Baku Mathematical Journal
2025, Vol. 4, No 1, PP. 120-132
https://doi.org/10.32010/j.bmj.2025.09

ON THE STOCHASTIC PROCESS WITH LIGHT AND HEAVY-TAILED GENERAL INTERFERENCE OF CHANCE

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Received: 09.01.2025 / Revised: 21.02.2025 / Accepted: 28.02.2025

Abstract. *In this study the stochastic process which describe so called probabilistic inventory control model is considered. The explicit formula for the* n^{th} *order moments of ergodic distribution of this process is obtained and the behavior of the ergodic distribution investigated, when the distribution of the interference of chance belongs to class of light-tailed distributions. The case, when the distribution of the interference of chance belongs to class of heavy-tailed distributions is also examined.*

Keywords: inventory control, ergodic distribution, light-tailed distribution, heavy-tailed distribution, overshoot, equilibrium distribution

Mathematics Subject Classification (2020): 60H30, 60K05, 60K10

1. Introduction

Stochastic processes with discrete interference of chance plays a key role in modeling real-world systems which affected by random disruptions. These processes are used in areas such as queueing theory, reliability theory and financial modeling, where specific random variables impact system behavior at specific times.

Several studies have examined these processes. In 2000 Levy and Taqqu [11] examined renewal reward processes with heavy-tailed interrenewal times and heavy-tailed rewards, contributing to the understanding of such stochastic processes in probability theory. Khaniyev et al. in 2008 [10] analyzed cases with exponential interference of chance, while Aliyev et al. in 2009 [5] expanded the study to gamma distributed interference of chance.

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Geluk and Frenk in 2011 [8] studied renewal theory for random variables with a heavy-tailed distribution and finite variance, providing insights into their statistical properties. Aliyev in 2011 [1] investigates a semi-Markov process with a component following a regularly varying tail distribution, examining its asymptotic properties. Also, in 2017 Aliyev [3] applies similar stochastic modeling techniques to an (s, S) inventory system, where demand is assumed to follow a heavy-tailed distribution.

While previous research has primarily focused on asymptotic approximations and moment expansions, our work differs by providing an exact formula for the ergodic distribution function, which includes the well-known equilibrium distribution. Additionally, we derive precise formulas for moments of ergodic distribution and analyze how different distributions of stock level, both light-tailed and heavy-tailed influence the system's ergodic distribution. This approach deepens understanding and practical applications of stochastic models with discrete interference of chance. By establishing explicit analytical results, our study expands the theoretical framework and practical applicability of stochastic process with discrete interference of chance.

2. Construction of Process

Let $\{\xi_n\}$, $\{\eta_n\}$, $\{\theta_n\}$ and $\{\zeta_n\}$, $n \geq 1$ are sequences of random variables defined on same probability space $(\Omega, \mathcal{F}, P)$, such that variables in each sequence independent and identically distributed. Suppose that ξ_n, η_n, θ_n and ζ_n can take only positive values and these distribution functions be denoted by

$$\Phi(t) = P\{\xi_1 \leq t\}, t > 0, F(x) = P\{\eta_1 \leq x\}, \ x > 0,$$

$$H(u) = P\{\theta_1 \leq u\}, u > 0, \ \pi(z) = P\{\zeta_1 \leq z\}, \ z > 0.$$

Define independent renewal sequence $\{T_n\}$ and $\{Y_n\}$, $n \geq 1$ as follows using the initial sequences of the random variables $\{\xi_n\}$ and $\{\eta_n\}$, $n \geq 1$ as:

$$T_n = \sum_{i=1}^{n} \xi_i, \quad Y_n = \sum_{i=1}^{n} \eta_i, \quad n = 1, 2, \ \ldots; \quad T_0 = Y_0 = 0.$$

Define also sequence of integer valued random variables:

$$N_0 = 0 \ ; \ N_1 = N(z) = \min\{k \geq 1 : z - Y_k < 0\},$$

$$N_n \equiv N_n(\zeta_{n-1}) = \min\left\{k \geq N_{n-1} + 1 : \zeta_{n-1} - \left(Y_k - Y_{N_{n-1}}\right) < 0\right\}, \quad n = 2, 3, \ldots \ .$$

Let the random variables τ_n represents the n^{th} time of the process drops below the level 0 and γ_n represents the n^{th} moment of exit from the level 0:

$$\tau_0 = 0, \ \tau_1 = T_{N_1}, \ \gamma_0 = 0, \ \gamma_1 = \tau_1 + \theta_1,$$

$$\ldots$$

$$\tau_n = \gamma_{n-1} + T_{N_n} - T_{N_{n-1}}, \quad \gamma_n = \tau_n + \theta_n, \ n = 1, 2, \ldots \ .$$

Define also the counting process $\nu(t)$ which describes the number of jumps of the process $X(t)$ in the interval $[0,\ t]$:

$$\nu(t) = max\{k \geq 0 : \ T_k \leq t\}.$$

Thus the following stochastic process can be constructed using these notations:

$$X(t) = \max\left\{0, \zeta_n - Y_{\nu(t)} + Y_{N_n}\right\}, \quad \gamma_n \leq t < \gamma_{n+1}, \quad n = 0, 1, 2, \ldots .$$

Here $X(0) \equiv \zeta_0 = z$.

One of the trajectories of the process $X(t)$ is given in the following picture:

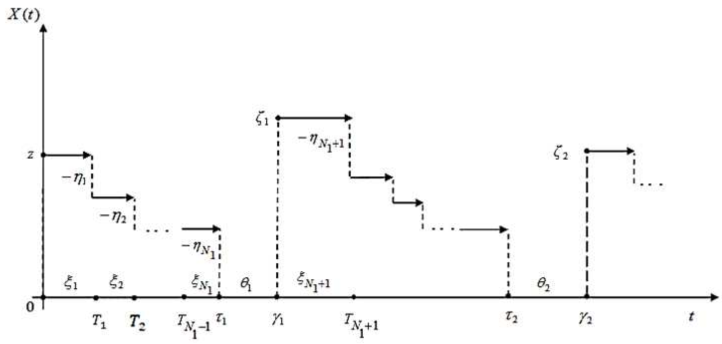

Fig. 1. One of the trajectories of the process $X(t)$

3. Ergodic Distribution of Process with General Interference of Chance and Exponential Demands and n^{th} Order Moments of Ergodic Distribution

Firstly, let's introduce the following general ergodic theorem for considered process.

Proposition. *Let initial sequence $\{\xi_n\}, \{\eta_n\}, \{\theta_n\}$ and $\{\zeta_n\}$, $n \geq 1$ – satisfies the following supplementary conditions:*

1) $E\xi_1 < \infty$;

2) $E\theta_1 < \infty$;

3) $E\zeta_1 < \infty$.

Then the process $X(t)$ is ergodic and ergodic distribution function has the following form:

$$Q_X(x) = 1 - \frac{EU(\zeta - x)}{EU(\zeta) + K}, \quad x \geq 0, \tag{1}$$

where $EU(\zeta) = \int_0^\infty U(z) d\pi(z)$, $K = \frac{E\theta_1}{E\xi_1}$ and $U(t) = \sum\limits_{n=0}^{\infty} F^{*n}(t)$ renewal function generated by random variable η_1.

Proof. Considered process belongs to a wide class of the processes which is called as "Processes with a discrete interference of chance" in literature. For this class, the general ergodic theorem is given in monograph Gihman and Skorohod (1975). Conditions 1)-3) of this theorem provide the fulfillment of the conditions of the general ergodic theorem. ◄

Theorem 1. *Let us the conditions of the Proposition be satisfied and also assume that random variable* η_1 *has the exponential distribution with parameter* $\mu > 0$*. Then ergodic distribution function of the process* $X(t)$ *has the following explicit form:*

$$Q_X(x) = C_1 R(x) - C_2 (1 - \pi(x)) + 1 - C_1, \tag{2}$$

where $R(x) = \frac{1}{E\zeta_1} \int_0^x (1 - \pi(z))\, dz$*- equilibrium distribution function (sometimes called the residual life distribution) associated with* $\pi(x)$*,* $K = \frac{E\theta_1}{E\xi_1}$ *is delay coefficient and*

$$C_1 = \frac{\mu E\zeta_1}{1 + \mu E\zeta_1 + K}, C_2 = \frac{1}{1 + \mu E\zeta_1 + K}.$$

Proof. It is easy to see that in case random variable η_1 has exponential distribution with parameter $\mu > 0$, renewal function $U(t)$ generated by random variable η_1 has the following form (see, for example, Gut [9])

$$U(t) = \mu t + 1.$$

This formula can be obtained from following renewal integral equation (see, for example, Feller [6]):

$$U(t) = F(x) + \int_0^t U(t - s) dF(s).$$

Consequently,

$$EU(\zeta - x) = \int_x^\infty U(z - x) d\pi(z) = \int_x^\infty (\mu(z - x) + 1)\, d\pi(z) =$$

$$= \mu \int_x^\infty (z - x) d\pi(z) + \int_x^\infty d\pi(z) = \mu \int_x^\infty (1 - \pi(z))\, dz + (1 - \pi(x)) =$$

$$= \mu \left(\int_0^\infty (1 - \pi(z))\, dz - \int_0^x (1 - \pi(z))\, dz \right) + (1 - \pi(z)) =$$

$$= \mu E\zeta_1 (1 - R(x)) + (1 - \pi(x)),$$

where $R(x) = \frac{1}{E\zeta_1} \int_0^x (1 - \pi(z))\, dz$.

It is not difficult to obtain that

$$EU(\zeta) = \int_0^\infty U(z) d\pi(z) = \int_0^\infty (\mu z + 1) d\pi(z) =$$

$$= \mu \int_0^\infty z d\pi(z) + \int_0^\infty d\pi(z) = 1 + \mu E\zeta_1.$$

Using these expressions, from (1) we can find that

$$Q_X(x) = 1 - \frac{1}{1+\mu E\zeta_1 + K}\left(\mu E\zeta_1\left(1-R(x)\right) + \left(1-\pi(x)\right)\right) =$$

$$= \frac{\mu E\zeta_1}{1+\mu E\zeta_1 + K} R(x) - \frac{1-\pi(x)}{1+\mu E\zeta_1 + K} + \left(1 - \frac{\mu E\zeta_1}{1+\mu E\zeta_1 + K}\right).$$

Theorem 1 is proved. ◄

Remark 1. The study of overshoot distributions plays an important role in probability theory. First in 1964 Rogozin [12] obtained the distribution of the overshoot for random walks. Analogical results are also obtained by proving a Rogozin-type theorem for stochastic processes with discrete interference of chance, particularly in the framework of semi-Markov processes (see, for example, [2], [4]). More recently, in 2025 Wong [13] further clarified the asymptotic behavior of overshoot in random walks, providing new insights into its limiting distribution and structural properties. But in this particular case we have Rogozin-type theorem with explicit distribution formula.

Example 1. Let assume that random variable ζ_1 has Weibull distribution with parameters (α, λ):

$$\pi(x) = 1 - e^{-\lambda x^{\alpha}}, \alpha > 0, \lambda > 0.$$

In this case, its obvious that $E\zeta_1 = \lambda^{-\frac{1}{\alpha}}\ \Gamma\left(1+\frac{1}{\alpha}\right)$ and

$$R(x) = \frac{1}{E\zeta_1}\int_0^x \left(1-\pi(z)\right) dz = \frac{\lambda^{\frac{1}{\alpha}}}{\Gamma\left(1+\frac{1}{\alpha}\right)}\int_0^x e^{-\lambda z^{\alpha}} dz =$$

$$= \frac{1}{\alpha\Gamma\left(1+\frac{1}{\alpha}\right)}\int_0^{\lambda x^{\alpha}} u^{\frac{1}{\alpha}-1} e^{-u} du = \frac{\gamma(\frac{1}{\alpha}, \lambda x^{\alpha})}{\Gamma\left(\frac{1}{\alpha}\right)},$$

here $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$ -is Euler's Gamma function and $\gamma(s,x) = \int_0^x t^{s-1} e^{-t} dt$ lower incomplete Gamma function.

In this case ergodic distribution function according to (2) is

$$Q_X(x) = \frac{\mu\Gamma\left(1+\frac{1}{\alpha}\right)}{\lambda^{\frac{1}{\alpha}} + \mu\Gamma\left(1+\frac{1}{\alpha}\right) + \lambda^{\frac{1}{\alpha}} K}\frac{\gamma\left(\frac{1}{\alpha}, \lambda x^{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)} - \frac{\lambda^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}} + \mu\Gamma\left(1+\frac{1}{\alpha}\right) + K} e^{-\lambda x^{\alpha}} +$$

$$+ \frac{\lambda^{\frac{1}{\alpha}} + \lambda^{\frac{1}{\alpha}} K}{\lambda^{\frac{1}{\alpha}} + \mu\Gamma\left(1+\frac{1}{\alpha}\right) + \lambda^{\frac{1}{\alpha}} K}.$$

Here you can see graph of ergodic distribution function for different values of λ, α, μ and K.

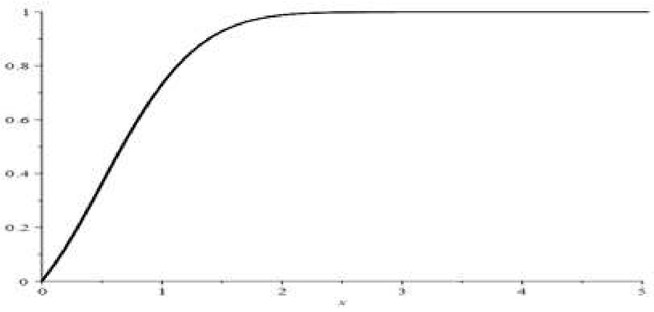

Fig. 2. The grahp of $Q_X(x)$ when $\lambda = 1, \alpha = 2, \mu = 1,\ K = 0$

Lets denote n^{th} order moments of ergodic distribution of considered process as

$$EX^n = \int_0^{+\infty} x^n dQ_X(x).$$

Now we will introduce explicit formula for EX^n.

Theorem 2. *Let us the conditions of the Proposition be satisfied and also assume that random variable η_1 has the exponential distribution with parameter $\mu > 0$. Then n^{th} order moments of ergodic distribution function has the following explicit form:*

$$EX^n = C_1 \frac{E\zeta_1^{n+1}}{(n+1)E\zeta_1} + C_2 E\zeta_1^n, \tag{3}$$

where $C_1 = \frac{\mu E\zeta_1}{1+\mu E\zeta_1 + K}, C_2 = \frac{1}{1+\mu E\zeta_1+K}$.

Proof. According to (2) we can write ergodic distribution function of process as

$$Q_X(x) = C_1 R(x) - C_2 \left(1 - \pi(x)\right) + 1 - C_1.$$

Its not difficult to see that

$$dQ_X(x) = C_1 dR(x) - C_2 d\left(1 - \pi(x)\right).$$

On the other hands

$$dR(x) = d\left(\frac{1}{E\zeta_1}\int_0^x \left(1 - \pi(z)\right) dz\right) = \frac{1}{E\zeta_1}\left(1 - \pi(x)\right) dx.$$

Consequently

$$dQ_X(x) = \frac{C_1}{E\zeta_1}\left(1 - \pi(x)\right) dx - C_2 d\left(1 - \pi(x)\right).$$

Taking into account expression of $dQ_X(x)$ we have:

$$EX^n = \int_0^{+\infty} x^n dQ_X(x) = \frac{C_1}{E\zeta_1}\int_0^{+\infty} x^n \left(1 - \pi(x)\right) dx -$$

$$-C_2\int_0^{+\infty} x^n d\left((1-\pi(x))\right) = \frac{C_1}{E\zeta_1}\int_0^{+\infty} x^n\left((1-\pi(x))\right)dx-$$

$$-C_2\left(x^n\left(1-\pi(x)\right)|_0^{+\infty} - n\int_0^{+\infty} x^{n-1}\left(1-\pi(x)\right)dx\right)=$$

$$=\frac{C_1}{(n+1)E\zeta_1}(n+1)\int_0^{+\infty} x^n\left(1-\pi(x)\right)dx + C_2\left(n\int_0^{+\infty} x^{n-1}\left(1-\pi(x)\right)dx\right)=$$

$$=C_1\frac{E\zeta_1^{n+1}}{(n+1)E\zeta_1}+C_2E\zeta_1^n.$$

Theorem 2 is proved. ◄

Remark 2. Note that term $\frac{E\zeta_1^{n+1}}{(n+1)E\zeta_1}$ in (3) is the n^{th} order moment of equilibrium distribution function $R(x)$.

4. On The Behavior of Ergodic Distribution in the Class Of Light and Heavy-Tailed Distributions

In probability theory and statistics, the concept of light-tailed and heavy-tailed distributions plays an important role in understanding the behavior of distributions, particularly in risk analysis and finance.

Let us introduce following classes of distributions.

Definition 1. *(Foss et.al, 2011, [7]). A distribution F on $\mathbb{R}^+$ is said to be heavy-tailed if and only if for all $t>0$,*

$$\int_0^{+\infty} e^{tx}dF(x)=\infty.$$

Otherwise the distribution F is called light-tailed. Clearly, for any light-tailed distribution F on the positive half-line $\mathbb{R}^+=[0,\infty)$, all moments are finite, that is for all $n>0$,

$$\int_0^{+\infty} x^n dF(x)<\infty.$$

We will denote class of all light-tailed and heavy-tailed distribution functions as respectively $\mathcal{L}$ and $\mathcal{H}$.

Definition 2. *(Foss et.al, 2011, [7]). A function $f \geq 0$ to be heavy-tailed if and only if for all $t>0$,*

$$\limsup_{x\to\infty} f(x)e^{tx}=\infty.$$

Otherwise the function $f\geq 0$ called light-tailed function.

Lemma. *(Foss et.al, 2011, [7]). For any distribution F the following assertions are true:*

(i) F is a heavy-tailed distribution;
(ii) The function $\overline{F}$ is heavy-tailed, where $\overline{F}(x) = 1 - F(x)$.

Remark 3. Note that according to Lemma

(i) F is a light-tailed distribution;
(ii) The function $\overline{F}$ is light-tailed.

Now let us introduce the following subclass of heavy-tailed distribution functions for $n \geq 1$,

$$\mathcal{H}_n = \left\{ F \in \mathcal{H} \middle| \int_0^{+\infty} x^n dF(x) < \infty \right\}.$$

According to well-known Lyapunov inequality (see, for example, Gut [9]) from probability theory it's easy to see that for $n \geq 2$,

$$\mathcal{H}_n \subset \mathcal{H}_{n-1}.$$

For example, $\mathcal{H}_1$ is the subclass of heavy-tailed functions with finite expectation.

Theorem 3. *Lets conditions of the Proposition be satisfied. Then*

(i) If distribution of ζ_1 belongs to class of light-tailed distribution functions, then ergodic distribution function of process $X(t)$ also belongs to class of ligh-tailed distribution functions:

$$\pi \in \mathcal{L} \Rightarrow Q_X \in \mathcal{L}.$$

(ii) If distribution of ζ_1 belongs to class of heavy-tailed distribution functions with finite expectation, then, ergodic distribution function of process $X(t)$ belongs to class of heavy-tailed distribution functions:

$$\pi \in \mathcal{H}_1 \Rightarrow Q_X \in \mathcal{H}.$$

(iii) If distribution of ζ_1 belongs to class of heavy-tailed distribution functions with finite n^{th} moment $(n \geq 2)$, then, ergodic distribution function of process $X(t)$ belongs to class of heavy-tailed distribution functions with finite $(n-1)^{th}$ moment:

$$\pi \in \mathcal{H}_n \Rightarrow Q_X \in \mathcal{H}_{n-1}, \ n \geq 2.$$

Proof. According to (2) tail distribution function of $Q_X(x)$ as,

$$\overline{Q}_X(x) = 1 - Q_X(x) = 1 - (C_1 R(x) - C_2 (1 - \pi(x)) + 1 - C_1) =$$
$$= C_1 (1 - R(x)) + C_2 (1 - \pi(x)) = C_1 \overline{R}(x) + C_2 \overline{\pi}(x). \tag{4}$$

In order to proof first statement of theorem, let us take $\pi \in \mathcal{L}$, in this case according to Lemma for some $t > 0$:

$$\limsup_{x \to \infty} e^{tx} \overline{\pi}(x) < \infty. \tag{5}$$

Now we want to show that

$$\limsup_{x\to\infty} e^{tx}\overline{Q}_X(x) < \infty.$$

In order to do that we need to show if $\pi\in \mathcal{L}$, then $R\in \mathcal{L}$. So we need to show for some $t > 0$:

$$\limsup_{x\to\infty} e^{tx}\overline{R}(x) < \infty.$$

Taking in the account (5) its obvious that there exist such $C > 0$ and $t > 0$ which for all x:

$$\overline{\pi}(x) \le Ce^{-tx}.$$

Therefore

$$\overline{R}(x) = \frac{1}{E\zeta_1}\int_x^\infty \overline{\pi}(z)dz \le \frac{1}{E\zeta_1}\int_x^\infty Ce^{-tz}dz = \frac{Ce^{-tx}}{tE\zeta_1}.$$

And

$$e^{-tx}\overline{R}(x) \le \frac{C}{tE\zeta_1}.$$

Taking limit from both sides we have that for some $t > 0$:

$$\limsup_{x\to\infty} e^{tx}\overline{R}(x) \le \frac{C}{tE\zeta_1} < \infty. \tag{6}$$

This means $R(x)$ also light tailed. Then taking in the account (5) and (6) in (4) we have for some $t > 0$

$$\limsup_{x\to\infty} e^{tx}\overline{Q}_X(x) = C_1 \limsup_{x\to\infty} e^{tx}\overline{R}(x) + C_2 \limsup_{x\to\infty} e^{tx}\overline{\pi}(x) < \infty.$$

We proved the first statement of theorem.

To proof second statement of the theorem, now lets take $\pi \in \mathcal{H}_1$, then according to Lemma for all $t > 0$:

$$\limsup_{x\to\infty} e^{tx}\overline{\pi}(x) = \infty. \tag{7}$$

In this case our aim is to show that all $t > 0$:

$$\limsup_{x\to\infty} e^{tx}\overline{Q}_X(x) = \infty.$$

In order to do that we need to show if $\pi \in \mathcal{H}_1$, then $R \in \mathcal{H}_1$. So we need to show for all $t > 0$:

$$\limsup_{x\to\infty} e^{tx}\overline{R}(x) = \infty.$$

Taking in the account (7) its obvious that for any $C > 0$ and $t > 0$ there exist such $x_0 > 0$ that:

$$\overline{\pi}(x) > Ce^{-tx},\ x > x_0.$$

Therefore

$$\overline{R}(x) = \frac{1}{E\zeta_1}\int_x^\infty \overline{\pi}(z)dz > \frac{1}{E\zeta_1}\int_x^\infty Ce^{-tz}dz = \frac{Ce^{-tx}}{tE\zeta_1}, x > x_0.$$

And any $C > 0$ and $t > 0$

$$e^{-tx}\overline{R}(x) > \frac{C}{tE\zeta_1}, x > x_0.$$

Taking limit from both sides we have that for any $C > 0$ and $t > 0$:

$$\limsup_{x\to\infty} e^{tx}\overline{R}(x) > \frac{C}{tE\zeta_1}.$$

Because $C > 0$ arbitrary

$$\limsup_{x\to\infty} e^{tx}\overline{R}(x) = \infty\,. \tag{8}$$

This means $R(x)$ also heavy tailed. Then taking in the account (7) and (8) in (4) we have for all $t > 0$

$$\limsup_{x\to\infty} e^{tx}\overline{Q}_X(x) = C_1 \limsup_{x\to\infty} e^{tx}\overline{R}(x) + C_2 \limsup_{x\to\infty} e^{tx}\overline{\pi}(x) = \infty.$$

We proved the second statement of the theorem.

Let us prove the last statement of theorem. The second statement of theorem already shows that if π heavy-tailed then Q_X also heavy-tailed. We only need to determine exact subclass of heavy-tailed distributions that Q_X belongs. And according to (3) for $n \geq 2$, $(n-1)^{th}$ order moment of ergodic distribution exist if and only if when n^{th} moment of π exist. In other word, the condition $\pi \in \mathcal{H}_n$ implies $Q_X \in \mathcal{H}_{n-1}$.

Theorem 3 is proved. ◀

Example 2. Let assume that random variable ζ_1 has has exponential distribution with parameter $\lambda > 0$, which is well-known light tailed distribution ($\pi \in \mathcal{L}$):

$$\pi(x) = 1 - e^{-\lambda x}, x \geq 0.$$

In this case $E\zeta_1 = \frac{1}{\lambda}$ and $R(x) = 1 - e^{-\lambda x}$.

Then according to (2)

$$Q_X(x) = \frac{\mu}{\lambda + \mu + \lambda K}\left(1 - e^{-\lambda x}\right) - \frac{\lambda}{\lambda + \mu + \lambda K}e^{-\mu x} + \frac{\lambda + \lambda K}{\lambda + \mu + \lambda K}.$$

Here we can see graphics of ergodic distribution function for different values of λ, μ and K.

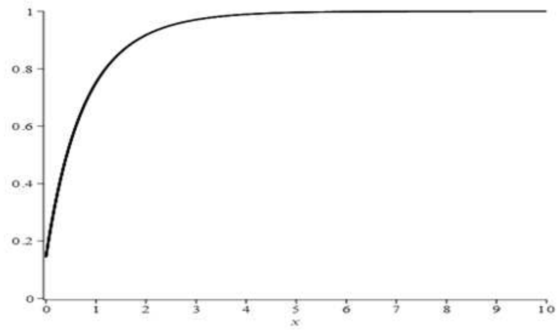

Fig. 3. The grahp of $Q_X(x)$ when $\lambda = 1, \alpha = 2, \mu = 1,\ K = 0$

Example 3. Let assume that random variable ζ_1 has the Pareto Type I distribution with parameters (α, λ), $\alpha > 1$ which is a well-known heavy-tailed distribution:

$$\pi(x) = 1 - \left(\frac{\lambda}{x}\right)^{\alpha}, \alpha > 1, x > \lambda > 0.$$

We consider $\alpha > 1$, because otherwise $E\zeta_1 = \infty$. In case that $\alpha > 1$, $E\zeta_1 = \frac{\alpha\lambda}{\alpha-1} < \infty$, consequently $\pi \in \mathcal{H}_1$ and

$$R(x) = \frac{1}{E\zeta_1}\int_0^x (1-\pi(z))\,dz = \frac{1}{E\zeta_1}\int_0^x (1-\pi(z))\,dz =$$

$$= \frac{1}{E\zeta_1}\left(\int_0^{\lambda} (1-\pi(z))\,dz + \int_{\lambda}^{x} (1-\pi(z))\,dz\right) =$$

$$= \frac{\alpha-1}{\alpha\lambda}\left(\lambda + \lambda^{\alpha}\left(\frac{x^{1-\alpha}}{1-\alpha} - \frac{\lambda^{1-\alpha}}{1-\alpha}\right)\right) = 1 - \frac{1}{\alpha}\left(\frac{\lambda}{x}\right)^{\alpha-1}.$$

Consequently according to (2)

$$Q_X(x) = \frac{\mu\alpha\lambda}{\mu\alpha\lambda + (\alpha-1)(1+K)}\left(1 - \frac{1}{\alpha}\left(\frac{\lambda}{x}\right)^{\alpha-1}\right) -$$

$$-\frac{\alpha-1}{\mu\alpha\lambda + (\alpha-1)(1+K)}\left(\frac{\lambda}{x}\right)^{\alpha} + \frac{(1+K)(\alpha-1)}{\mu\alpha\lambda + (\alpha-1)(1+K)}.$$

Here you can see graph of ergodic distribution function for different values of λ, α, μ and K.

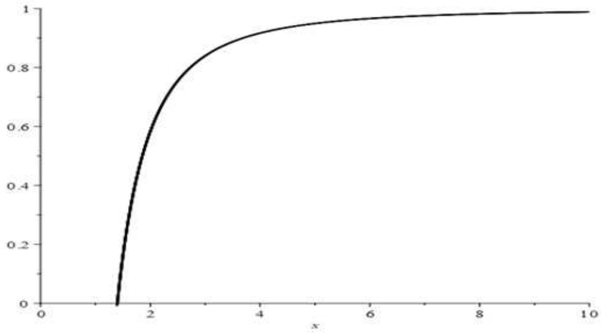

Fig. 4. The grahp of $Q_X(x)$ when $\lambda = 2, \alpha = 3, \mu = 1.5,\ K = 0.5$

5. Conclusion

In this study, we investigate stochastic processes with general discrete interference of chance and provided exact formulations for the ergodic distribution function and n^{th} order moments of ergodic distribution. Our findings extend previous research by considering both light-tailed and heavy-tailed interference distributions and analyzing their effects on behavior of ergodic distribution of process. These results contribute to a deeper understanding of stochastic models and their practical applications in various fields, including reliability theory and financial modeling. Future research can build on this work by exploring additional interference distributions and their impact on ergodicity of process.

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