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NECESSARY CONDITIONS FOR THE EXTREMUM IN NON-SMOOTH PROBLEMS OF VARIATIONAL CALCULUS WITH DELAY

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In memory of M. G. Gasymov on his 85th birthday

Abstract. *In this article, with the help of a special variation of the Weierstrass type, the necessary condition for a minimum is obtained. As consequences, the analogue of the Weierstrass condition and its local modification, as well as the analogue of the Legendre condition are obtained.*

Keywords: calculus of variations, strong (weak) local minimum, necessary conditions, non-smooth problem with delay

Mathematics Subject Classification (2020): 49J52, 49K99

1. Introduction

As is known (see, for example, [5]), extremal problems described with a delay reflect many problems of real processes. Considering this, in this paper, in contrast to [8], a variational problem with a delay in the following form is considered:

$$S\left(x\left(\cdot\right)\right)=\int_{t_0}^{t_1} L\left(t, x\left(t\right), x\left(t-h\right), \dot{x}\left(t\right), \dot{x}\left(t-h\right)\right) dt \to \min_{x(\cdot)}, \tag{1}$$

$$x\left(t\right)=\varphi\left(t\right), \quad t\in\left[t_0-h,\ t_0\right],\ x\left(t_1\right)=x^{*}\in\mathbb{R}^n. \tag{2}$$

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Here $\mathbb{R}^n-$ is n- dimensional Euclidean space, x^*, t_0, t_1- given points, $h = \text{const} > 0$, $t_1 - t_0 > h$, $x(t) \in KC^1\left(\hat{I}, \mathbb{R}^n\right)$, where $\hat{I} = [t_0 - h, \, t_1]$ and $KC^1\left(\hat{I}, \mathbb{R}^n\right)$ is a class of piecewise-smooth function [8]. Then, the function $L(t, x, y, \dot{x}, \dot{y})$: $I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} := (-\infty, +\infty)$ is continuous in the totality of variables, and $L(t, x, y, \dot{x}, \dot{y}) = 0$, with variables $(t, x, y, \dot{x}, \dot{y}) \in (t_1, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, and the function $\varphi(t) \in C^1([t_0 - h, t_0], \mathbb{R}^n)$, where $y = y(t) = x(t-h)$, $\dot{y} = \dot{y}(t) = \dot{x}(t-h)$, $t \in I := [t_0, t_1]$.

The functions $x(\cdot) \in KC^1\left(\hat{I}, \mathbb{R}^n\right)$, satisfying the boundary conditions (2) are said to be admissible.

Recall the notions that are introduced, for example, in [13]. An admissible function $\bar{x}(\cdot)$ is said to be a strong (weak) local minimum in the problem (1), (2), if there exists a number $\bar{\delta} > 0$ $\left(\hat{\delta} > 0\right)$ such that the inequality $S(x(\cdot)) \geq S(\bar{x}(\cdot))$ holds for all admissible functions $\bar{x}(\cdot)$, for which

$$\|x(\cdot) - \overline{x}(\cdot)\|_{C\left(\hat{I}, \mathbb{R}^n\right)} = \overline{\delta}\Big(max\left\{\|x(\cdot) - \overline{x}(\cdot)\|_{C\left(\hat{I}, \mathbb{R}^n\right)}, \left\|\dot{x}(\cdot) - \dot{\overline{x}}(\cdot)\right\|_{L_\infty\left(\hat{I}, \mathbb{R}^n\right)}\right\} = \widehat{\delta}\Big).$$

In these cases, we will say that the admissible function $\overline{x}(\cdot)$ affords a strong (weak) local minimum in the problem (1), (2) with $\overline{\delta}$ $\left(\widehat{\delta}\right)$ - neighborhood.

Let us also recall [1], [3] while obtaining necessary conditions for classical calculus of variation, as a rule, at least it is assumed that the integrant $L(\cdot)$ is continuously-differentiable in some domain U of space $\mathbb{R}^{4n+1}$.

The variational problem is said to be non-smooth if the integrant $L(\cdot)$ of this problem is non-differentiable with respect to at least one of its arguments.

Non-smooth extremal problems arise in various problems of nonlinear mechanics, economic planning theory, computer science, theory of optimal processes, etc.

It should be noted that the theory of optimal control arose from the needs of modern science and technology; in its content it belongs to the class of non-smooth extremal problems, for example, due to non-functional restrictions on control variables.

Analyzing early and more recent published studies [3], [4], [8], [9] and etc. dedicated to smooth and non-smooth variational problems, we can note that the qualitative research of the problem (1), (2) still remains an important task today. Naturally, it becomes more important to obtain results for the problem (1) , (2) that do not follow as the consequence of the general theory of optimal control (see for example [6], [7], [10]-[12]). The latter proposal is implemented in this work.

Our goal was to obtain an analogue of the fundamental theorem of work [8] for the problem (1), (2).

2. Necessary Conditions for a Minimum in the Problem (1), (2)

Let $\bar{x}(\cdot)$ be some admissible function in the problem (1), (2) . In addition, let I_1, I_2 and I_3 be the sets of break points of the functions $\dot{\overline{x}}(t), t \in I$, $\dot{\overline{x}}(t-h)$, $t \in I$ and $\dot{\overline{x}}(t+h)$, $t \in [t_0, t_1 - h]$, respectively. Since $\overline{x}(\cdot) \in KC^1\left(\hat{I}, \mathbb{R}^n\right)$, it is clear that, the set $I_1 \cup I_2 \cup I_3$ is finite. Consider the set $\overline{I} = I \backslash (I_1 \cup I_2 \cup I_3)$ and define the following function corresponding to the integrant $L(t, x, y, \dot{x}, \dot{y})$ and the function $\overline{x}(\cdot)$:

$$Q(t, \lambda, \xi; \dot{x}(\cdot)) = \lambda \left[\Delta_{\dot{x}} \overline{L}(t, \xi) + \Delta_{\dot{y}} \overline{L}(t+h, \xi)\right] +$$

$$(1-\lambda) \left[\Delta_{\dot{x}} \overline{L}\left(t, \frac{\lambda}{\lambda-1}\xi\right) + \Delta_{\dot{y}} \overline{L}\left(t+h, \frac{\lambda}{\lambda-1}\xi\right)\right], \quad t \in [t_0, t_1 - h) \cap \overline{I}, \qquad (3)$$

$$Q(t, \lambda, \xi; \dot{x}(\cdot)) = \lambda \Delta_{\dot{x}} \overline{L}(t, \xi) + (1-\lambda) \Delta_{\dot{x}} \overline{L}\left(t, \frac{\lambda}{\lambda-1}\xi\right), t \in [t_1 - h, t_1] \cap \overline{I},$$

where $\lambda \in [0, 1)$, $\xi \in \mathbb{R}^n$,

$$\Delta_{\dot{x}} \overline{L}(t, \eta) = L\left(t, \overline{x}(t), \overline{y}(t), \dot{\overline{x}}(t) + \eta, \dot{\overline{y}}(t)\right) - \overline{L}(t), t \in [t_0, t_1] \cap \overline{I}, \qquad (4)$$

$$\Delta_{\dot{y}} \overline{L}(t+h, \eta) = L\big(\tau, \overline{x}(\tau), \overline{y}(\tau), \dot{\overline{x}}(\tau), \dot{\overline{y}}(\tau) + \eta\big)\big|_{t=t+h} - \overline{L}(\tau + h),$$

$$t \in [t_0, t_1 - h] \cap \overline{I}, \eta \in \left\{\xi, \frac{\lambda}{\lambda-1}\xi\right\},$$

$$\overline{L}(\nu) = L\left(\nu, \overline{x}(\nu), \overline{y}(\nu), \dot{\overline{x}}(\nu), \dot{\overline{y}}(\nu)\right), \qquad \nu \in \{t, t+h\}.$$

Theorem. *Let the integrant $L(\cdot)$ is continuous in the totality of variables. Then:*

i. if the admissible function $\bar{x}(\cdot)$ is a strong local minimum in the problem (1), (2), *then the following inequality is fulfilled:*

$$Q\left(t, \lambda, \xi; \dot{\overline{x}}(\cdot)\right) \geq 0, \quad \forall (t, \lambda, \xi) \in \overline{I} \times [0, 1) \times \mathbb{R}^n; \qquad (5)$$

ii. if the admissible function $\overline{x}(\cdot)$ is the weak local minimum in the problem (1), (2), *then there exist a number $\delta > 0$ such that the following inequality is fulfilled:*

$$Q(t, \lambda, \xi; \overline{x}(\cdot)) \geq 0, \quad \forall (t, \lambda, \xi) \in \overline{I} \times \left[0, \frac{1}{2}\right] \times B_\delta(0), \qquad (6)$$

where the function $Q(\cdot; \overline{x}(\cdot))$ is determined by (3) *and a set $B_\delta(0) -$ a closed ball of radius δ centered at $0 \in \mathbb{R}^n$.*

Proof. Firstly, let us prove the part (i) of Theorem at $t \in [t_0, t_1 - h] \cap \overline{I}$ (note that at $t \in [t_1 - h, t_1]$ proof of (5) is given quite similarly to Theorem 2.1 of [8]). Let $c := (\theta, \lambda, \xi) \in [t_0, t_1 - h) \cap \overline{I} \times [0, 1) \times \mathbb{R}^n -$ be an arbitrary fixed point. Let us define a special function of the form [8]:

$$q(t; c, \varepsilon) = \begin{cases} (t - \theta)\xi, & t \in [\theta, \theta + \lambda\varepsilon), \\ \frac{\lambda}{\lambda-1}(t - \theta - \varepsilon)\xi, & t \in [\theta + \lambda\varepsilon, \theta + \varepsilon), \\ 0, & t \in \widehat{I} \backslash [\theta, \theta + \varepsilon). \end{cases} \qquad (7)$$

Here $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 = \min\{h, t_1 - \theta - h\}$.

By virtue of (7) the estimation is valid

$$\|q(t;c,\varepsilon)\|_{C\left(\hat{I},\mathbb{R}^n\right)} = M_c\varepsilon, \quad M_c = const > 0, \quad \varepsilon \in (0,\varepsilon_0)\,. \tag{8}$$

Further, it is clear that for any $\varepsilon \in (0,\varepsilon_0]$ the function $q(\cdot;c,\varepsilon)$ is an element of space $KC^1\left(\widehat{I},\mathbb{R}^n\right)$ and its derivative $\dot{q}(\cdot;c,\varepsilon)$ is calculated by the formula

$$\dot{q}(t;c,\varepsilon) = \begin{cases} \xi, \quad t \in [\theta,\theta+\lambda\varepsilon]\,, \\ \frac{\lambda}{\lambda-1}\xi, \quad t \in [\theta+\lambda\varepsilon,\theta+\varepsilon]\,, \\ 0, \quad t \in \widehat{I}\setminus(\theta,\theta+\varepsilon)\,. \end{cases} \tag{9}$$

Here, it is considered that the derivative $\dot{q}(\cdot;c,\varepsilon)$ at the points θ, $\theta+\lambda\varepsilon$ and $\theta+\varepsilon$ is calculated both on the right and the left.

Let us consider a special variation of the function $\overline{x}(\cdot)$ of the form:

$$x(t;c,\varepsilon) = \overline{x}(t) + q(t;c,\varepsilon)\,, \quad t \in \widehat{I}, \quad \varepsilon \in (0,\varepsilon_0)\,, \tag{10}$$

where $q(\cdot;c,\varepsilon)$ is determined by (7).

Since $\theta \in [t_0,t_1-h) \cap \overline{I}$ and for all $\varepsilon \in (0,\varepsilon_0)$, we have the inclusion $q(\cdot;c,\varepsilon) \in KC^1\left(\widehat{I},\mathbb{R}^n\right)$, then, taking into account (9), we get the validity of the following statement: there exists a number $\bar{\varepsilon} \in (0,\varepsilon_0)$ such that $\overline{x}(\cdot;c,\varepsilon)$ is admissible for every $\varepsilon \in (0,\bar{\varepsilon}]$ and the function $\dot{\overline{x}}(\cdot)$ is continuous on the segments $[\theta-h,\theta-h+\varepsilon]\,,[\theta,\theta+\varepsilon]$ and $[\theta+h,\theta+h+\varepsilon]$ and this statement is called property (A).

Let us continue to the proof of Theorem, we calculate the increment $S(x(\cdot;c,\varepsilon)) - S(\overline{x}(\cdot)) =: \Delta_\varepsilon S(\overline{x}(\cdot);c)$ of the functional (1), taking into account (9), where $\varepsilon \in (0,\bar{\varepsilon}]$. Taking into account (7)-(10) and the designation (4), we have

$$\Delta_\varepsilon S(\bar{x}(\,\cdot\,);c) = S_1(\varepsilon) + S_2(\varepsilon)\,, \varepsilon \in (0,\bar{\varepsilon}]\,, \tag{11}$$

where

$$S_1(\varepsilon) = \int_\theta^{\theta+\lambda\varepsilon} \left[\Delta_{\dot{x}}\bar{L}(t,\xi) + \Delta_{\dot{y}}\bar{L}(t+h,\xi)\right]dt,$$

$$S_2(\varepsilon) = \int_{\theta+\lambda\varepsilon}^{\theta+\varepsilon} \left[\Delta_{\dot{x}}\bar{L}\left(t,\frac{\lambda}{\lambda-1}\xi\right) + \Delta_{\dot{y}}\bar{L}\left(t+h,\frac{\lambda}{\lambda-1}\xi\right)\right]dt.$$

Considering the continuity (with respect to all variables) of the integrant $L(\cdot)$ and the property (A) of (11), using the Mean Value Theorem for definite integrals, we obtain:

$$\Delta_\varepsilon S(\overline{x}(\cdot);c) = \varepsilon\left\{\lambda\left[\Delta_{\dot{x}}\overline{L}(\tau_{1,\varepsilon},\xi) + \Delta_{\dot{y}}\overline{L}(\tau_{1,\varepsilon}+h,\xi)\right] + \right.$$

$$\left. +(1-\lambda)\left[\Delta_{\dot{x}}\overline{L}\left(\tau_{2,\varepsilon},\frac{\lambda}{\lambda-1}\xi\right) + \Delta_{\dot{y}}\overline{L}\left(\tau_{2,\varepsilon}+h,\frac{\lambda}{\lambda-1}\xi\right)\right]\right\}, \quad \varepsilon \in (0,\bar{\varepsilon}]\,, \tag{12}$$

where $\tau_{1,\varepsilon} \in (\theta,\theta+\lambda\varepsilon)$ and $\tau_{2,\varepsilon} \in (\theta+\lambda\varepsilon,\theta+\varepsilon)$.

In addition, by virtue of (7), (8), (10) and the definition of the points $\tau_{i,\varepsilon},\ i=1,2,$ and also the property (A) we have

$$\begin{gathered}\lim_{\varepsilon\to+0}\tau_{i,\varepsilon}=\theta,\ \lim_{\varepsilon\to+0}x\left(\tau_{i,\varepsilon};c,\varepsilon\right)=\lim_{\varepsilon\to+0}\overline{x}\left(\tau_{i,\varepsilon}\right)=\overline{x}\left(\theta\right),\\ \lim_{\varepsilon\to+0}\dot{\overline{x}}\left(\tau_{i,\varepsilon}\right)=\dot{\overline{x}}\left(\theta\right),\ \lim_{\varepsilon\to+0}y\left(\tau_{i,\varepsilon};c,\varepsilon\right)=\lim_{\varepsilon\to+0}\overline{y}\left(\tau_{i,\varepsilon}\right)=\overline{y}\left(\theta\right),\\ \lim_{\varepsilon\to+0}x\left(\tau_{i,\varepsilon}+h;c,\varepsilon\right)=\lim_{\varepsilon\to+0}\overline{x}\left(\tau_{i,\varepsilon}+h\right)=\overline{x}\left(\theta+h\right),i=1,2.\end{gathered}\tag{13}$$

Let the function $\overline{x}\left(\cdot\right)$ affords a strong local minimum in the problem (1), (2) with $\overline{\delta}-$ neighborhood. Let us choose a number $\widehat{\varepsilon}=\min\left\{\frac{\overline{\delta}}{M_c},\overline{\varepsilon}\right\}$, where the inequality M_c is determined by (8). Then, by virtue of (7), (8), and (10), there is an estimation for all $\varepsilon\in(0,\widehat{\varepsilon}]$

$$\left\|x\left(t;c,\varepsilon\right)-\overline{x}\left(t\right)\right\|_{C\left(\widehat{I},\mathbb{R}^n\right)}\leq\overline{\delta}.$$

Therefore, by virtue of the definition of a strong local minimum, for all $\varepsilon\in(0,\widehat{\varepsilon}]$, the inequality $\frac{1}{\varepsilon}\Delta_\varepsilon S\left(\overline{x}\left(\cdot\right),c\right)\geq0$ is valid, where $\Delta_\varepsilon S\left(\overline{x}\left(\cdot\right),c\right)$ is determined by (12). Due to (3), (12), (13) and the continuity of the integrant $L\left(\cdot\right)$, passing to the limit at $\varepsilon\to+0$, in the last inequality we obtain the validity of inequality (5) for all $\theta\in[t_0,t_1-h]\cap\overline{I}$. If, $\theta\in[t_1-h,t_1]\cap\overline{I}$, then the proof of inequality (5) is given quite similarly to Theorem 2.1 of [8]. Consequently, (i) part of the Theorem is proven.

Now we present the proof of (ii) part of Theorem. Let the admissible function $\overline{x}\left(\cdot\right)$ afford a weak local minimum in the problem (1), (2) with a $\widehat{\delta}-$ neighborhood. In addition, let $B_{\widehat{\delta}}\left(0\right)=\left\{\xi:\|\xi\|_{\mathbb{R}^n}=\widehat{\delta}\right\}$ and $\widehat{c}=(\theta,\lambda,\xi)$ be an arbitrary fixed point, where $\theta\in\overline{I}$, $(\lambda,\xi)\in\left(0,\frac{1}{2}\right)\times B_{\widehat{\delta}}\left(0\right)$.

Let us consider a variation of a function $\overline{x}\left(\cdot\right)$ of the form

$$x\left(t;\widehat{c},\varepsilon\right)=\overline{x}\left(t\right)+q\left(t;\hat{c},\varepsilon\right),\quad t\in\hat{I},\varepsilon\in(0,\hat{\varepsilon}],\tag{14}$$

where $q\left(\cdot;\widehat{c},\varepsilon\right)$ is determined by (7) by replacing c by $\widehat{c}$ and the number is defined above.

Considering the equality (14), we claim that the relations (7)-(13) are valid for each $\widehat{c}=(\theta,\lambda,\xi)\in\overline{I}\times[0,\frac{1}{2})\times B_{\widehat{\delta}}\left(0\right)$ and for all $\varepsilon\in(0,\widehat{\varepsilon}]$. For example, similarly to (8), we have the inequality $\left\|q\left(\cdot;\widehat{c},\varepsilon\right)\right\|_{C\left(\widehat{I},\mathbb{R}^n\right)}=M_{\widehat{c}}\varepsilon,\ \forall\ \varepsilon\in(0,\widehat{\varepsilon}]$, where $M_c=const>0$. Then, considering (14) for all $(\lambda,\xi)\in\left[0,\frac{1}{2}\right]\times B_{\widehat{\delta}}\left(0\right)$ and $\varepsilon\in(0,\varepsilon^*]$, where $\varepsilon^*=\min\left\{\frac{\widehat{\delta}}{M_{\widehat{c}}},\widehat{\varepsilon}\right\}$, the following inequalities are valid:

$$\left\|x\left(\cdot;\widehat{c},\varepsilon\right)-\overline{x}\left(\cdot\right)\right\|_{C\left(\hat{I},\mathbb{R}^n\right)}=\widehat{\delta},\quad\left\|\dot{x}\left(\cdot;\widehat{c},\varepsilon\right)-\dot{\overline{x}}\left(\cdot\right)\right\|=\|\xi\|_{\mathbb{R}^n}=\widehat{\delta}.$$

Here, the inequality $\left|\frac{\lambda}{\lambda-1}\right|\leq1$, $\lambda\in\left[0,\frac{1}{2}\right]$ is taken into account. Considering these inequalities and the definition of a weak local minimum, for the increment $\Delta_\varepsilon S\left(\overline{x}\left(\cdot\right),\widehat{c}\right)$ the following inequality is fulfilled:

$$\frac{1}{\varepsilon}\Delta_\varepsilon S\left(\overline{x}\left(\cdot\right),\hat{c}\right)\geq0,\forall\ \varepsilon\in\left(0,\varepsilon^*\right],$$

where $\Delta_\varepsilon S\left(\overline{x}\left(\cdot\right),\hat{c}\right)$, is determined from (12) replacing c by $\widehat{c}$.

From the last inequality, by means of the the argument given at the end of the proof of the part (i) of Theorem, we obtain the validity of inequality (6) for an arbitrary fixed $\widehat{c} = (\theta, \lambda, \xi) \in \overline{I} \times \left[0, \frac{1}{2}\right] \times B_{\widehat{\delta}}(0)$. Therefore, assuming the arbitrariness $\widehat{c}$ and setting $\delta = \widehat{\delta}$, we get the proof of (ii) part of Theorem. ◀

Let us now some theoretically and practically valuable consequences of Theorem.

Corollary 1. *Let, in addition to the conditions of Theorem, the integrant $L(\cdot)$ be continuously differentiated with respect to $\dot{x}$ and $\dot{y}$. Then:*

(i) if an admissible function is a strong local minimum in the problem (1), (2), *inequality is satisfied:*

$$\Delta_{\dot{x}}\overline{L}(t,\xi) - \overline{L}_{\dot{x}}^{T}(t)\,\xi + \Delta_{\dot{y}}\overline{L}(t+h,\xi) - \overline{L}_{\dot{y}}^{T}(t+h,\xi) \geq 0, \forall t \in \overline{I} \cap [t_0, t_1 - h)\,,\ \xi \in \mathbb{R}^n, \quad (15)$$

$$\Delta_{\dot{x}}\overline{L}(t,\xi) - \overline{L}_{\dot{x}}^{T}(t)\,\xi \geq 0, \forall t \in \overline{I} \cap [t_1 - h, t_1]\,, \xi \in \mathbb{R}^n; \quad (16)$$

(ii) if an admissible function $\overline{x}(\cdot)$ is a weak local minimum in the problem (1), (2), *then there is a number $\delta > 0$, in which for each $(t,\xi) \in \overline{I} \cap [t_0, t_1 - h] \times B_\delta(0)$ and $(t,\xi) \in \overline{I} \cap [t_1 - h, t_1] \times B_\delta(0)$ inequalities* (15) *and* (16) *are satisfied, respectively, where $\Delta_{\dot{x}}\overline{L}(\,\cdot\,)$ and $\Delta_{\dot{y}}\bar{L}(\,\cdot\,)$ are determined by* (4).

Proof. Firstly, we prove (i) part of Corollary 1.

Since by the assumption for $L(t,x,y,\dot{x},\dot{y})$ the equality $L(t,x,y,\dot{x},\dot{y}) = 0$, holds for $(t,x,y,\dot{x},\dot{y}) \in (t_1, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, it is clear that it is sufficient to show the validity of the inequality (15). By virtue of the assumption of Corollary 1, by the Taylor formula and taking into account (4), we have

$$\Delta_{\dot{x}}\overline{L}\left(t, \frac{\lambda}{\lambda - 1}\xi\right) = \frac{\lambda}{\lambda - 1}\overline{L}_{\dot{x}}^{T}(t)\,\xi + o(\lambda)\,,$$

$$\Delta_{\dot{y}}\overline{L}\left(t+h, \frac{\lambda}{\lambda - 1}\xi\right) = \frac{\lambda}{\lambda - 1}\overline{L}_{\dot{y}}^{T}(t+h)\,\xi + o(\lambda)\,, \quad (17)$$

where $\lambda^{-1}o(\lambda) \to 0$ at $\lambda \to +0$, $t \in \overline{I} \cap [t_0, t_1 - h]\,,\ \ \xi \in \mathbb{R}^n$.

According to (17), the function $Q(\,\cdot\,; \overline{x}(\cdot))$ determined by (3) takes the form

$$Q(t,\lambda,\xi;\overline{x}(\cdot)) = \lambda\left[\Delta_{\dot{x}}\overline{L}(t,\xi) + \Delta_{\dot{y}}\overline{L}(t+h,\xi)\right] +$$

$$+\lambda\left[-\overline{L}_{\dot{x}}^{T}(t)\,\xi - \overline{L}_{\dot{y}}^{T}(t+h)\,\xi + (1-\lambda)\,\lambda^{-1}o(\lambda)\right], \quad (18)$$

where $t \in \overline{I} \cap [t_0, t_1 - h]\,,\ \ \xi \in \mathbb{R}^n,\ \lambda \in (0,1)$.

Since the function $\overline{x}(\cdot)$ is a strong local minimum in the problem (1), (2), then from statement (5) of Theorem, taking into account (18) and $\lambda \in (0,1)$, we have $\frac{1}{\lambda}Q(t,\lambda,\xi,\bar{x}(\,\cdot\,)) \geq 0$. Hence, passing to the limit $\lambda \to +0$ we obtain inequality (15). This means that part (i) of Corollary 1 has been proven.

Continuing the proof of Corollary 1, let the function $\bar{x}(\cdot)$ be a weak local minimum in the problem (1), (2). Then, by virtue of equality (18), statement (6) of Theorem directly follows the validity of part (ii) of Corollary 1. Consequently, Corollary 1 is completely proven. ◀

Corollary 2. *Let the admissible function $\overline{x}(\cdot)$ be a weak local minimum in the problem (1), (2). We suppose that, in addition to the conditions of Theorem, the integrant $L(\cdot) = L(t,x,y,\dot{x},\dot{y})$ is twice differentiable with respect to $\dot{x}$ and $\dot{y}$. Then the inequality holds:*

$$\xi^T \left[\overline{L}^T_{\dot{x}\,\dot{x}}(t) + \overline{L}^T_{\dot{y}\dot{y}}(t+h)\right]\xi \geq 0, \;\; \forall\, t \in \overline{I} \cap [t_0, t_1 - h], \;\; \forall\, \xi \in \mathbb{R}^n, \tag{19}$$

$$\xi^T \overline{L}^T_{\dot{x}\dot{x}}(t)\,\xi \geq 0, \;\; t \in \bar{I} \cap [t_1 - h, t_1]. \tag{20}$$

Proof. Let $\theta \in \bar{I} \bigcap [t_0, t_1 - h]$ and $\eta \in \mathbb{R}^n$ be arbitrary fixed points. Under the assumptions of Corollary 2, inequality (15) is satisfied. Let us choose $\varepsilon_0 > 0$ such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the inclusion $\varepsilon\eta =: \xi \in B_{\widehat{\delta}}(0)$ holds. Then, considering the smoothness conditions imposed on the integrant $L(\cdot)$, according to the Taylor formula and taking into account (4) for the point $(\theta, \varepsilon\eta)$ we have

$$\Delta_{\dot{x}}\overline{L}(\theta, \varepsilon\eta) = \varepsilon\overline{L}^T_{\dot{x}}(\theta)\,\eta + \frac{\varepsilon^2}{2}\eta^T\overline{L}^T_{\dot{x}\dot{x}}(\theta)\,\eta + o\left(\varepsilon^2\right), \tag{21}$$

$$\Delta_{\dot{y}}\overline{L}(t+h, \varepsilon\eta) = \varepsilon\overline{L}^T_{\dot{y}}(\theta+h)\,\eta + \frac{1}{2}\varepsilon^2\eta^T\overline{L}^T_{\dot{y}\dot{y}}(\theta+h)\,\eta + o\left(\varepsilon^2\right), \tag{22}$$

where $\varepsilon^{-2}o\left(\varepsilon^2\right) \to 0$ at $\varepsilon \to 0$

Further, the inequality (15) is valid for $\xi = \lambda\eta$. Therefore, by virtue of (21) and (22), inequality (15) takes the form

$$\frac{1}{2}\varepsilon^2\eta^T\left[\overline{L}^T_{\dot{x}\dot{x}}(\theta) + \overline{L}^T_{\dot{y}\dot{y}}(t+h)\right]\eta + o\left(\varepsilon^2\right) \geq 0, \forall\varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

From here, taking into the arbitrariness θ and η, the validity of inequality (19) follows.

Similarly, from (16) the validity of inequality (20) follows, since by assumption the equality $L(t,x,y,\dot{x},\dot{y}) = 0, \forall t \in (t_1, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$. Corollary 2 has been proven. ◀

Example. To illustrate the meaningfulness of Theorem, let us consider the following example.

Consider the problem

$$S(x(\cdot)) = \int_0^3 \left[\dot{x}^3(t) + |\dot{x}(t)| + |\,(1 + x(t))\,\dot{x}(t-1)\,|\,\right] dt \to \min_{x(\cdot)}, \tag{23}$$

$$x(t) = 0, \quad t \in [-1, 0], \;\; x(3) = 0, \tag{24}$$

where $h = 1, L(t,x,y,\dot{x},\dot{y}) = \dot{x}^3 + |\dot{x}| + |(1+x)\,\dot{y}|, y = x(t-1), x(t) \in KC^1([-1,3], \mathbb{R})$, i.e. the admissible functions are scalar.

Study the minimum of the admissible function $\overline{x}(t) = 0, t \in [-1,3]$. Among this function considering (4) we have

$$\bar{L}(t) = 0, \;\; \Delta_{\dot{x}}\bar{L}(t,\xi) = \xi^3 + |\xi|, \;\; \Delta_{\dot{y}}\bar{L}(t+h,\xi) = |\xi|,$$

$$\Delta_{\dot{x}}\bar{L}\left(t, \frac{\lambda}{\lambda-1}\xi\right) = \left(\frac{\lambda}{\lambda-1}\right)^3\xi^3 + \left|\frac{\lambda}{\lambda-1}\xi\right|, \;\; \Delta_{\dot{y}}\bar{L}\left(t+h, \frac{\lambda}{\lambda-1}\xi\right) = \left|\frac{\lambda}{\lambda-1}\xi\right|.$$

In this case, the function $Q(\cdot;\overline{x}(\cdot))$ defined by (3) takes the form

$$Q(t,\lambda,\varepsilon;\overline{x}(\cdot)) = \lambda\left[\xi^3 + 2|\xi|\right] + (1-\lambda)\left[\left(\frac{\lambda}{\lambda-1}\right)^3\xi^3 + 2\left|\frac{\lambda}{\lambda-1}\xi\right|\right].$$

Let $\lambda \in (0,\frac{1}{2}]$ and $\xi \in [-1,1]$, then we have $\frac{1-2\lambda}{(1-\lambda)^2}\xi \in [-1,1]$ and $4|\xi| \geq 4\xi^2$. Therefore the following inequality is fulfilled:

$$Q(t,\lambda,\xi;\overline{x}(\cdot)) = \lambda\left[\frac{1-2\lambda}{(1-\lambda)^2}\xi^3 + 4|\xi|\right] \geq \lambda\left[\frac{1-2\lambda}{(1-\lambda)^2}\xi^3 + 4\xi^2\right] =$$

$$= \lambda\xi^2\left[\frac{1-2\lambda}{(1-\lambda)^2}\xi + 4\right] \geq 0.$$

From this we obtain that for the function $\overline{x}(\cdot) = 0$ the necessary condition (6) is satisfied and, therefore, it can be a weak local minimum in the problem (23), (24) with a $\delta = 1-$ neighborhood. Continuing the study, we assert that in this problem the function $\overline{x}(\cdot) = 0$ is a weak local minimum. This follows from the fact that for an arbitrary admissible function $x(t)$, $t \in [-1, 3]$ for which $\|\dot{x}(t)\|_{L_\infty([-1,3],\mathbb{R})} \leq 1$ the following inequality holds:

$$\int_0^3 \left[\dot{x}^3(t) + |\dot{x}(t)| + |(1+x(t))\,\dot{x}(t-1)|\right] dt \geq$$

$$\geq \int_0^3 \left[\dot{x}^3(t) + |\dot{x}(t)|\right] dt \geq \int_0^3 \dot{x}^2(t)\left[\dot{x}(t)+1\right] dt \geq 0.$$

In the problem (23), (24) the admissible function is not a strong local minimum, since $Q\left(t,\frac{2}{3},2;\overline{x}(\cdot)\right) = 8 - \frac{56}{3} < 0$, i.e. the necessary condition (5) is violated.

We especially note that to study the problem (23), (24), any necessary conditions of the classical calculus of variations (see for example [1], [3]) and even analogues of Pontryagin's maximum principle [2], [14] are not applicable, since the integrant is not differentiable with respect to the variables in the problem (23), (24).

3. Conclusion

If in the problem (1), (2) $h = 0$, then all the statements we obtained coincide with the corresponding statements in [8].

The statements of Corollary 1 and Corollary 2 are analogues of the Weierstrass condition and the Legendre condition for the problem (1), (2), respectively. We also note that the results of our work can be generalized to the case of more general variational problems with many delays.

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