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ROUGH PROJECTIVE REPRESENTATIONS OF ROUGH GROUPS

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Abstract. *In this paper, we introduce the concepts of rough projective representations of rough groups and discuss their relations with rough 2-cocycle. In particular, we deduce that any rough projective representation associated with a rough multiplier is equivalent to a rough representation if and only if a rough multiplier is a rough coboundary.*

Keywords: rough set, rough group, rough multiplier, rough projective representation

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1. Introduction

In 1982, Z. Pawlak introduced the concept of rough set [10] to analyze and to model vague and uncertain data. More precisely, technics from the rough set theory are used to describe a set of objects to which values are assigned, to find dependency between the attributes, to identify the most significant ones and to reduce the superfluous ones. The notion of rough set appears to be powerful with an important applications in quantum mechanics, software engineering, computer systems, decision analysis, electrical engineering, finance, chemistry, computer engineering, economics, neurology, medicine, statistics, etc [9], [14]. The notion of rough set has been later extended to the group theory setting [2], [7].

Besides, the notion of projective representation was initiated by I. Schur since 1904 [11], [12], [13]. The theory of projective representations involves homomorphisms into

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projective linear groups. Some interesting references on the subject are [6], [8]. In 2000, B. Bagchi and G. Misra established that there is a natural correspondence between the projective representations of homology group and a class of usual representations of its universal cover [1]. In [3], C. Cheng developed a character theory for projective representations of finite groups. He determined the number of distinct irreducible projective representations (up to isomorphism) of a finite group with a given associated Schur multiplier and derived properties on the degrees of these projective representations.

The main purpose of this work is to bring together rough groups and projective representations. More precisely, we introduce projective representations of rough groups and investigate some of their main properties.

The rest of the paper is organized as follows. Section 2. collects some definitions and results that we may needed. Section 3. states the mains results.

2. Preliminaries

In this section, we give some well-known definitions.

Definition 1. *[2] Let U be a non-empty set (called the* universe*). Let R be an equivalence relation on U. The pair (U, R) is called an* approximation space*.*

Let (U, R) be an approximation space. For $x \in U$, the equivalence class of x is denoted by $[x]$. For $X \subset U$, set

$$\overline{X} = \{x \in U \,:\, [x] \cap X \neq \emptyset\} \text{ and } \underline{X} = \{x \in U \,:\, [x] \subset X\}.$$

The sets $\overline{X}$ and $\underline{X}$ are called the *upper approximation* and *lower approximation* of X in (U, R) respectively. We have $\underline{X} \subset X \subset \overline{X}$.

Example 1. Let us consider the non-empty set $U = \{0, 1, 2, ..., 50\}$. We define the relation R on U such that xRy if $x - y = 5k, k \in \mathbb{Z}$. It is an equivalence relation on U. From this equivalence relation, we have five equivalence classes, namely:
$E_0 = \overline{0} = \{0, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50\}$;
$E_1 = \overline{1} = \{1, 6, 11, 16, 21, 26, 31, 36, 41, 46\}$;
$E_2 = \overline{2} = \{2, 7, 12, 17, 22, 27, 32, 37, 42, 47\}$;
$E_3 = \overline{3} = \{3, 8, 13, 18, 23, 28, 33, 38, 43, 48\}$;
$E_4 = \overline{4} = \{4, 9, 14, 19, 24, 29, 34, 39, 44, 49\}$.
In other words, $U/R = \{E_0, E_1, E_2, E_3, E_4\}$. Since $U \neq \emptyset$ and R is an equivalence relation on U, then (U, R) is a space d 'approximation. We consider a subset X of U given by $X = \{10, 11, 12, 13, 14\}$. We obtain the upper and lower approximations of X, as follows $\overline{X} = E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4 = U$ and $\underline{X} = \emptyset$.

Assume that U is endowed with a binary operation $U \times U \longrightarrow U$. The product of two elements x and y is denoted by xy. The inverse of x (if it exists) is denoted by x^{-1}.

Definition 2. *[2] Let (U, R) be an approximation space. Assume that there is a binary operation on U. A subset G of U is called a* rough group *if the following properties hold:*

1. $\forall\, x, y \in G,\ xy \in \overline{G}$,
2. $\forall\, x, y, z \in \overline{G},\ (xy)z = x(yz)$,
3. $\exists\, e \in \overline{G},\ \forall x \in G,\ ex = xe = x$, *(e is called a* rough identity element of G*).*
4. $\forall\, x \in G, \exists\, y \in G, xy = yx = e$ *(y is called the* rough inverse of x and denoted x^{-1}*).*

Definition 3. *[2] A non-empty subset H of rough group G is called a rough subgroup of G if the two conditions are satisfied:*

1. $\forall x, y \in H,\ xy \in \overline{H}$,
2. $\forall x \in H,\ x^{-1} \in H$.

There is only one guaranteed trivial rough subgroup of rough group G, that is G itself. A necessary and sufficient condition for the set $\{e\}$ to be a trivial rough subgroup of the rough group G is that $e \in G$.

Example 2. Consider the approximation space (Q_8, R) where $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is a group of quaternions. Let $*$ be the multiplicative law of quaternions.
Let R be the equivalence relation on Q_8 such that $Q_8/R = \{\{\pm 1\}, \{\pm i\}, \{\pm j, \pm k\}\}$.
Let $G = \{\pm i, -1\}$. Then, we have $\overline{G} = \{\pm i, \pm 1\}$ and $\underline{G} = \{\pm i\}$. According to the Definition 2, the following conditions are verified:

1. $\forall x, y \in G, x * y \in \overline{G}$,
2. the associativity property of the $*$ law is verified,
3. $1 * (\pm i) = (\pm i) * 1 = (\pm i)$ and $1 * (-1) = (-1) * 1 = -1$ then 1 is the rough identity element of G,
4. $(-i)^{-1} = i \in G$ and $(-1)^{-1} = -1 \in G$.

Then, G is a rough group.

Definition 4. *[7] Let (U_1, R_1) and (U_2, R_2) be two approximation spaces with binary operations on U_1 and U_2. Suppose that $G_1 \subset U_1, G_2 \subset U_2$ are rough groups. If the mapping $\varphi : \overline{G_1} \to \overline{G_2}$ satisfies $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in G_1$, then φ is called a rough homomorphism.*

Let F be a field of scalars with identity element denoted by 1. Let us denote by F^* the multiplicative group of non-zero elements of F.

Definition 5. *[5] A map $\alpha : \overline{G} \times \overline{G} \to F^*$ is called a rough multiplier or a rough 2-cocycle of G over F^* if $\forall x, y, z \in \overline{G}$,*

1. $\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z)$,
2. $\alpha(x, e) = \alpha(e, x) = 1$.

The set of rough multipliers or rough 2-cocycles of G over F^* is denoted by $Z_r^2(G, F^*)$.

Definition 6. *[4] A* rough representation *of G on E is a homomorphism $\sigma : x \longmapsto \sigma(x)$ from $\overline{G}$ into $GL(E)$, that is,*

$$\forall x, y \in G,\ \sigma(xy) = \sigma(x)\sigma(y).$$

Definition 7. *[5] Let $f : \overline{G} \to F^*$ be a map such that $f(e) = 1$. A map $\nu : \overline{G} \times \overline{G} \to F^*$ defined by*

$$\nu(x, y) = f(x)f(y)f(xy)^{-1},\ \forall\, x, y \in \overline{G}.$$

is called a rough coboundary

3. Main Results

Let E be a vector space over the field F. Denote by $GL(E)$ the set of invertible linear operators on E with identity operator denoted by I_E.

The set $F^*I_E = \{\lambda I_E, \lambda \neq 0, \lambda \in F\}$ is a normal subgroup of $GL(E)$. The quotient set $\mathrm{PGL}(E) = \mathrm{GL}(E)/F^*I_E$ is the projective general linear group. Let G be a rough group.

Definition 8. *A mapping $\pi : \overline{G} \to GL(E)$ is called a* rough projective representation *of G over F if there exists a mapping $\alpha : \overline{G} \times \overline{G} \to F^*$ such that the following two properties hold :*

(i) $\forall\, x, y \in \overline{G}$, $\pi(x)\pi(y) = \alpha(x, y)\pi(xy)$,
(ii) $\pi(e) = I_E$,

where α is called a Schur multiplier.

Example 3. We consider the set $C_4 = \{1, a, b, c\}$ provided with the multiplication law defined by the following Cayley table:

$\curvearrowright$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Let R be the equivalence relation canonically associated with $U_i, i \in \{1, 2\}$ on C_4 such that $C_4/R = \{U_1, U_2\}$ where $U_1 = \{1, a\}, U_2 = \{b, c\}$.

Let $G = \{a, b\}$. Then, $\overline{G} = U_1 \cup U_2 = C_4$ and $\underline{G} = \emptyset$. One may check that the conditions of Definition 2 are satisfied. Therefore, G is a rough group.

Denote by $\mathcal{M}_2(\mathbb{C})$ the space of 2×2 matrices with complex entries. Let us consider the map $\pi : \overline{G} = C_4 \longrightarrow \mathcal{M}_2(\mathbb{C})$ defined by

$$\pi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pi(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pi(b) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \pi(c) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and put $\alpha(1, x) = \alpha(x, 1) = \alpha(x, x) = 1, x \in \overline{G}$,
$\alpha(a, b) = \alpha(b, c) = \alpha(c, a) = i$,
$\alpha(a, c) = \alpha(b, a) = \alpha(c, b) = -i$.

Finally, it is straightforward to verify that $\alpha(x, y)\pi(xy) = \pi(x)\pi(y)$ for $x, y \in \overline{G}$. Therefore, the map $\pi : \overline{G} \longrightarrow \mathcal{M}_2(\mathbb{C})$ is a rough projective representation of G.

Definition 9. *Let E_1 and E_2 be vector spaces over F. Two rough projective representations $\pi_1 : \overline{G} \to GL(E_1)$ and $\pi_2 : \overline{G} \to GL(E_2)$ are said to be rough projective equivalent if there exists a map $\mu : \overline{G} \to F^*$ with $\mu(e) = 1$ and a vector space isomorphism $\phi : E_1 \to E_2$ such that $\pi_2(x) = \mu(x)\phi \circ \pi_1(x) \circ \phi^{-1}$ for all $x \in \overline{G}$.*

Proposition. *If $\pi : \overline{G} \to GL(E)$ is a rough projective representation of G associated with α, then α is a rough multiplier of G.*

Proof. Since $\pi : \overline{G} \to GL(E)$ is a rough projective representation of G associated with α, then for $x, y, z \in \overline{G}$, we have

$$\begin{aligned} &\pi((xy)z) = \pi(x(yz)) \\ \Rightarrow\ &\alpha(xy, z)\pi(xy)\pi(z) = \alpha(x, yz)\pi(x)\pi(yz) \\ \Rightarrow\ &\alpha(xy, z)\alpha(x, y)\pi(x)\pi(y)\pi(z) = \alpha(x, yz)\alpha(x, y)\pi(x)\pi(y)\pi(z) \\ \Rightarrow\ &\alpha(x, yz)\alpha(y, z) = \alpha(x, y)\alpha(xy, z). \end{aligned}$$

In addition, $\pi(e) = I_E$ and for all $x \in \overline{G}$, $\pi(x) = \pi(xe) = \alpha(x, e)\pi(x)\pi(e) = \alpha(x, e)\pi(x)$.

That means, $\alpha(x, e)I_E = \pi(x)\pi(x)^{-1} = I_E$. Then, $\alpha(x, e) = 1$. Similarly, we can prove that $\alpha(e, x) = 1$. We deduce that $\alpha(x, e) = \alpha(e, x) = 1$. Hence, α is a rough multiplier of G. ◀

Theorem 1. *If $\pi : \overline{G} \to GL(E)$ is a rough projective representation of G and $\sigma : GL(E) \to PGL(E)$ is the natural homomorphism, then $\sigma \circ \pi : \overline{G} \to PGL(E)$ is a rough homomorphism. Conversely, if $h : \overline{G} \to PGL(E)$ is a rough homomorphism such that for any $x \in \overline{G}, \pi(x) \in GL(E)$ is an element of the coset $h(x)$, then $\pi : \overline{G} \to GL(E)$ with $\pi(e) = I_E$, is a rough projective representation of G.*

Proof. Let $\sigma : GL(E) \to PGL(E)$ defined by $\sigma(h) = F^*h$ be the natural homomorphism of $GL(E)$ onto $PGL(E)$. Let $x, y \in \overline{G}$. Since π is rough projective representation, then we have

$$\sigma \circ \pi(x)\sigma \circ \pi(y) = \sigma(\pi(x)\pi(y)) = F^*\pi(x)\pi(y) = F^*\alpha(x, y)\pi(xy) =$$

$$= F^*\pi(xy) = \sigma(\pi(xy)) = \sigma \circ \pi(xy).$$

Hence, $\sigma \circ \pi : \overline{G} \to PGL(E)$ is a rough homomorphism.

Conversely, let us define $\pi : \overline{G} \to GL(E)$ such that for each $x \in \overline{G}, \pi(x)$ is a fixed element of the coset $h(x)$. Then, there exists a canonical projection $\varphi : GL(E) \to PGL(E)$ defined by $Y \mapsto YF^*I_E$ such that $\varphi \circ \pi(x) = h(x)$. For $x, y \in \overline{G}$, we have

$$\varphi(\pi(xy)) = h(xy) = h(x)h(y) = \varphi(\pi(x))\varphi(\pi(x)).$$

On the one hand, we have

$$\varphi(\pi(xy)) = \pi(xy)F^*I_E \tag{1}$$

and the other hand,

$$\varphi(\pi(x))\varphi(\pi(x)) = \pi(x)\pi(y)F^*I_E. \tag{2}$$

We deduce from the equations (1) and (2) that there exists an element $\beta \in K^*$ such that $\pi(x)\pi(y) = \beta I_E \pi(x)\pi(y)$. From there, we have $\pi(x)\pi(y) = \beta\pi(xy)$.

Set $\alpha(x, y) := \beta \in K^*$ such that $\pi(x)\pi(y) = \beta\pi(xy)$. We obtain an application $\alpha : \overline{G}\times\overline{G} \to K^*$ such that $\alpha(x, y)\pi(xy) = \pi(x)\pi(y)$ and $\pi(e) = I_E$. We conclude that $\pi : \overline{G} \to GL(E)$ is a rough projective representation of G. ◀

Theorem 2. *Let E_1 and E_2 be two vector spaces, $\pi_1 : \overline{G} \to GL(E_1)$ and $\pi_2 : \overline{G} \to GL(E_2)$ be two rough projective representations of G associated with α_1 and α_2 respectively. If there exists a map $\mu : \overline{G} \to F^*$ with $\mu(e) = 1$ and a vector space isomorphism $\phi : E_1 \to E_2$ for which*

$$\forall\, x \in \overline{G},\ \pi_2(x) = \mu(x)\phi\pi_1(x)\phi^{-1}, \tag{3}$$

then $\alpha_2 = \nu\alpha_1$, where ν is a rough 2-cocycle i.e $\nu(x, y) = \mu(x)\mu(y)\mu(xy)^{-1}$, for all $x, y \in \overline{G}$.

Proof. Let $x, y \in \overline{G}$. Since π_1 and π_2 are rough projective representations of G on E_1 and E_2 respectively, then we have

$$\begin{aligned}
\alpha_2(x, y)\pi_2(xy) = \pi_2(x)\pi_2(y) &= (\mu(x)\phi\pi_1(x)\phi^{-1})(\mu(y)\phi\pi_1(y)\phi^{-1})\\
&\qquad \text{(by applying equation (3) to } \pi_2(x) \text{ and } \pi_2(y))\\
&= \mu(x)\mu(y)\phi\pi_1(x)\phi^{-1}\phi\pi_1(y)\phi^{-1}\\
&= \mu(x)\mu(y)\phi\pi_1(x)\pi_1(y)\phi^{-1}\\
&= \nu(x, y)\mu(xy)\phi\alpha_1(x, y)\pi_1(xy)\phi^{-1}\\
&= \nu(x, y)\alpha_1(x, y)\mu(x, y)\phi\pi_1(xy)\phi^{-1}\\
&= \nu(x, y)\alpha_1(x, y)\pi_2(xy) \text{ (as } \alpha_2(x, y)\pi_2(xy) = \pi_2(x)\pi_2(y)).
\end{aligned}$$

Hence, $\alpha_2(x, y) = \nu(x, y)\alpha_1(x, y)$. ◀

Theorem 3. *Let $\alpha \in Z^2(G, F^*)$. Any rough projective representation associated with α is equivalent to a rough representation if and only if α is a rough coboundary.*

Proof. Suppose that the rough projective representation π associated with α is equivalent to a rough representation σ. Then, there exists an isomorphism $\phi : E_1 \to E_2$ and a map $\mu : \overline{G} \to F^*$ with $\mu(e) = 1$ such that $\pi(x) = \mu(x)\phi\sigma(x)\phi^{-1}$, $\forall\, x \in \overline{G}$. Since π is a rough projective representation associated with α, i.e $\alpha(x, y)\pi(xy) = \pi(x)\pi(y)$, for all $x, y \in \overline{G}$. Now, we have

$$\begin{aligned}
\alpha(x, y)\mu(xy)\phi\sigma(xy)\phi^{-1} &= (\mu(x)\phi\sigma(x)\phi^{-1})(\mu(y)\phi\sigma(y)\phi^{-1})\\
&= \mu(x)\mu(y)\phi\sigma(x)\sigma(y)\phi^{-1}\\
&= \mu(x)\mu(y)\phi\sigma(xy)\phi^{-1}.
\end{aligned}$$

Hence, $\alpha(x, y)\mu(xy) = \mu(x)\mu(y)$, since $\phi\sigma(xy)\phi^{-1} \neq 0$ (because $\phi \neq 0$ and $\sigma(x) \in GL(E)$). We obtain, $\alpha(x, y)\mu(xy) = \mu(x)\mu(y) \Rightarrow \alpha(x, y) = \mu(x)\mu(y)\mu(xy)^{-1}$.

Hence, α is a rough coboundary.

Conversely, α is assumed to be a rough coboundary. Then, there is a map $\mu : G \to F^*$ such that

$$\alpha(x, y) = \mu(x)\mu(y)\mu(xy)^{-1}, \forall\ (x, y) \in \overline{G}^2. \tag{4}$$

Since, π be rough projective representation associated with α. We have

$$\pi(x)\pi(y) = \alpha(x, y)\pi(xy). \tag{5}$$

By combining equations (4) and (5), we have,

$$\pi(x)\pi(y) = \mu(x)\mu(y)\mu(xy)^{-1}\pi(xy),$$

$$(\mu(x)^{-1}\pi(x))(\mu(y)^{-1}\pi(y)) = \mu(xy)^{-1}\pi(xy).$$

Let us take $\sigma : \overline{G} \to GL(E)$ defined by $\sigma(x) = \mu(x)^{-1}\pi(x)$. According to equation (5), it is clear that $\sigma(xy) = \sigma(x)\sigma(y),\ \forall x, y \in \overline{G}$ and $\sigma(e) = \mu(e)^{-1}\pi(e) = I_E$. Hence, σ is a rough representation. Now, we have, $\forall\, x \in \overline{G},\ \ \sigma(x) = \mu(x)^{-1}\pi(x)$. This implies $\pi(x) = \mu(x)\sigma(x)$ and $\pi(x) \circ I_E = \mu(x)I_E \circ \sigma(x)$.

Let us take $\phi = I_E$. Then, ϕ is an isomorphism. Hence, σ is a rough representation (and it is equivalent to a π). ◀

4. Conclusion

The rough projective representation theory concerns both the rough group algebraic structure and the notion of projective representation. From the two concepts combined, we shown that the Schur multiplier is rough 2-cocycle and we established that any rough projective representation associated with a rough multiplier is equivalent to a rough representation if and only if a rough multiplier is a rough coboundary.

In the future, we plan to investigate the relationship between the set of rough projective representations and a family of modules associated to the multiplier. We also intend to scrutnize the rough supra-topological groups notion.

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