Baku Mathematical Journal
2024, Vol. 3, No 2, PP. 186-201
https://doi.org/10.32010/j.bmj.2024.18

UNIQUENESS CRITERIA FOR INVERSE SCATTERING PROBLEM IN TERMS OF TRANSMISSION MATRIX IN BOUNDARY CONDITION FOR A FIRST ORDER SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

M.I. ISMAILOV*, **C. SABAZ**

Received: 08.04.2024 / Revised: 11.06.2024 / Accepted: 19.06.2024

In memory of M. G. Gasymov on his 85th birthday

Abstract. *The inverse scattering problem (ISP) involves the recovery of the matrix coefficient of a first-order system on the half-line from its scattering matrix. Specifically, when the matrix coefficient exhibits a triangular structure, the system possesses a Volterra-type integral transformation operator at infinity. This transformation operator facilitates the determination of the scattering matrix on the half-line through matrix Riemann-Hilbert factorization. Solving the ISP on the half-line entails reducing it to an ISP on the whole line for the considered system. This reduction involves extending the coefficients to the whole line by zero. The uniqueness criteria in terms of transmission matrix in boundary condition for the ISP is also established.*

Keywords: first order system on semi-axis, scattering matrix, transmission matrix, inverse scattering problem

Mathematics Subject Classification (2020): 34A55, 34L25, 34M50, 45Q05

1. Introduction

We consider the following system of ordinary differential equations (ODEs) on the half-line:

$$-i\frac{dy}{dx} + Q(x)y = \lambda\sigma y,\ 0 \le x < +\infty \tag{1}$$

* Corresponding author.

Mansur I. Ismailov
Gebze Technical University, Kocaeli, Türkiye
E-mail: mismailov@gtu.edu.tr

Cihan Sabaz
Gebze Technical University, Kocaeli, Türkiye
E-mail: cihansabaz@gtu.edu.tr

with the complex parameter λ. It is assumed that $\sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$ is a $2n \times 2n$ order diagonal matrix with constant diagonal elements, where $\sigma_1 = diag\,(\xi_1, ..., \xi_n)$, $\sigma_2 = diag\,(\xi_{n+1}, ..., \xi_{2n})$ with $\xi_1 \geq \xi_2 \geq ... > 0 > \xi_{n+1} \geq ... \geq \xi_{2n}$ and Q is a $2n \times 2n$ order matrix function with measurable complex valued rapidly decreasing entries. The matrix $Q(x)$ is called as potential.

Let $y(x,\lambda) = \begin{bmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{bmatrix}$, where $y_1(x,\lambda)$ and $y_2(x,\lambda)$ are n dimensional vector functions. Consider the system (1) under the boundary condition at point $x = 0$ of the following form:

$$y_2(0,\lambda) = Hy_1(0,\lambda), \;\; \det H \neq 0. \tag{2}$$

In the case $n = 1$, the (1) is called the Dirac system and the ISP is satisfactorily studied in various literature, see for example [1], [10] and references therein. The general case of Dirac system (1) in the case $1 = \xi_1 = ... = \xi_n > 0 > \xi_{n+1} = ... = \xi_{2n} = -1$ the ISP on the half-line are studied in [2], [3], [8] by reducing it the Gelfand-Levitan-Marchenko equation. The system (1) in the case of different characteristic numbers $\xi_1 < ... < \xi_n < 0 < \xi_{n+1} < ... < \xi_{2n}$ the ISP on the half-line is studied in [5] by reducing it the matrix Riemann-Hilbert problem.

We will consider the system (1) on the half-line in the case of characteristic numbers $\xi_1 > ... > \xi_n > 0 > \xi_{n+1} = ... = \xi_{2n}$ and the special form of potential $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$ with Q_{11} is strictly lower triangular matrix, Q_{12} and Q_{21} are lower triangular matrices and $Q_{22} = 0$. Let us call such a matrix potential formally by *CM-canonical* potential. For simplicity we will consider the case $n = 2$, which the non-stationary situation is considered in [6].

For the solution of ISP on the half-line, we concretize the rapidly decreasing of the potential by the Neimark-type condition [9]:

$$\|Q(x)\| \leq Ce^{-\varepsilon x}, \text{ for some } \varepsilon > 0. \tag{3}$$

The suitability of this type a condition in theory of ISP on the half-line for the equations with the non-self-adjoint potential is presented in [7]. This condition assumes the analyticity of the scattering matrix in the strip $|Im\lambda| < \varepsilon_0$ for some ε_0, and it guarantees also that the point spectrum and spectral singularities remain discrete and do not accumulate on the real axis.

The paper is organized as follows: In the next section, we determine Volterra-type integral transformation operator for the system (1) at infinity, when the matrix coefficients $Q(x)$ of the system are in the special triangular structure. The scattering matrix on the half-line is defined and some is analytic properties are studied. In Sect. 3, the ISP for the system (1) on the half-axis is formulated and it reduced to the ISP on the whole axis for the system with the coefficients which are zero for $x < 0$. The matrix Riemann-Hilbert (RH) problems corresponding to ISP on the half-line, when the contour is real-line, normalization is canonical and all the partial indices are zero, also are given in this section. Under the conditions of unique solvability of these matrix Riemann-Hilbert (RH) problems, the uniqueness of the solution of the ISP on the half-line is obtained from the uniqueness of the solution of the ISP on the whole line. The examples are given, on the non-uniqueness in the ISP is discussed, when there are some violations on the conditions. In Sect. 4, the conclusion is given for a future perspective of the ISP on the half-axis for the system of first order ODEs.

2. Scattering Problem on the Half-Axis

2. 1. Transformation operator at infinity

In solving ISPs the Volterra-type integral representation of the solution plays an important role. Such a representation for the CM-canonical system (1) can be taken from transformation operator for that system with a boundary condition at infinity. More precisely, if the system (1) is CM-canonical, then it has the solution in form of Volterra integral operator at infinity.

We first prove the following lemma.

Lemma 1. *Let λ be a real number and the potential $Q(x)$ satisfy the condition (3). For a bounded solution $y(x,\lambda) = \begin{bmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{bmatrix}$ of the system (1), the asymptotic relations*

$$y_1(x,\lambda) = e^{i\lambda\sigma_1 x}A(\lambda) + o(1), \;\; x \to +\infty, \tag{4}$$

$$y_2(x,\lambda) = e^{i\lambda\sigma_2 x}B(\lambda) + o(1), \;\; x \to +\infty,$$

hold, where

$$\begin{aligned} e^{i\lambda\sigma_1 x} &= diag(e^{i\lambda\xi_1 x}, e^{i\lambda\xi_2 x}), \\ e^{i\lambda\sigma_2 x} &= diag(e^{i\lambda\xi_3 x}, e^{i\lambda\xi_4 x}). \end{aligned}$$

The proof of this lemma is omitted because the Lemma 1 in [5].

Lemma 2. *Let λ be a real number and $y(x,\lambda) = \begin{bmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{bmatrix}$ be a bounded solution of the CM-canonical system (1) with the potential $Q(x)$ satisfying the condition (3). Then the representation*

$$\begin{aligned} y_1(x,\lambda) &= e^{i\lambda\sigma_1 x}A(\lambda) + \int_x^{+\infty} M_{11}(x,t)e^{i\lambda\sigma_1 t}A(\lambda)dt + \int_x^{+\infty} M_{12}(x,t)e^{i\lambda\sigma_2 t}B(\lambda)dt, \\ y_2(x,\lambda) &= e^{i\lambda\sigma_2 x}B(\lambda) + \int_x^{+\infty} M_{21}(x,t)e^{i\lambda\sigma_1 t}A(\lambda)dt + \int_x^{+\infty} M_{22}(x,t)e^{i\lambda\sigma_2 t}B(\lambda)dt, \end{aligned} \tag{5}$$

holds, where M_{11}, M_{12} and M_{21} are lower triangular 2×2 matrices

$$M_{11} = \begin{bmatrix} m_{11} & 0 \\ m_{21} & m_{22} \end{bmatrix}, \; M_{12} = \begin{bmatrix} m_{13} & 0 \\ m_{23} & m_{24} \end{bmatrix}, \; M_{21} = \begin{bmatrix} m_{31} & 0 \\ m_{41} & m_{42} \end{bmatrix}, \; M_{22} = \begin{bmatrix} m_{33} & 0 \\ 0 & m_{44} \end{bmatrix},$$

$A(\lambda)$ and $B(\lambda)$ are the vectors which are mentioned in Lemma 1. These kernels are related to the matrix potential $Q(x)$ by

$$\begin{gathered} q_{13}(x)=i\tfrac{\xi_3-\xi_1}{\xi_3}m_{13}(x,x),\ \ q_{21}(x)=i\tfrac{\xi_1-\xi_2}{\xi_1}m_{21}(x,x), \\ q_{23}(x)=i\tfrac{\xi_3-\xi_2}{\xi_3}m_{23}(x,x),\ \ q_{24}(x)=i\tfrac{\xi_4-\xi_2}{\xi_4}m_{24}(x,x), \\ q_{31}(x)=i\tfrac{\xi_1-\xi_3}{\xi_1}m_{31}(x,x),\ \ q_{41}(x)=i\tfrac{\xi_1-\xi_4}{\xi_1}m_{41}(x,x), \\ q_{42}(x)=i\tfrac{\xi_2-\xi_4}{\xi_2}m_{42}(x,x), \end{gathered} \tag{6}$$

and they have the estimate

$$\|M_{ij}(x,t)\|\le ce^{-\varepsilon(x+\theta(t-x))},\ \ t\ge x\ge 0,\quad i,j=1,2, \tag{7}$$

where c is constant,

$$\theta=\min(\theta_1,\theta_2,\theta_3,\theta_4),\ \ \theta_1=\min_{k>j}\frac{\xi_j}{\xi_j-\xi_k},\ \ \theta_2=\min_{k+j>2}\frac{\xi_{2+j}}{\xi_{2+j}-\xi_k},$$

$$\theta_3=\min_{k+j<4}\frac{\xi_j}{\xi_j-\xi_{2+k}},\ \ \theta_4=\min_{k<j}\frac{\xi_{2+j}}{\xi_{2+j}-\xi_{2+k}}.$$

Proof. Let $y(x,\lambda)$ be a bounded solution of CM-canonical system (1). Then the asymptotic relation (4) holds according to Lemma 1. It is clear that the solution $y(x,\lambda)$ of the system (1) with the boundary condition (4) satisfies the system of integral equations

$$\begin{gathered} y_1(x,\lambda)=e^{i\lambda\sigma_1 x}A(\lambda)+i\int_x^{+\infty}e^{i\lambda\sigma_1(x-s)}\left[Q_{11}(s)y_1(s,\lambda)+Q_{12}(s)y_2(s,\lambda)\right]ds, \\ y_2(x,\lambda)=e^{i\lambda\sigma_2 x}B(\lambda)+i\int_x^{+\infty}e^{i\lambda\sigma_2(x-s)}Q_{21}(s)y_1(s,\lambda)ds. \end{gathered} \tag{8}$$

By starting (5) and (8), we obtain the system of integral equation with respect to matrix kernel $m_{ij}(x,t)$ $(i,j=1,2,3,4)$:

$$m_{11}(x,t)=-i\int_x^{+\infty}q_{13}(s)m_{31}(s,t-x+s)ds,$$

$$m_{21}(x,t)=-i\frac{\xi_1}{\xi_1-\xi_2}q_{21}(\frac{\xi_1}{\xi_1-\xi_2}t-\frac{\xi_2}{\xi_1-\xi_2}x)$$

$$-i\int_x^{\frac{\xi_1 t-\xi_2 x}{\xi_1-\xi_2}}\left[q_{21}(s)m_{11}(s,t-\frac{\xi_2}{\xi_1}(x-s))+q_{23}(s)m_{31}(s,t-\frac{\xi_2}{\xi_1}(x-s))+\right.$$

$$\left.+q_{24}(s)m_{41}(s,t-\frac{\xi_2}{\xi_1}(x-s))\right]ds,$$

$$m_{22}(x,t) = -i\int\limits_x^{+\infty} q_{24}(s)m_{42}(s,t-x+s)ds,$$

$$m_{31}(x,t) = -i\frac{\xi_1}{\xi_1-\xi_3}q_{31}(\frac{\xi_1}{\xi_1-\xi_3}t-\frac{\xi_3}{\xi_1-\xi_3}x) - i\int\limits_x^{\frac{\xi_1 t-\xi_3 x}{\xi_1-\xi_3}} q_{31}(s)m_{11}(s,t-\frac{\xi_3}{\xi_1}(x-s))ds,$$

$$m_{41}(x,t) = -i\frac{\xi_1}{\xi_1-\xi_4}q_{41}(\frac{\xi_1}{\xi_1-\xi_4}t-\frac{\xi_4}{\xi_1-\xi_4}x)$$

$$-i\int\limits_x^{\frac{\xi_1 t-\xi_4 x}{\xi_1-\xi_4}} \left[q_{41}(s)m_{11}(s,t-\frac{\xi_4}{\xi_1}(x-s)) + q_{42}(s)m_{21}(s,t-\frac{\xi_4}{\xi_1}(x-s))\right] ds,$$

$$m_{42}(x,t) = -i\frac{\xi_2}{\xi_2-\xi_4}q_{42}(\frac{\xi_2}{\xi_2-\xi_4}t-\frac{\xi_4}{\xi_2-\xi_4}x) -$$

$$-i\int\limits_x^{\frac{\xi_2 t-\xi_4 x}{\xi_2-\xi_4}} q_{42}(s)m_{22}(s, \quad t-\frac{\xi_4}{\xi_2}(x-s))ds, \quad t \geq x; \tag{9}$$

$$m_{13}(x,t) = -i\frac{\xi_3}{\xi_3-\xi_1}q_{13}(\frac{\xi_3}{\xi_3-\xi_1}t-\frac{\xi_1}{\xi_3-\xi_1}x) - i\int\limits_x^{\frac{\xi_3 t-\xi_1 x}{\xi_3-\xi_1}} q_{13}(s)m_{33}(s,t-\frac{\xi_1}{\xi_3}(x-s))ds,$$

$$m_{23}(x,t) = -i\frac{\xi_3}{\xi_3-\xi_2}q_{23}(\frac{\xi_3}{\xi_3-\xi_2}t-\frac{\xi_2}{\xi_3-\xi_2}x)$$

$$-i\int\limits_x^{\frac{\xi_3 t-\xi_2 x}{\xi_3-\xi_2}} \left[q_{21}(s)m_{13}(s,t-\frac{\xi_2}{\xi_3}(x-s)) + q_{23}(s)m_{33}(s,t-\frac{\xi_2}{\xi_3}(x-s))\right] ds,$$

$$m_{24}(x,t) = -i\frac{\xi_4}{\xi_4-\xi_2}q_{24}(\frac{\xi_4}{\xi_4-\xi_2}t-\frac{\xi_2}{\xi_4-\xi_2}x) - i\int\limits_x^{\frac{\xi_4 t-\xi_2 x}{\xi_4-\xi_2}} q_{24}(s)m_{44}(s,t-\frac{\xi_2}{\xi_4}(x-s))ds,$$

$$m_{33}(x,t) = -i\int\limits_x^{+\infty} q_{31}(s)m_{13}(s,t-x+s)ds,$$

$$m_{44}(x,t) = -i\int\limits_x^{+\infty} q_{42}(s)m_{24}(s,t-x+s)ds, \quad t \geq x. \tag{10}$$

When the function $Q(x)$ satisfies the estimate (3), then there exists a unique solution of the system of integral equations (9) and (10) in the class of bounded functions, since the mentioned systems are Volterra-type integral equations. In addition, the estimation (7) in $t \geq x \geq 0$ at the equalities (6) are valid for these solutions. Conversely, if the functions $M_{ij}(x,t)$ $(i,j=1,2)$ satisfy the systems (9) and (10), then the formula (5) gives the bounded solution of the system (8) for real λ. ◄

2. 2. Scattering matrix on half-axis and its properties

From the representation (5) of the bounded solution $y(x,\lambda)=\begin{bmatrix} y_1(x,\lambda) \\ y_2(x,\lambda)\end{bmatrix}$ of the CM-canonical system (1) and the boundary condition (2), we have for real values of λ:

$$M_{21-}(\lambda)A(\lambda)+(I+M_{22+}(\lambda))\,B(\lambda)\,H\left[(I+M_{11-}(\lambda))\,A(\lambda)+M_{12+}(\lambda)\,B(\lambda)\right], \quad (11)$$

where

$$M_{k1-}(\lambda)=\int\limits_0^{+\infty} M_{k1}(0,t)\,e^{i\lambda\sigma_1 t}dt,\ \ M_{k2+}(\lambda)=\int\limits_0^{+\infty} M_{k2}(0,t)\,e^{i\lambda\sigma_2 t}dt.$$

By denoting

$$M_{H+}(\lambda)=M_{22+}(\lambda)-HM_{12+}(\lambda),$$
$$M_{H-}(\lambda)=HM_{11-}(\lambda)\,H^{-1}-M_{21-}(\lambda)\,H^{-1}.$$

The formula (11) has the form

$$[I+M_{H+}(\lambda)]\,B(\lambda)=[I+M_{H-}(\lambda)]\,HA(\lambda). \quad (12)$$

We introduce the matrix function

$$S_H(\lambda)=[I+M_{H+}(\lambda)]^{-1}\,[I+M_{H-}(\lambda)]\,,\ \lambda\in\mathbb{R}. \quad (13)$$

By analogy with the case $n=1$, we call $S_H(\lambda)$,$\lambda\in\mathbb{R}$ the scattering matrix on the half-line for the CM-canonical system with the boundary condition (2).

The following lemma is true.

Lemma 3. *The matrix functions $I+M_{H-}(\lambda)$ and $I+M_{H+}(\lambda)$ are analytic for $Im\lambda<-\frac{\theta}{\xi_1}\epsilon$ and $Im\lambda>-\frac{\theta}{\xi_4}\epsilon$, respectively. The following asymptotic relations also hold as $|\lambda|\to+\infty$:*

$$\begin{aligned}\det[I+M_{H-}(\lambda)]=1+o(1)\,,\ \ Im\lambda<-\tfrac{\theta}{\xi_1}\varepsilon,\\ \det[I+M_{H+}(\lambda)]=1+o(1)\,,\ \ Im\lambda>-\tfrac{\theta}{\xi_4}\varepsilon.\end{aligned} \quad (14)$$

Proof. From the estimation (7) of the matrix kernels $M_{ij}(x,t)$ $(i,j=1,2)$ follows that $\|M_{ij}(0,t)\|\le\widetilde{C}e^{-\theta t}$. It means that the matrix functions $M_{k1-}(\lambda)=\int\limits_0^{+\infty}M_{k1}(0,t)\,e^{i\lambda\sigma_1 t}dt$ and $M_{k2+}(\lambda)=\int\limits_0^{+\infty}M_{k2}(0,t)\,e^{i\lambda\sigma_2 t}dt$ are analytic for $Im\lambda<-\frac{\theta}{\xi_1}\varepsilon$ and $Im\lambda>-\frac{\theta}{\xi_4}\varepsilon$ respectively, and tend to zero as $|\lambda|\to+\infty$ in the domains of analyticity. ◄

According to Lemma 3, the functions

$$\det\left[I + M_{H-}(\lambda)\right] \text{ and } \det\left[I + M_{H+}(\lambda)\right]$$

have a finite number of zeros. We define $\varepsilon_0 > 0$ by the relation

$$\varepsilon_0 = \min\left\{\varepsilon_1, -\frac{\theta}{\xi_1}\varepsilon, -\frac{\theta}{\xi_4}\varepsilon\right\},$$

where ε_1 is the distance from the real axis to the non-real zeros of the functions $\det\left[I + M_{H+}(\lambda)\right]$ and $\det\left[I + M_{H-}(\lambda)\right]$. Then the relations

$$\det\left[I + M_{H+}(\lambda)\right] \neq 0, \ \det\left[I + M_{H-}(\lambda)\right] \neq 0 \tag{15}$$

hold for $0 < |Im\lambda| < \varepsilon_0$.

The next theorem about the properties of scattering matrix $S_H(\lambda)$ follows from (15) and Lemma 3.

Theorem 1. *The matrix functions $S_H(\lambda)$ and $S_H^{-1}(\lambda)$ are meromorphic in the strip $|Im\lambda| < \varepsilon_0$, and they have no non-real poles and as $|\lambda| \to +\infty$*

$$S_H(\lambda) = I + o(1), \ S_H^{-1}(\lambda) = I + o(1).$$

3. Inverse Scattering Problem on the Half-Axis

3. 1. Matrix Riemann-Hilbert problems

The inverse scattering problem (ISP) on the half-line for the system (1) consists in recovering the matrix potential $Q(x)$ from a given matrix function $S_H(\lambda)$. The exact solvable examples show that one scattering problem is not enough for the unique restoration of the potential from two scattering matrices which correspond the different boundary conditions in form of (2).

Let $S_{H_1}(\lambda)$ and $S_{H_2}(\lambda)$ be two scattering matrices on the half line for the CM-canonical system (1), where

$$\det(H_1 - H_2) \neq 0. \tag{16}$$

By the definition of scattering matrix on the half-line we get

$$\left[I + M_{H_k+}(\lambda)\right] S_{H_k}(\lambda) = \left[I + M_{H_k-}(\lambda)\right], \ \lambda \in \mathbb{R}, \ k = 1, 2, \tag{17}$$

where

$$M_{H_k+}(\lambda) = M_{22+}(\lambda) - H_k M_{12+}(\lambda),$$
$$M_{H_k-}(\lambda) = H_k M_{11-}(\lambda) H_k^{-1} - M_{21-}(\lambda) H_k^{-1}. \tag{18}$$

If these matrices are known, then relations (17) are matrix Riemann-Hilbert problems, where the contour is real line, normalization is canonical, and all the partial indices are zero. We will call these problems as Riemann-Hilbert problems of the ISP for the CM-canonical system (1) on the half-line.

The ISP for the system (1) on the half-line closely is related with the ISP on the whole line. For this reason, we introduce the matrix $P(\lambda)$, $\lambda \in \mathbb{R}$ for the bounded solutions $y(x,\lambda)$ of the CM-canonical system as follows

$$P(\lambda)\begin{bmatrix} A(\lambda) \\ B(\lambda) \end{bmatrix} = \begin{bmatrix} y_1(0,\lambda) \\ y_2(0,\lambda) \end{bmatrix}, \ \lambda \in \mathbb{R}. \tag{19}$$

By the uniqueness of the solution of the Cauchy problem at the point $x = 0$, for the system (1) we have $y(x,\lambda) = 0$ when $y_1(0,\lambda) = y_2(0,\lambda) = 0$. Then $A(\lambda) = B(\lambda) = 0$, by the formula (5). It means that the matrix $P(\lambda)$ is invertible. We will call the matrix $\Pi(\lambda) = P^{-1}(\lambda)$ the transmission matrix.

Now, consider the system of ODE on the whole line

$$-i\frac{dy}{dx} + \widetilde{Q}(x)y = \lambda\sigma y, \tag{20}$$

with the potential $\widetilde{Q}(x) = \begin{cases} Q(x), \ x \geq 0 \\ 0, \ x < 0 \end{cases}$.

By comparing the definition of the transmission matrix $\Pi(\lambda)$ (19) with the definition of scattering matrix on the whole line [4], [12] (see also [15]), it is easily seen that matrix $\Pi(\lambda)$ is the scattering matrix for the system (20) on the whole line. The operator transforming the potential $\widetilde{Q}(x)$ of the system (20) to its scattering matrix $\Pi(\lambda)$:

$$L_1(\mathbb{R}) \ni \widetilde{Q}(x) \to \Pi(\lambda)$$

is continuous [14]. The restriction of this operator to small neighborhood of zero in $L_1(\mathbb{R})$ is one-to-one transformation (Lemma 3.3 in [14]). This fact implies the uniqueness of the ISP for the system (20) in the whole line in the case of "small" potential. The uniqueness of the ISP for the system (20) with the finite potential is shown in [13]. Another scattering data for the ISP on the whole line for the system of first-order ODE 's are given in [11].

Thus we obtain the following result about the ISP for the CM-canonical system (1) on the half-line.

Theorem 2. *Let $S_{H_1}(\lambda)$ and $S_{H_2}(\lambda)$ be two scattering matrices on the half-line for the CM-canonical system (1) with potential $Q(x)$ satisfying the condition (3). Let the matrices H_1 and H_2 satisfy the condition (16). Then, the matrix $Q(x)$ is uniquely determined from matrices $S_{H_1}(\lambda)$ and $S_{H_2}(\lambda)$ when the Riemann-Hilbert problems (17) are uniquely solvable.*

Proof. First, let us show that the transmission matrix $\Pi(\lambda)$ is uniquely determined from $S_{H_1}(\lambda)$ and $S_{H_2}(\lambda)$ when the Riemann-Hilbert problems (17) are uniquely solvable. Applying the representation (7) to definition (19) of $P(\lambda)$, it can be easily seen the following block structure of $P(\lambda)$:

$$P(\lambda) = \begin{bmatrix} I + M_{11-}(\lambda) & M_{12+}(\lambda) \\ M_{21-}(\lambda) & I + M_{22+}(\lambda) \end{bmatrix}.$$

When the Riemann-Hilbert problems (17) are uniquely solvable, i.e. if matrices $M_{H_k+}(\lambda)$ and $M_{H_k-}(\lambda)$ are uniquely determined from $S_{H_k}(\lambda)$ $(k=1,2)$ in domain which they are analytical, then matrices $M_{11-}(\lambda), M_{21-}(\lambda), M_{12+}(\lambda)$ and $M_{22+}(\lambda)$ are uniquely expressed from $M_{H_k+}(\lambda)$ and $M_{H_k-}(\lambda)$ $(k=1,2)$ by formula (18) and condition (16):

$$M_{12+}(\lambda) = (H_1 - H_2)^{-1}\left[M_{H_2+}(\lambda) - M_{H_1+}(\lambda)\right],$$

$$M_{11-}(\lambda) = (H_1 - H_2)^{-1}\left[M_{H_1-}(\lambda) H_1 - M_{H_2-}(\lambda) H_2\right],$$

$$M_{22+}(\lambda) = M_{H_1+}(\lambda) + H_1 M_{12+}(\lambda) = M_{H_2+}(\lambda) + H_2 M_{12+}(\lambda),$$

$$M_{21-}(\lambda) = H_1 M_{11-}(\lambda) - M_{H_1-}(\lambda) H_1 = H_2 M_{11-}(\lambda) - M_{H_2-}(\lambda) H_1.$$

As already is known that the transmission matrix $\Pi(\lambda)$ is closely related with the scattering matrix for the system of ordinary differential equation on the whole axis. Indeed, if we take the coefficients zero for $x<0$, then we obtain the system (20) on the whole axis and the transmission matrix for the system (1) coincides with the scattering matrix on the whole line for the system (20). The uniqueness of the ISP for system (20) with the potential (3) implies that the potential $Q(x)$ is uniquely determined from $\Pi(\lambda)$. ◀

The matrix Riemann-Hilbert problems which is mentioned in Theorem 2 are in the form of

$$\left[I + M_{H+}(\lambda)\right] S_H(\lambda) = \left[I + M_{H-}(\lambda)\right], \ \lambda \in \mathbb{R}$$

with the boundary conditions (14), that is,

$$M_{H\pm}(\infty) = 0$$

where $M_{H+}(\lambda)$ and $M_{H-}(\lambda)$ are $n \times n$ matrices which are analytic in upper and lower complex $\lambda-$plane and the components of $M_{H\pm}(\lambda)$ belong to set $Ç^{\pm}$, which denotes the set consisting of functions of the form $\int_0^{+\infty} f(x) e^{\pm i\lambda x} dx$, $\lambda \in \mathbb{R}$, where $f(x)$ is continuous and $f(x) \in L_1$, that is, $\int_0^{\pm\infty} |f(x)|\, dx$ exists.

It is shown that the determinants

$$\det\left[I + M_{H+}(\lambda)\right] \text{ and } \det\left[I + M_{H-}(\lambda)\right]$$

have a finite number of zeros in their domains of analyticity. If the matrix functions $I + M_{H+}(\lambda)$ and $I + M_{H-}(\lambda)$ degenerate nowhere in their domains of analyticity, i.e. $\det\left[I + M_{H+}(\lambda)\right] \neq 0$, $\det\left[I + M_{H-}(\lambda)\right] \neq 0$, then the Riemann-Hilbert problem is said to be regular. The solution of a regular Riemann-Hilbert problem under canonical normalization of unique ([11], p. 155). By matrix analogue of the Wiener theorem [12], under the condition $\det\left[I + M_{H+}(\lambda)\right] \neq 0$, there exists a matrix $B_{H+}(\lambda)$ with the components belonging to $Ç^{+}$, such that $\left[I + M_{H+}(\lambda)\right]^{-1} = I + B_{H+}(\lambda)$. Thus, in the regular case the matrix RH problem reduces to left canonical factorization problem of the matrix $S_H(\lambda)$ ([4], p. 31-37). Because the factorization factors are uniquely determined in the left canonical factorization problem ([4], p. 35-37), the matrices $M_{H+}(\lambda)$ and $M_{H-}(\lambda)$ are uniquely determined by $S_H(\lambda)$.

Therefore, we obtain the following corollary of Theorem 2.

Corollary. *If the Riemann-Hilbert problems (17) are regular, then the matrix potential* $Q(x)$ *of the CM-canonical system (1) is uniquely determined from its scattering matrices* $S_{H_1}(\lambda)$ *and* $S_{H_2}(\lambda)$ *with* $\det(H_1 - H_2) \neq 0$.

3. 2. Examples

Consider the following CM-canonical system on the half-axis $x \geq 0$

$$\begin{cases} -iy_{1,x} + q_{13}y_3 = \lambda\xi_1 y_1, \\ -iy_{2,x} + q_{21}y_1 + q_{23}y_3 = \lambda\xi_2 y_2, \\ -iy_{3,x} = \lambda\xi_3 y_3, \\ -iy_{4,x} + q_{41}y_1 = \lambda\xi_4 y_4, \end{cases} \tag{21}$$

where $\xi_1 > \xi_2 > 0 > \xi_3 = \xi_4$.

Consider the system (21) under the boundary condition

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = H_1 \begin{bmatrix} y_3(0) \\ y_4(0) \end{bmatrix}, \tag{22}$$

where $H_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

It is easy to check that the system (21) with the asymptotic

$$y_k(x) = a_k e^{i\lambda\xi_k x} + o(1),\ x \to +\infty,$$
$$y_{k+2} = a_{k+2} e^{i\lambda\xi_{k+2}x} + o(1),\ x \to +\infty,\ k = 1, 2$$

has the solution

$$y_1 = a_1 e^{i\lambda\xi_1 x} + ia_3 \int_x^{+\infty} q_{13}(s) e^{i\lambda(\xi_3 - \xi_1)s} ds e^{i\lambda\xi_1 x},$$

$$y_2 = a_2 e^{i\lambda\xi_2 x} + ia_1 \int_x^{+\infty} q_{21}(s) e^{i\lambda(\xi_1 - \xi_2)s} ds e^{i\lambda\xi_2 x}$$

$$-a_3 \int_x^{+\infty} q_{21}(\tau) \int_\tau^{+\infty} q_{13}(s) e^{i\lambda(\xi_3 - \xi_1)s} ds e^{i\lambda(\xi_1 - \xi_2)\tau} d\tau e^{i\lambda\xi_2 x} + ia_3 \int_x^{+\infty} q_{23}(s) e^{i\lambda(\xi_3 - \xi_2)s} ds e^{i\lambda\xi_2 x},$$

$$y_3 = a_3 e^{i\lambda\xi_3 x},$$

$$y_4 = a_4 e^{i\lambda\xi_4 x} + ia_1 \int_x^{+\infty} q_{41}(s) e^{i\lambda(\xi_1 - \xi_4)s} ds e^{i\lambda\xi_4 x}$$

$$-a_3\int\limits_x^{+\infty} q_{41}(\tau)\int\limits_\tau^{+\infty} q_{13}(s)e^{i\lambda(\xi_3-\xi_1)s}dse^{i\lambda(\xi_1-\xi_4)\tau}d\tau e^{i\lambda\xi_4 x}.$$

Taking into account the boundary conditions (22), we obtain the following relations between column vectors $A=(a_1,a_2)$ and $b=(a_3,a_4)$:

$$B=S_{H_1}(\lambda)\,H_1A,$$

where

$$S_{H_1}(\lambda)=\begin{bmatrix} s_{11}(\lambda) & s_{12}(\lambda)\\ s_{21}(\lambda) & s_{22}(\lambda)\end{bmatrix}, \tag{23}$$

where

$$s_{11}(\lambda)=\left(1-i\int\limits_0^{+\infty} q_{13}(s)e^{i\lambda(\xi_3-\xi_1)s}ds\right)^{-1},\ s_{12}(\lambda)=0,\ s_{22}(\lambda)=1,$$

$$s_{21}(\lambda)=i\int\limits_0^{+\infty} q_{21}(s)e^{i\lambda(\xi_1-\xi_2)s}ds-i\int\limits_0^{+\infty} q_{41}(s)e^{i\lambda(\xi_1-\xi_4)s}ds$$

$$+\left(-1-\int\limits_0^{+\infty} q_{21}(\tau)\int\limits_\tau^{+\infty} q_{13}(s)e^{i\lambda(\xi_3-\xi_1)s}dse^{i\lambda(\xi_1-\xi_2)\tau}d\tau+i\int\limits_0^{+\infty} q_{23}(s)e^{i\lambda(\xi_3-\xi_2)s}ds+\right.$$

$$\left.+\int\limits_0^{+\infty} q_{41}(\tau)\int\limits_\tau^{+\infty} q_{13}(s)e^{i\lambda(\xi_3-\xi_1)s}dse^{i\lambda(\xi_1-\xi_4)\tau}d\tau\right)s_{11}(\lambda).$$

The coefficient q_{13} can be easily found from the equation $s_{11}(\lambda)$:

$$\frac{i}{\xi_3-\xi_1}q_{13}\left(\frac{\tau}{\xi_3-\xi_1}\right)=\frac{1}{2\pi i}\int\limits_{-\infty+ix}^{+\infty+ix}\left(S_{11}^{-1}-1\right)e^{i\lambda s}ds,\ x<\frac{\varepsilon}{\xi_3-\xi_1}.$$

Let us denote

$$\int\limits_\tau^{+\infty} q_{13}(s)e^{i\lambda(\xi_3-\xi_1)s}ds=P(\tau).$$

Now, consider the ISP for the system (21), i.e. the problem of recovering the coefficients q_{13}, q_{21}, q_{23}, q_{41} of the system (1) from its scattering matrix $S_{H_1}(\lambda)$ on the half-axis. As is shown, the coefficients of the system (21) and scattering matrix $S_{H_1}(\lambda)$ are related with the relation (23).

Denoting

$$C_+(\lambda)=\int\limits_0^{+\infty} c_+(\tau)e^{i\lambda\tau}d\tau,$$

$$c_+(\tau) = i\left[\frac{1}{\xi_1-\xi_2}q_{21}\left(\frac{\tau}{\xi_1-\xi_2}\right) - \frac{1}{\xi_1-\xi_4}q_{41}\left(\frac{\tau}{\xi_1-\xi_4}\right)\right]$$

and

$$C_-\left(\lambda\right) = \left(-1 + c_-(\tau)\right)s_{11}(\lambda),$$

$$c_-(\tau) = -\frac{1}{\xi_1-\xi_2}q_{21}(\frac{\tau}{\xi_1-\xi_2})p\left(\frac{\tau}{\xi_1-\xi_2}\right) + \frac{i}{\xi_3-\xi_2}q_{23}(\frac{\tau}{\xi_3-\xi_2}) +$$
$$+\frac{1}{\xi_1-\xi_4}q_{41}(\frac{\tau}{\xi_1-\xi_4})p\left(\frac{\tau}{\xi_1-\xi_4}\right)$$

the formula (23) can be written in following form

$$s_{21}\left(\lambda\right) = C_+\left(\lambda\right) + C_-\left(\lambda\right). \tag{24}$$

Then we conclude that the function $C_+\left(\lambda\right)$ is analytic in half-plane $Im\lambda > -\frac{\varepsilon}{\xi_1-\xi_2}$, and $C_-\left(\lambda\right)$ is analytic in half-plane $Im\lambda < -\frac{\varepsilon}{\xi_3-\xi_2}$. In addition, the functions $C_\pm\left(\lambda\right)$ tend to zero as $Im\lambda \to \infty$ in the domains of analyticity.

Thus, the ISP for the system (21) on the half-axis can be solvable by Wiener-Hopf method. Actually, it is possible to determine the functions $C_-\left(\lambda\right)$ and $C_+\left(\lambda\right)$ of a complex variable λ, which are analytic, respectively, in the half-plane $Im\lambda < -\frac{\varepsilon}{\xi_3-\xi_2}$ and $Im\lambda > -\frac{\varepsilon}{\xi_1-\xi_2}$, tend to zero as $Im\lambda \to \infty$ in both domains of analyticity and satisfy in the strip $-\frac{\varepsilon}{\xi_1-\xi_2} < Im\lambda < -\frac{\varepsilon}{\xi_3-\xi_2}$ the equation (24).

Since the function $s_{21}(\lambda)$ is analytic in the strip $-\frac{\varepsilon}{\xi_1-\xi_2} < Im\lambda < -\frac{\varepsilon}{\xi_3-\xi_2}$, then the following representation is possible in the given strip

$$s_{21}\left(\lambda\right) = s_{21+}\left(\lambda\right) + s_{21-}\left(\lambda\right), \tag{25}$$

when $s_{21}(\lambda)$ tends uniformly to zero as $|\lambda| \to +\infty$ in this strip (see [15], p. 293). Here the functions $s_{21+}\left(\lambda\right)$ and $s_{21-}\left(\lambda\right)$ are analytic in $Im\lambda > -\frac{\varepsilon}{\xi_1-\xi_2}$ and $Im\lambda < -\frac{\varepsilon}{\xi_3-\xi_2}$, respectively.

From (24) and (25) we get the following formula

$$C_+\left(\lambda\right) - s_{21+}\left(\lambda\right) = -C_-\left(\lambda\right) + s_{21-}\left(\lambda\right) \tag{26}$$

in the strip $-\frac{\varepsilon}{\xi_1-\xi_2} < Im\lambda < -\frac{\varepsilon}{\xi_3-\xi_2}$.

The left side of (26) is a function which is analytic in half-plane $Im\lambda > -\frac{\varepsilon}{\xi_1-\xi_2}$ and the right side of (26) is analytic in half-plane $Im\lambda < -\frac{\varepsilon}{\xi_3-\xi_2}$. From the equality of these functions in the strip $-\frac{\varepsilon}{\xi_1-\xi_2} < Im\lambda < -\frac{\varepsilon}{\xi_3-\xi_2}$ it follows that there exist a unique entire function $P\left(\lambda\right)$ coinciding, respectively, with the left and right sides of (26) in the domains of their analyticity. Since the function $C_\pm\left(\lambda\right) - S_{21\pm}$ tends zero as $Im\lambda \to \infty$ in the domain of analyticity, then $P\left(\lambda\right) = 0$. So that $C_-\left(\lambda\right) = s_{21-}\left(\lambda\right)$ and $C_+\left(\lambda\right) = s_{21+}\left(\lambda\right)$. In means that the functions $s_{21-}\left(\lambda\right)$ and $s_{21+}\left(\lambda\right)$ are Laplace transformations of the functions $c_-(\tau)$ and $c_+(\tau)$ respectively. Then

$$c_+\left(\tau\right) = \frac{1}{2\pi i}\int\limits_{-\infty+ix}^{+\infty+ix} s_{21+}e^{i\lambda\tau}d\tau,\ x > -\frac{\varepsilon}{\xi_1-\xi_2},$$

$$c_{-}(\tau)=\frac{1}{2\pi i}\int\limits_{-\infty+ix}^{+\infty+ix}(1+s_{21-})\,s_{11}^{-1}e^{i\lambda\tau}d\tau,\ x<-\frac{\varepsilon}{\xi_3-\xi_2}.$$

Thus we obtain the following system of linear algebraic equations with respect to q_{13}, q_{21}, q_{23}, q_{41} :

$$\frac{1}{\xi_1-\xi_2}q_{21}(\frac{\tau}{\xi_1-\xi_2})P\left(\frac{\tau}{\xi_1-\xi_2}\right)+\frac{1}{\xi_3-\xi_2}q_{23}(\frac{\tau}{\xi_3=\xi_2})-$$

$$-\frac{1}{\xi_1-\xi_4}q_{41}(\frac{\tau}{\xi_1-\xi_4})P\left(\frac{\tau}{\xi_1-\xi_4}\right)=-ic_{-}(\tau),$$

$$\frac{1}{\xi_1-\xi_2}q_{21}\left(\frac{\tau}{\xi_1-\xi_2}\right)-\frac{1}{\xi_1-\xi_4}q_{41}\left(\frac{\tau}{\xi_1-\xi_4}\right)=-ic_{+}(\tau).\tag{27}$$

This example shows that one scattering operator is nor enough for unique restoration of potential

Let us consider the system (21) under the another boundary condition

$$\begin{bmatrix}y_1(0)\\y_2(0)\end{bmatrix}=H_2\begin{bmatrix}y_3(0)\\y_4(0)\end{bmatrix},$$

where $H_2=\begin{bmatrix}2\ 0\\0\ 1\end{bmatrix}$.

The components of the scattering matrix $S_{H_2}(\lambda)=\begin{bmatrix}r_{11}(\lambda)\ r_{12}(\lambda)\\r_{21}(\lambda)\ r_{22}(\lambda)\end{bmatrix}$ are as follows:

$$r_{11}(\lambda)=\left(2-i\int\limits_0^{+\infty}q_{13}(s)e^{i\lambda(\xi_3-\xi_1)s}ds\right)^{-1},\ r_{12}(\lambda)=0,\ r_{22}(\lambda)=1,$$

$$r_{21}(\lambda)=i\int\limits_0^{+\infty}q_{21}(s)e^{i\lambda(\xi_1-\xi_2)s}ds-i\int\limits_0^{+\infty}q_{41}(s)e^{i\lambda(\xi_1-\xi_4)s}ds$$

$$+\left(-\int\limits_0^{+\infty}q_{21}(\tau)\int\limits_\tau^{+\infty}q_{13}(s)e^{i\lambda(\xi_3-\xi_1)s}dse^{i\lambda(\xi_1-\xi_2)\tau}d\tau+i\int\limits_0^{+\infty}q_{23}(s)e^{i\lambda(\xi_3-\xi_2)s}ds+\right.$$

$$\left.+\int\limits_0^{+\infty}q_{41}(\tau)\int\limits_\tau^{+\infty}q_{13}(s)e^{i\lambda(\xi_3-\xi_1)s}dse^{i\lambda(\xi_1-\xi_4)\tau}d\tau\right)r_{11}(\lambda).$$

The coefficient q_{13} can be easily found from the equation $r_{11}(\lambda)$:

$$\frac{i}{\xi_3-\xi_1}q_{13}\left(\frac{\tau}{\xi_3-\xi_1}\right)=\frac{1}{2\pi i}\int\limits_{-\infty+ix}^{+\infty+ix}\left(r_{11}^{-1}-2\right)e^{i\lambda s}ds,\ x<\frac{\varepsilon}{\xi_3-\xi_1}.$$

Let us denote

$$\int_{\tau}^{+\infty} q_{13}(s)e^{i\lambda(\xi_3-\xi_1)s}ds = G(\tau).$$

With similar arguments as in the previous example, the following equations are obtained

$$C_+(\lambda) = \int_0^{+\infty} c_+(\tau)e^{i\lambda\tau}d\tau,$$

$$c_+(\tau) = i\left[\frac{1}{\xi_1-\xi_2}q_{21}\left(\frac{\tau}{\xi_1-\xi_2}\right) - \frac{1}{\xi_1-\xi_4}q_{41}\left(\frac{\tau}{\xi_1-\xi_4}\right)\right]$$

and

$$C_-(\lambda) = c_-(\tau)r_{11}(\lambda),$$

$$c_-(\tau) = -\frac{1}{\xi_1-\xi_2}q_{21}(\frac{\tau}{\xi_1-\xi_2})G\left(\frac{\tau}{\xi_1-\xi_2}\right) + \frac{i}{\xi_3-\xi_2}q_{23}(\frac{\tau}{\xi_3-\xi_2})+$$

$$+\frac{1}{\xi_1-\xi_4}q_{41}(\frac{\tau}{\xi_1-\xi_4})G\left(\frac{\tau}{\xi_1-\xi_4}\right).$$

Then

$$c_+(\tau) = \frac{1}{2\pi i}\int_{-\infty+ix}^{+\infty+ix} r_{21+}(\lambda)\, e^{i\lambda\tau}d\tau,\ x > -\frac{\varepsilon}{\xi_1-\xi_2},$$

$$c_-(\tau) = \frac{1}{2\pi i}\int_{-\infty+ix}^{+\infty+ix} r_{21-}(\lambda)\, r_{11}^{-1}(\lambda)\, e^{i\lambda\tau}d\tau,\ x < -\frac{\varepsilon}{\xi_3-\xi_2}.$$

Thus we obtain the following system of linear algebraic equations with respect to q_{13}, q_{21}, q_{23}, q_{41} :

$$\frac{i}{\xi_1-\xi_2}q_{21}(\frac{\tau}{\xi_1-\xi_2})G\left(\frac{\tau}{\xi_1-\xi_2}\right) + \frac{1}{\xi_3-\xi_2}q_{23}(\frac{\tau}{\xi_3-\xi_2})-$$

$$-\frac{i}{\xi_1-\xi_4}q_{41}(\frac{\tau}{\xi_1-\xi_4})G\left(\frac{\tau}{\xi_1-\xi_4}\right) = -ic_-(\tau),$$

$$\frac{1}{\xi_1-\xi_2}q_{21}\left(\frac{\tau}{\xi_1-\xi_2}\right) - \frac{1}{\xi_1-\xi_4}q_{41}\left(\frac{\tau}{\xi_1-\xi_4}\right) = -ic_+(\tau). \tag{28}$$

It is easy to see that, the uniqueness of the solution of the system (27), (28) is violated if $\det(H_1 - H_2) = 0$. In this case, the ISP for the system (21) has also not a unique solution.

4. Conclusion

The paper investigates the inverse scattering problem (ISP) for the first-order CM-canonical differential system of size 2n on the half-line, considering a general boundary condition. It begins by introducing Jost-type solutions in a conventional manner, outlining some properties of the scattering matrix, and subsequently delves into the problem of reconstructing the potential from the scattering matrix. The primary theorem (Theorem 2) asserts that the potential can be uniquely determined by two scattering matrices pertaining to the system subject to two distinct boundary conditions, namely $y_2(0) = H_i y_1(0)$ for $i = 1, 2$, provided that $H_1 - H_2$ is nonsingular. The paper presents examples demonstrating that

(a) a single scattering matrix is inadequate for unambiguous reconstruction;

(b) the condition $\det(H_1 - H_2) \neq 0$ is indispensable.

The another criteria in terms of boundary transmission matrix is expected for more general first order ODEs, which suggests a line for further investigation.

References

1. Chadan K., Sabatier P.C. *Inverse Problems in Quantum Scattering Theory.* Springer-Verlag, New York, 1989.
2. Gasymov M.G. The inverse scattering problem for a system of Dirac equations of order $2n$. *Soviet Physics Dokl.*, 1966, **11**, pp. 676-678.
3. Gasymov M.G. The inverse scattering problem for a system of Dirac equations of order $2n$. *Trans. Moscow Math. Soc.*, 1968, **19**, pp. 41-119.
4. Gohberg I.C., Krein M.G. Systems of integral equations on a half line with kernels depending on the difference of arguments. *Amer. Math. Soc. Transl. (2)*, 1960, **14**, pp. 217-287.
5. Ismailov M.I. Inverse scattering on the half-line for a first-order system with a general boundary condition. *Ann. Henri Poincaré*, 2017, **18** (8), pp. 2621-2639.
6. Ismailov M.I. Inverse scattering problem for linear system of four-wave interaction problem on the half-line with a general boundary condition. *J. Math. Phys. Anal. Geom.*, 2023, **19** (2), pp. 443-455.
7. Lyantse V.È. An analog of the inverse problem of scattering theory for a nonselfadjoint operator. *Math. USSR-Sb.*, 1967, **1** (4), pp. 485-504.
8. Malamud M.M. Questions of uniqueness in inverse problems for systems of differential equations on a finite interval. *Trans. Moscow Math. Soc.*, 1999, **60**, pp. 173-224.
9. Naimark M.A. Investigation of the spectrum and the expansion in eigenfunctions of a non-selfadjoint differential operator of the second order on a semi-axis. *Amer. Math. Soc. Transl. (2)*, 1960, **16**, pp. 103-193.
10. Nizhnik L.P., Vu P.L. An inverse scattering problem on the semi-axis with a non-selfadjoint potential matrix. *Ukr. Math. J.*, 1975, **26**, pp. 384-398.
11. Novikov S., Manakov S.V., Pitaevskii L.P., Zakharov V.E. *Theory of Solitons. The Inverse Scattering Method*. Plenum, New York, 1984.
12. Paley R.C., Wiener N. *Fourier Transforms in the Complex Domain.* Amer. Math. Soc., New York, 1934.

13. Shabat A.B. Inverse-scattering problem for a system of differential equations. *Funct. Anal. Appl.*, 1975, **9** (3), pp. 244-247.
14. Shabat A.B. An inverse scattering problem. *Differ. Uravn.*, 1979, **15** (10), pp. 1824-1834 (in Russian).
15. Sveshnikov A.G., Tikhonov A.N. *The Theory of Functions of a Complex Variable.* Mir, Moscow, 1971.