta}{r(a)},$$

$$v(t) = \gamma \int_t^{\sigma(b)} \frac{1}{r(\tau)} \Delta\tau + \frac{\delta}{r(\sigma(b))}.$$

**Example 5.** Consider BV-P as

$$-\left[t^2 y^{\Delta}(t)\right]^{\Delta} = 0,$$

$$\alpha y(a,s) - \beta y^{\Delta}(a,s) = 0,$$

$$\gamma y(\sigma(b),s) + \delta y^{\Delta}(\sigma(b),s) = 0.$$

Let's create Green's-f on different time scales for arbitrary selection of constants as $\alpha = \beta = \gamma = \delta = 1$ and $a = 1$, $b = 2$.

Firstly, let $\mathbb{T}=\mathbb{R}$. In this case, we get

$$\begin{aligned} d &= \frac{7}{4}, \\ u(t) &= 2-\frac{1}{t}, \\ v(t) &= \frac{1}{t}-\frac{1}{2}, \end{aligned}$$

and

$$G(t,s)=\begin{cases} \frac{4}{7}(\frac{2t-1}{t})(\frac{2-s}{2s}), & \text{if} \quad t\le s, \\ \frac{4}{7}(\frac{2-t}{2t})(\frac{-1+2s}{s}), & \text{if} \quad s\le t. \end{cases}$$

Secondly, if $\mathbb{T}=\mathbb{Z}$, it yields

$$\begin{aligned} d &= \frac{85}{36}, \\ u(t) &= 1+\sum_{k=1}^{t-1}\frac{1}{k^2}, \\ v(t) &= \frac{1}{9}+\sum_{k=t}^{2}\frac{1}{k^2}, \end{aligned}$$

and

$$G(t,s)=\begin{cases} \frac{36}{85}\left(1+\sum\limits_{k=1}^{t-1}\frac{1}{k^2}\right)\left(\frac{1}{9}+\sum\limits_{k=s+1}^{2}\frac{1}{k^2}\right), & \text{if} \quad t\le s, \\ \frac{36}{85}\left(\frac{1}{9}+\sum\limits_{k=t}^{2}\frac{1}{k^2}\right)\left(1+\sum\limits_{k=1}^{s}\frac{1}{k^2}\right), & \text{if} \quad s\le t. \end{cases}$$

Finally, when $\mathbb{T}=2^{\mathbb{N}_0}$, Green's-f and the related computations are given as follows:

$$\begin{aligned} d &= \frac{41}{16}, \\ u(t) &= 1+\sum_{k\in[1,t)}\frac{1}{k}, \\ v(t) &= \frac{1}{16}+\sum_{k\in[t,4)}\frac{1}{k^2}, \end{aligned}$$

and

$$G(t,s)=\begin{cases} \frac{16}{41}\left(1+\sum_{k\in[1,t)}\frac{1}{k}\right)\left(\frac{1}{16}+\sum_{k\in[2s,4)}\frac{1}{k^2}\right), & \text{if} \quad t\le s, \\ \frac{16}{41}\left(1+\sum_{s\in[1,2s)}\frac{1}{k}\right)\left(\frac{1}{16}+\sum_{k\in[t,4)}\frac{1}{k^2}\right), & \text{if} \quad s\le t. \end{cases}$$

This example clearly shows how solutions change as the time scale changes.

# 4. Conclusion

How effective Green's-f is important to solve BV-P's is known for classical case. Here, this method is tested by using different types of samples on time scales. As the time scale changes, Green's-f changes according to nature of the problem. As a result, Green's-f has proven its effectiveness when applied to difficult-type problems on time scales. Previously known results are reinforced by interesting examples.

**Acknowledgement** The authors are very grateful to the reviewers for their useful comments.

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