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# $B$-RIESZ POTENTIAL IN THE LOCAL COMPLEMENTARY GENERALIZED $B$-MORREY SPACES

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**Abstract.** *In this paper we consider the Riesz potential* $I_{\alpha,\gamma}$ *(B-Riesz potential), associated with the Laplace-Bessel differential operator* $\Delta_B = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum\limits_{i=1}^{k} \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$. *We prove that the B-Riesz potential* $I_{\alpha,\gamma}$ *is bounded from the local complementary"generalized B-Morrey space* ${}^{\complement}\mathcal{M}_{\{0\}}^{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ *to* ${}^{\complement}\mathcal{M}_{\{0\}}^{q,\omega_2,\gamma}(\mathbb{R}^n_{k,+})$, *where* $0 < \alpha < n+|\gamma|$, $\alpha/(n+|\gamma|) = 1/p - 1/q$, $1 < p < (n+|\gamma|)/\alpha$, $\frac{1}{p} + \frac{1}{p'} = 1$.

**Keywords**: $B$-maximal operator, $B$-Riesz potential, generalized $B$-Morrey space, local "complementary" generalized $B$-Morrey space

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## 1. Introduction

In the theory of partial differential equations, Morrey spaces $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ play an important role. They were introduced by C. Morrey in 1938 [20] and defined as follows: for $0 \le \lambda \le n$, $1 \le p < \infty$, $f \in \mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ if $f \in L_p^{\rm loc}(\mathbb{R}^n)$ and

$$\|f\|_{\mathcal{M}^{p,\lambda}} \equiv \|f\|_{\mathcal{M}^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty.$$

If $\lambda = 0$, then $\mathcal{M}^{p,\lambda}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$; if $\lambda = n$, then $\mathcal{M}^{p,\lambda}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$; if $\lambda < 0$ or $\lambda > n$, then $\mathcal{M}^{p,\lambda}(\mathbb{R}^n) = \Theta$, where $\Theta$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

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These spaces appeared to be quite useful in the study of the local behaviour of the solutions to elliptic partial differential equations, apriori estimates, and other topics in the theory of partial differential equations.

Given $f \in WL_p^{\rm loc}(\mathbb{R}^n)$, $W\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ denotes the weak Morrey space, and

$$\|f\|_{W\mathcal{M}^{p,\lambda}} \equiv \|f\|_{W\mathcal{M}^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(\mathbb{R}^n)$ denotes the weak $L_p(\mathbb{R}^n)$ spaces.

F. Chiarenza and M. Frasca [7] studied the boundedness of the maximal operator $M$ in Morrey spaces $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ (see, also [5], [6]). D. R. Adams [1] studied the boundedness of the Riesz potential in Morrey spaces and proved the follows statement (see, also [6]):

If in place of the power function $r^\lambda$ in the definition of $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ we consider any positive measurable weight function $\omega(r)$, then it becomes generalized Morrey spaces $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$.

**Definition 1.** *Let $\omega(r)$ positive measurable weight function on $(0,\infty)$ and $1 \le p < \infty$. We denote by $\mathcal{M}^{p,\omega}(\mathbb{R}^n)$ the generalized Morrey spaces, the spaces of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{\mathcal{M}^{p,\omega}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{r^{-\frac{n}{p}}}{\omega(r)} \|f\|_{L_p(B(x,r))}.$$

T. Mizuhara [19], E. Nakai [22] and V.S. Guliyev [9] obtained sufficient conditions on weights $\omega_1$ and $\omega_2$ ensuring the boundedness of $T$ from $\mathcal{M}^{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}^{p,\omega_2}(\mathbb{R}^n)$.

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at $x$ of radius $r$.

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. The maximal operator $M$ and the Riesz potential $I^\alpha$ are defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)| dy,$$

$$I^\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \qquad 0 < \alpha < n,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$.

The local Morrey-type spaces $\mathcal{M}^{p,\omega_1}(\mathbb{R}^n)$ and the complementary local Morrey-type spaces ${}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\mathbb{R}^n)$ were inten-sively studied during the last decades. In [9] local "complementary" generalized Morrey spaces ${}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\mathbb{R}^n)$, the space of all functions $f \in L_p(\mathbb{R}^n \backslash B(x_0,r))$, $r > 0$ by the norm

$$\|f\|_{{}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\mathbb{R}^n)} = \sup_{r>0} \frac{r^{\frac{n}{p'}}}{\omega(r)} \|f\|_{L_p(\mathbb{R}^n \backslash B(x_0,r))}$$

were introduced and studied.

Note that the maximal operator, potential and related topics associated with the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{k} \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}, \quad \gamma_1 > 0, \dots, \gamma_k > 0$$

have been investigated by many researchers, see B. Muckenhoupt and E. Stein [21], E. Stein [25], I. Kipriyanov [16], K. Trimeche [27], L. Lyakhov [18], K. Stempak [26], A.D. Gadjiev and I.A. Aliev [8], V.S. Guliyev [10], [11], V.S. Guliyev and J.J. Hasanov [12], J.J. Hasanov [14], R. Ayazoglu and J.J. Hasanov [3], C. Aykol and J.J. Hasanov [4], J.J. Hasanov, R. Ayazoglu, S. Bayrakci [15], A. Serbetci, I. Ekincioglu [23], E.L. Shishkina [24], L.R. Aliyeva, S. Esen Almali, Z.V. Safarov [2] and others.

In this paper we consider the generalized shift operator, generated by the Laplace-Bessel differential operator $\Delta_B$ in terms of which the $B$-Riesz potential is investigated in the local complementary generalized $B$-Morrey space.

## 2. Preliminaries

Let $\mathbb{R}^n_{k,+}$ be the part of the Euclidean space $\mathbb{R}^n$ of points $x = (x_1, ..., x_n)$ defined by the inequalities $x_1 > 0, \dots, x_k > 0$, $1 \le k \le n$, $(x')^\gamma = x_1^{\gamma_1} \cdot \ldots \cdot x_k^{\gamma_k}$, $\gamma = (\gamma_1, ..., \gamma_k)$ is a multi-index consisting of fixed positive numbers.

In this paper we realize some estimations of the $B$-Riesz potential generated by the generalized shift operator ([17]) of the form

$$T^y f(x) = C_{\gamma,k} \int_0^\pi \dots \int_0^\pi f\left((x', y')_\beta, x'' - y''\right) \, d\nu(\beta),$$

where $(x_i, y_i)_{\beta_i} = (x_i^2 - 2x_i y_i \cos\beta_i + y_i^2)^{\frac{1}{2}}$, $1 \le i \le k$, $(x', y')_\beta = ((x_1, y_1)_{\beta_1}, \dots, (x_k, y_k)_{\beta_k})$, $d\nu(\beta) = \prod_{i=1}^{k} \sin^{\gamma_i - 1}\beta_i \, d\beta_1 \dots d\beta_k$, $1 \le k \le n$ and

$$C_{\gamma,k} = \pi^{-\frac{k}{2}} \, \Gamma^{-1}\left(\frac{|\gamma|}{2}\right) \prod_{i=1}^{k} \Gamma\left(\frac{\gamma_i + 1}{2}\right) = \frac{2^{k-1}|\gamma|}{\pi} \left(\frac{|\gamma|}{2} + 1\right) \omega(2, k, \gamma).$$

Note that the generalized shift operator $T^y$ is closely related to the $\Delta_B$ Laplace-Bessel differential operator ([16]). Furthermore, $T^y$ generates the corresponding $B$-convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y) \, T^y g(x) \, (y')^\gamma dy,$$

for which the Young inequality

$$\|f \otimes g\|_{L_{r,\gamma}} \le \|f\|_{L_{p,\gamma}} \|g\|_{L_{q,\gamma}}, \quad 1 \le p, q \le r \le \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1,$$

holds.

Let $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ be the space of measurable functions on $\mathbb{R}^n_{k,+}$ with finite norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})} = \left( \int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx \right)^{1/p}, \quad 1 \le p < \infty.$$

For $p = \infty$ the space $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ is defined by means of the usual modification

$$\|f\|_{L_{\infty,\gamma}} = \underset{x \in \mathbb{R}^n_{k,+}}{ess\ sup}\, |f(x)|.$$

**Definition 2.** *[10] Let $1 \le p < \infty$, $0 \le \lambda \le Q$. We denote by $B$-Morrey space $\mathcal{M}^{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, associated with the Laplace-Bessel differential operator the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_{k,+}$, with the finite norm*

$$\|f\|_{\mathcal{M}^{p,\lambda,\gamma}} = \sup_{t>0,\, x \in \mathbb{R}^n_{k,+}} \left( t^{-\lambda} \int_{E_t} T^y [|f|\ ]^p(x) (y')^\gamma dy \right)^{1/p}.$$

Consider the $B$-Riesz potential

$$I_{\alpha,\gamma} f(x) = \int_{\mathbb{R}^n_{k,+}} T^y[|f|\ ](x) |y|^{\alpha - Q} (y')^\gamma dy, \quad 0 < \alpha < Q.$$

**Theorem 1.** *[13] Let $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$. Then the operator $I_{\alpha,\gamma}$ is bounded from $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\gamma}(\mathbb{R}^n_{k,+})$.*

Let $1 \le p < \infty$, $\omega$ positive measurable function. The norm in the spaces $\mathcal{M}^{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ defined by

$$\|f\|_{\mathcal{M}^{p,\omega,\gamma}} = \sup_{x \in \mathbb{R}^n_{k,+},\, t>0} \frac{t^{-\frac{Q}{p}}}{\omega(t)} \left( \int_{E_t} T^y [|f|\ ]^p(x) (y')^\gamma dy \right)^{1/p},$$

the local "complementary" generalized $B$-Morrey space ${}^{\complement}\mathcal{M}^{p,\omega,\gamma}_{\{0\}}(\mathbb{R}^n_{k,+})$ is defined by the norm

$$\|f\|_{{}^{\complement}\mathcal{M}^{p,\omega,\gamma}_{\{0\}}(\mathbb{R}^n_{k,+})} = \sup_{t>0} \frac{t^{\frac{Q}{p'}}}{\omega(t)} \left( \int_{\mathbb{R}^n_{k,+} \setminus E(0,t)} T^y [|f|\ ]^p(x) (y')^\gamma dy \right)^{1/p}.$$

If $\omega(t) \equiv r^{-\frac{Q}{p}}$, then $\mathcal{M}^{p,\omega,\gamma}(\mathbb{R}^n_{k,+}) \equiv L_{p,\gamma}(\mathbb{R}^n_{k,+})$; if $\omega(t) \equiv t^{\frac{\lambda - Q}{p}}$, $0 \le \lambda < Q$, then $\mathcal{M}^{p,\omega,\gamma}(\mathbb{R}^n_{k,+}) \equiv \mathcal{M}^{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

## 3. $B$-Riesz Potentials in the Spaces ${}^{\complement}\mathcal{M}^{p,\omega,\gamma}_{\{0\}}(\mathbb{R}^n_{k,+})$

**Theorem 2.** *Let $0<\alpha<Q$, $1<p<\frac{Q}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q}$. If the integral*

$$\int_0^1 r^{\frac{Q}{p'}-1}\|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\backslash E(0,r)}dr$$

*is convergent, then*

$$\|I_{\alpha,\gamma}f\|_{L_{q,\gamma}(\mathbb{R}^n_{k,+}\backslash E(0,t)}\leq Ct^{-\frac{Q}{p'}}\int_0^t s^{\frac{Q}{p'}-1}\|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\backslash B(x_0,s))}ds \tag{1}$$

*for every $f\in L_{p,\gamma}(\mathbb{R}^n_{k,+}\backslash E(0,t)$ and $C$ does not depend on $f,x_0$ and $t\in(0,\infty)$.*

*Proof.* We represent $f$ as

$$f=f_1+f_2,\quad f_1(y)=f(y)\chi_{\mathbb{R}^n_{k,+}\backslash E(0,t)}(y)\quad f_2(y)=f(y)\chi_{E(0,t)}(y).$$

So that

$$\|I_{\alpha,\gamma}f\|_{L_{q,\gamma}(\mathbb{R}^n_{k,+}\backslash E(0,t))}\leq\|I_{\alpha,\gamma}f_1\|_{L_{q,\gamma}(\mathbb{R}^n_{k,+}\backslash E(0,t))}+\|I_{\alpha,\gamma}f_2\|_{L_{q,\gamma}(\mathbb{R}^n_{k,+}\backslash E(0,t))}.$$

Since $f_1\in L_{p,\gamma}(\mathbb{R}^n_{k,+})$, from Theorem 1 we have

$$\|I_{\alpha,\gamma}f_1\|_{L_{q,\gamma}(\mathbb{R}^n_{k,+}\backslash E(0,t))}\leq\|I_{\alpha,\gamma}f_1\|_{L_{q,\gamma}(\mathbb{R}^n_{k,+})}\leq C\|f_1\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})}=C\|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\backslash E(0,t)}.$$

From the monotonicity of the norm $\|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\backslash E(0,r)}$ with respect to $r$ we have

$$\|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\backslash E(0,t)}\leq Ct^{-\frac{Q}{p'}}\int_0^t r^{\frac{Q}{p'}-1}\|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\backslash E(0,r)}dr$$

and then

$$\|I_{\alpha,\gamma}f_1\|_{L_{q,\gamma}(\mathbb{R}^n_{k,+}\backslash E(0,t))}\leq Ct^{-\frac{Q}{p'}}\int_0^t s^{\frac{Q}{p'}-1}\|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\backslash E(0,s))}ds. \tag{2}$$

Now we will show that $I_{\alpha,\gamma}f_2$ is bounded for every $x\in\mathbb{R}^n_{k,+}\backslash E(0,2r)$. By $x\in\mathbb{R}^n_{k,+}\backslash E(0,2r)$ and $y\in E(0,r)$, it follows that $T^y|x|\geq\frac{1}{2}|x|\geq r$, then we get

$$\begin{aligned}|I_{\alpha,\gamma}f_2(x)|&\leq\int\limits_{E(0,t)}T^y|f(x)|\,|y|^{\alpha-Q}(y')^\gamma dy\\&\leq Cr^{\alpha-Q}\int\limits_{E(0,r)}T^y|f(x)|(y')^\gamma dy.\end{aligned}$$

For $\beta>\frac{Q}{p'}-1$, we have

$$\int\limits_{|y|\leq r}T^y|f(x)|(y')^\gamma dy=\int_{\mathbb{S}_{k,+}}\sum_{i=1}^n\xi_i^2\xi_n^\gamma d\sigma(\xi)\int_0^r T^{t\xi}|f(x)|t^{Q-1}dt=$$

$$= (\beta+1)\int_{\mathbb{S}^+_{k,+}} \sum_{i=1}^n \xi_i^2\xi_n^\gamma d\sigma(\xi) \int_0^r t^{Q-1}T^{t\xi}|f(x)|t^{-\beta-1}dt \int_0^t s^\beta ds =$$

$$= (\beta+1)\int_{\mathbb{S}^+_{k,+}} \sum_{i=1}^n \xi_i^2\xi_n^\gamma d\sigma(\xi) \int_0^r s^\beta ds \int_s^r T^{t\xi}|f(x)|t^{Q-1-\beta-1}dt \le$$

$$\le C \int\limits_{\mathbb{S}^+_{k,+}} \sum_{i=1}^n \xi_i^2\xi_n^\gamma d\sigma(\xi) \int\limits_0^r s^{\beta-\frac{Q}{p'}-\beta-1} \left(\int\limits_s^r T^{t\xi}|f(x)|^p t^{Q-1}dt\right)^{1/p} ds =$$

$$= C\int\limits_0^r s^{\frac{Q}{p'}-1}ds \int\limits_{\mathbb{S}_{k,+}+} \sum_{i=1}^n \xi_i^2\xi_n^\gamma d\sigma(\xi) \left(\int\limits_s^r T^{t\xi}|f(x)|^p t^{Q-1}dt\right)^{1/p} \le$$

$$\le C\int\limits_0^r s^{\frac{Q}{p'}-1} \left(\int\limits_{\mathbb{S}_{k,+}} \int\limits_s^r T^{t\xi}|f(x)|^p t^{Q-1} \sum_{i=1}^n \xi_i^2\xi_n^\gamma dt d\sigma(\xi)\right)^{1/p} ds =$$

$$= C\int\limits_0^r s^{\frac{Q}{p'}-1} \left(\int\limits_{s\le|y|\le r} T^y|f(x)|^p (y')^\gamma dy\right)^{1/p} ds \le$$

$$\le C\int\limits_0^r s^{\frac{Q}{p'}-1} \|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus B(x_0,s))} ds.$$

Therefore

$$|I_{\alpha,\gamma}f_2(x)| \le C\int\limits_0^r s^{\frac{Q}{p'}-1}\|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus B(x_0,s))}ds. \tag{3}$$

It remains to make use of (3) and obtain

$$\|I_{\alpha,\gamma}f_2\|_{L_{q,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,t))} \le Ct^{-\frac{Q}{p'}}\int_0^t s^{\frac{Q}{p'}-1}\|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus B(x_0,s))}ds. \tag{4}$$

From (2) and (4) we arrive at (1). ◄

**Theorem 3.** *Let $0<\alpha<Q$, $1<p<\frac{Q}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q}$, the functions $\omega_1(r)$ and $\omega_2(r)$ fulfill the condition*

$$\int_0^t \omega_1(r)\frac{dr}{r} \le C\,t^{-\alpha}\omega_2(t), \tag{5}$$

*where $C$ does not depend on $t$. Then the operator $I_{\alpha,\gamma}$ is bounded from ${}^{\complement}\mathcal{M}^{p,\omega_1,\gamma}_{\{0\}}(\mathbb{R}^n_{k,+})$ to ${}^{\complement}\mathcal{M}^{q,\omega_2,\gamma}_{\{0\}}(\mathbb{R}^n_{k,+})$.*

*Proof.* It suffices to prove the boundedness of the operator $I_{\alpha,\gamma}$, since $M^{\alpha}f(x) \leq CI_{\alpha,\gamma}|f|(x)$.

Let $f \in {}^{\complement}\mathcal{M}_{\{0\}}^{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$. We have

$$\|I_{\alpha,\gamma}f\|_{{}^{\complement}\mathcal{M}_{\{0\}}^{q,\omega_2,\gamma}(\mathbb{R}^n_{k,+})} = \sup_{t>0} \frac{t^{\frac{Q}{q'}}}{\omega_2(t)} \|\chi_{\mathbb{R}^n_{k,+}\setminus E(0,t)} T^{\cdot} I_{\alpha,\gamma}f\|_{L_{q,\gamma}(\mathbb{R}^n_{k,+})}. \tag{6}$$

We estimate $\|\chi_{\mathbb{R}^n_{k,+}\setminus E(0,t)} I_{\alpha,\gamma}f\|_{L_{q,\gamma}(\mathbb{R}^n_{k,+})}$ in (6) by means of Theorem 2. We obtain

$$\|I_{\alpha,\gamma}f\|_{{}^{\complement}\mathcal{M}_{\{0\}}^{q,\omega_2,\gamma}(\mathbb{R}^n_{k,+})} \leq C \sup_{t>0} \frac{t^{-\frac{Q}{p'}+\frac{Q}{q'}}}{\omega_2(t)} \int_0^t r^{\frac{Q}{p'}-1} \|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,r)}dr$$

$$\leq C\|f\|_{{}^{\complement}\mathcal{M}_{\{0\}}^{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})} \sup_{t>0} \frac{t^{\alpha}}{\omega_2(t)} \int_0^t \frac{\omega_1(r)}{r} dr.$$

It remains to make use of the condition (5). ◀

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