Baku Mathematical Journal
202X, Vol. XX, No X, PP. XX-XX
https://doi.org/10.32010/j.bmj.202X.XX

ON THE BOUNDEDNESS OF THE RESOLVENT OF THE OPERATOR GENERATED BY PARTIAL OPERATOR-DIFFERENTIAL EXPRESSIONS OF HIGHER ORDER IN HILBERT SPACE

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Received: date / Revised: date / Accepted: date

In memory of M. G. Gasymov on his 85th birthday

Abstract. *In the paper we consider the boundedness of the resolvent of a differential operator generated by partial differential-operator expression higher even order in Hilbert space. The main theorem on the boundedness of the operator* $(L - \lambda E)^{-1}$ *for rather large values of the parameter lying on some ray* $\lambda \in l$, $|\lambda| \geq \lambda_0$ *was proved.*

Keywords: Hilbert space, operator-differential equation, resolvent, Fourier transformation, sef-adjoint operator, completely continuous operator, compactness, positive-definite operator

Mathematics Subject Classification (2020): 35B40, 35R20, 47F05

1. Introduction

Let $H_0, H_1, ...H_{2m}$ be Hilbert spaces, $H_{i+1} \subset H_i, i = 0, 1, 2, ..., 2m-1$, where all embeddings are compact .

Let us consider the differential expression

$$L(x, D)u = \sum_{|\alpha| \leq 2m} A_\alpha(x) D^\alpha u, \quad x \in \mathbb{R}^n,$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), |\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}$.The function $u(x) \in H_{2m}$ is such that $D^\alpha u \in H_{2m-|\alpha|}$. It is assumed that for each $x \in \mathbb{R}^n$, $A_\alpha(x) \neq 0 : H_{2m-|\alpha|} \to H_0$ are bounded operators. $A_0(x) = A_0 + \gamma(x)$, where $A_0 : H_0 \to H_0$ is such a positive definite self-adjint operator that A_0^{-1} is completely continuous. The complex-valued function $\gamma(x)$ is assumed to be measurable and locally bounded.

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2. Estimation of the Resolvent of the Operator $L(x, D)$

Before finding our main goal, the estimation of the resolvent of the operator $L(x, D)$ we formulate the conditions to which the coefficients of the operator $L(x, D)$ must satisfy in future.

We denote

$$R_0(x,\xi) = \left[\sum_{|\alpha|=2m} A_\alpha(x)(i\xi)^\alpha \right]^{-1,}, \quad \xi = (\xi_1, \xi_2, ..., \xi_n), \quad \xi^\alpha = \xi_1^{\alpha_1}, \xi_2^{\alpha_2}, ..., \xi_{n1}^{\alpha_n}.$$

Assume that

I . $R_0(x, \xi)$ for all $\xi \in \mathbb{R}^n \backslash \{0\}$ is a bounded operator $H_0 \to H_{2m}$, moreover

$$\sum_{i=0}^{2m} |\xi|^i \, \|R_0\|_{H_0 \to H_{2m-i}} \le \delta_1,$$

where $\delta_1 = const$ is independent of x, ξ.

II. There exists such a ray $l = \{\lambda : \arg\lambda = \beta\}$ of a complex plane λ that the operator $R(x,\xi,\lambda) = \left[\sum\limits_{|\alpha|\le 2m} A_\alpha(x)(i\xi)^\alpha - \lambda E \right]^{-1,}$ is a bounded operator $H_0 \to H_{2m}$ for $\lambda \in l$, $\xi \in \mathbb{R}^n$, $|x| > c$ and

$$(|\gamma(x)| + |\lambda|) \, \|R_0\|_{H_0 \to H_{2m}} + \sum_{i<2m} |\xi|^{2m-i} \, \|R_0\|_{H_0 \to H_{2m-i}} \le \delta_2.$$

III. The quantities $\sup\limits_{\|x-x_0\|\le h} \|A_\alpha(x) - A_\alpha(x_0)\|$, $\sup \left| \frac{\gamma(x)-\gamma(x_0)}{\gamma(x_0)} \right|$ tend to zero as $h \to 0$ uniformly with respect to n, x_0, $|\alpha| = 2m$.

IV. $\sum\limits_{0<|\alpha|\le 2m} \|A_\alpha(x)\| < \delta_3$.

Denote by $\tilde{H}_i$ a space with a scalar product

$$(f, g)_{\tilde{H}_i} = \int_{\mathbb{R}^n} (f(x), g(x))_{H_i} \, dx, \quad f, g \in H_i, \quad i = 0, 1, ..., 2m.$$

We have the following theorem.

Theorem 1. *Let conditions I-IV be fulfilled, and*

$$\sum_{|\alpha|\le 2m} \int_{\mathbb{R}^n} \|D^\alpha u\|_{H_0}^2 \, dx + \int_{\mathbb{R}^n} \gamma^2(x) \, \|u\|_{H_0}^2 \, dx < \infty$$

for $\lambda \in l$, $|\lambda| \ge \lambda_0$. *Then*

$$\sum_{|\alpha|\le 2m} \|D^\alpha u\|_{H_0}^2 \, dx \le c_1 \, \|(L - \lambda E) \, u\|_{H_0}^2 \, ,$$

where $c_1 = const$, *is independent of* λ.

Proof. Denote $(L - \lambda E)\, u = f$. Divide $\mathbb{R}^n$ into the system of cubes with such ribs h that combination of their interiors coincide with $\mathbb{R}^n$ and each point is overlapped by finitely many cubes. Let S_i be any of the cubes of this system, P_i be the center of S_i. Let us consider partition of unity

$$\sum_{i=1}^{\infty} \theta_i(x) \equiv 1, \;\; \theta_i(x) \in \overset{\circ}{C}{}^{\infty}(S_i).$$

It is easy to see that

$$(L - \lambda E)\, \theta_i(x) u = f_i, \tag{1}$$

where

$$f_i = \sum_{\substack{0<|\alpha|\leq 2m \\ |\alpha'|>0, \alpha'+\beta'=\alpha}} A_\alpha(x) C_{\alpha'\beta'} D^{\alpha'} \theta_i(x) D^{\beta'} u + f\theta_i(x), \quad C_{\alpha'\beta'} = const.$$

Hence it follows

$$A_0(P_i)\theta_i u + \sum_{|\alpha|=2m} A_\alpha(P_i)\, D^{\alpha'} \theta_i u - \lambda\theta_i u = f_i - \sum_{|\alpha|<2m} A_\alpha(x) D^{\alpha'} \theta_i u -$$
$$- \sum_{|\alpha|=2m} \left(A_\alpha(x) - A_\alpha(P_i)\right) D^{\alpha'} \theta_i u = F_i(x). \tag{2}$$

Denote $\theta_i u = v_i$. To the both sides of equality (2) we apply Fourier transform with respect to x. It can be done since $v_i(x)$ has a compact support. As a result we obtain:

$$\left[A_0(P_i) + \sum_{|\alpha|=m} A_\alpha(P_i)(i\xi)^\alpha - \lambda \right] \tilde{v}_i(\lambda) = \tilde{F}_i(\lambda).$$

From assumption II for $\lambda \in l$ we have:

$$(|\gamma(P_i)| + |\lambda|)\, \|\tilde{v}_i(\lambda)\|_{H_0} + \|\tilde{v}_i(\lambda)\|_{H_{2m}} \leq C_2 \left\|\tilde{F}_i(\lambda)\right\|_{H_0} \leq$$
$$\leq C_3 \left\|\tilde{f}_i(\lambda)\right\|_{H_0} + C_3 \|\tilde{v}_i(\lambda)\|_{H_{2m-1}} + \varepsilon(h)\, \|\tilde{v}_i(\lambda)\|_{H_{2m}},$$

where $\varepsilon(h) \to 0$ as $h \to 0$.

Since the embedding's $H_i \subset H_{i-1}$ are compact we have the following estimate:

$$\|w\|_{H_{2m-1}} \leq \varepsilon \|w\|_{H_{2m}} + C(\varepsilon)\, \|W\|_{H_0} \text{ for all } \varepsilon > 0. \tag{3}$$

Having chosen $\varepsilon = \frac{1}{2}\varepsilon_3$, for rather small h from (3) we obtain

$$(|\gamma(P_i)| + |\lambda|)\, \|\tilde{v}_i(\lambda)\|_{H_0} + \|\tilde{v}_i(\lambda)\|_{H_{2m}} \leq C_4 \left\|\tilde{f}_i(\lambda)\right\|_{H_0},$$

if is $|\lambda|$ rather large.

Hence and from the Parseval equality we have:

$$\left(\left|\gamma^2(P_i)\right| + |\lambda|^2\right) \int_{S_i} \|v_i\|^2_{H_0}\, dx + \int_{S_i} \|v_i\|^2_{H_{2m}}\, dx \le C \int_{S_i} \|f_i\|^2_{H_0}\, dx.$$

This inequality yields:

$$\left(|\gamma(x)|^2 + |\lambda|^2\right) \|v_i(\lambda)\|^2_{H_0}\, dx + \int_{S_i} \|v_i(\lambda)\|^2_{H_{2m}}\, dx \le$$

$$\le C(h) \int_{S_i} \|u\|^2_{H_{2m-1}}\, dx + c \int_{S_i} \|f\|^2_{H_0}\, dx. \qquad (4)$$

We sum inequality (4) over all $i = 1, 2, \ldots$.

As a result we obtain:

$$\int_{\mathbb{R}^n} \left(|\gamma(x)|^2 + |\lambda|^2\right) \|u\|^2_{H_0}\, dx + \int_{\mathbb{R}^n} \|u(x)\|^2_{H_{2m}}\, dx \le$$

$$\le c(h) \int_{\mathbb{R}^n} \|u\|^2_{H_{2m-1}}\, dx + c \int_{\mathbb{R}^n} \|f\|^2_{H_0}\, dx. \qquad (5)$$

To estimate the first addend in the right hand side of inequality (5), we use inequality (3)

As a result, having taken in (3) $\varepsilon > 0$ so that $\varepsilon \cdot c(h) = \frac{1}{2}$, we obtain

$$\int_{\mathbb{R}^n} |\gamma(x)|^2 + |\lambda|^2 \|u\|^2_{H_0}\, dx + \int_{\mathbb{R}^n} \|u(x)\|^2_{H_{2m}}\, dx \le$$

$$\le c \int_{\mathbb{R}^n} \|f\|^2_{H_0}\, dx + c(\varepsilon, h) \int_{\mathbb{R}^n} \|u\|^2_{H_0}\, dx. \qquad (6)$$

If $|\lambda|^2 > 2c(\varepsilon, h)$, then the required inequality (1) follows from (6). ◀

Theorem 2. *If the conditions of Theorem 1 are fulfilled, then the operator* $(L - \lambda E)^{-1} : H_0 \to H_0$ *is a bounded operator for each* $\lambda \in l$, $|\lambda| \ge \lambda_0$.

Proof. To prove the theorem, it suffices to prove the existence of the solution to the equation

$$L(x, D)u - \lambda u = f(x) \text{ for any } f(x) \in H_0.$$

Let $\theta_i(x)$ be the same partition of unity that was used when proving Theorem 1, $f_i = \theta_i f$.

Let us consider the equation

$$\sum_{|\alpha|=2m} A_\alpha(P_i) D^\alpha w_i + A_0(P_i) w_i = f_i(x).$$

Its solution exists and can be determined by applying Fourier transform by the following formula:

$$\tilde{w}_i = \left[\sum_{|\alpha|=2m} A_\alpha(P_i)(i\xi)^\alpha + A_0(P_i) - \lambda \right]^{-1} \tilde{f}_i(\lambda).$$

Let $\sigma_i(x)$ be such a partition of unity that $\sigma_i(x)\psi_i(x) \equiv \psi_i$.We determine the function

$$w(x) = \sum_{i=1}^{\infty} \sigma_i(x) w_i(x) = L(f).$$

Then we have:

$$(L - \lambda E)\, w = \sum_{i=1}^{\infty} \left[\sum_{|\alpha|=2m} \sigma_i(x) A_\alpha(x) D^\alpha w_i + \sum_{\substack{0<|\alpha|\le 2m \\ |\alpha'|>0,\alpha'+\beta'=\alpha}} d_{\alpha'\beta'} A_\alpha(x) D^{\beta'} w_i + \right.$$

$$\left. + A_0(x)\sigma_i w_i - \lambda \sigma_i(x) w_i \right] =$$

$$= \sum_{i=1}^{\infty} \left[\sigma_i f_i + \sum_{|\alpha|=2m} \sigma_i \left[A_\alpha(x) - A_\alpha(x_i) \right] D^\alpha w_i + \right.$$

$$\left. + \sum_{\alpha'+\beta'=\alpha} A_\alpha(x) d_{\alpha'\beta'}(x) D^{\beta'} w_i - \left(A_0(x) - A_0(x_i) \right) \sigma_i w_i \right] =$$

$$= f + \sum_{i=1}^{\infty} \left[\sum_{|\alpha|=2m} \sigma_i(x) [A_\alpha(x) - A_\alpha(x_0)] D^\alpha w_i + \right.$$

$$\left. + \sum_{\alpha'+\beta'=\alpha} A_\alpha(x) d_{\alpha'\beta'}(x) D^{\beta'} w_i + \left(A_0(x) - A_0(x_0) \right) \sigma_i w_i \right] = f + Tf. \tag{7}$$

Here $d_{\alpha'\beta'}(x)$ are infinitely differentiable functions with a support in S_i, T is an operator $H_0 \to H_0$. Estimate $\|T\|$. For $|\alpha| = 2m$ we have

$$\left\| \sigma_i(x) \left(A_\alpha(x) - A_\alpha(x_0) \right) D^\alpha w_i \right\|_{H_0} \le \varepsilon \int_{S_i} \left\| D^\alpha w_i \right\|_{H_0}^2 dx.$$

If h is rather small, hen condition III yields

$$\left(|\gamma(x)| + |\lambda| \right) \|w\|_{H_0} + \|w\|_{H_{2m}} \le c \|f\|_{H_0}.$$

This means that

$$\left\| \sigma_i(x) \left[A_\alpha(x) - A_\alpha(x_i) \right] D^\alpha w_i \right\|_{H_0} \le c\varepsilon \|f\|_{H_0}. \tag{8}$$

Furthermore,

$$\left\| A_\alpha(x) d_{\alpha'\beta'}(x) D^{\beta'} w_i \right\|_{H_0} \le c \|w\|_{H_{2m-1}} \le$$

$$\le \varepsilon \|w\|_{H_{2m}} + \|w\|_{H_0} \le \varepsilon \|f\|_{H_0} + (1+|\lambda|)^{-1} c_\varepsilon \|f_i\|_{H_0}. \quad (9)$$

Finally we obtain

$$\|A_0(x) - A_0(x_i)\sigma_i w_i\|_{H_0} \le c\varepsilon \|f_i\|_{H_0}. \quad (10)$$

From (8)-(10) it follows that $\|T\| \le \frac{1}{2}$ if h is rather small.

So, $(L - \lambda E)\, w = f + Tf$.

By the Banach theorem there exists such φ that $T\varphi + \varphi = f$.

Hence we obtain that $(L - \lambda E)\, \varphi = T\varphi + \varphi = f$.

The, the solvability of equation (7) is proved. ◄

We observe that Cauchy problem and existence of solutions of boundary value problems and asymptotic properties of solutions for ordinary operator-differential equations was studying by M.G. Gasymov [6], Yu.A. Dubinskii [5], B.A. Plamenevskii [10], S.S. Mirzoev [9], A.A. Shkalikov [13] and others. In comporve with the ordinary operator-differential equations the partial operator-differential equations was small investigated. In this direction we can refer to the works of S. Agmon, A. Douglis, L. Nirenberg [1], G.I. Aslanov [2]-[4], A.A. Shkalikov [13], V.B. Shakhmurov [11], V.B. Shakhmurov and Azad A. Babaev [12] and others. In general the studying of the solutions of operator-differential equations we refer to detail to fundamental monographies S.G. Krein [7], J.L. Lions and E. Magenes [8] and S.Ya. Yakubov [14].

References

1. Agmon S., Douglis A., Nirenberg L. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. *Commun. Pure Appl. Math.*, I, 1959, **12** (4), pp. 623-727; II, 1964, **17** (1), pp. 35-92.
2. Aslanov G.I. On the solubility and asymptotic behaviour of solutions of differential equations in Hilbert space. *International Conference on Differential Equations and Related Questions (15th joint session of the Petrovskii Seminar and the Moscow Mathematical Society, 19-22 January 1993)*, Russ. Math. Surv., 1993, **48** (4), pp. 190-191.
3. Aslanov G.I. Differential equations with operator coefficients in Hilbert spaces. *Math. Notes*, 1993, **53** (3), pp. 351-353.
4. Aslanov G.I. Differential equations with unbounded operator coefficients in Hilbert spaces. *Dokl. Math.*, 1995, **50** (1), pp. 5-9.
5. Dubinskii Yu.A. On some differential-operator equations of arbitrary order. *Math. USSR-Sb.*, 1973, **19** (1), pp. 1-21.
6. Gasymov M.G. The solvability of boundary value problems for a class of operatordifferential equations. *Dokl. Akad. Nauk SSSR*, 1977, **235** (3), pp. 505-508 (in Russian).
7. Krein S.G. *Linear Differential Equations in Banach Space.* Amer. Math. Soc., Providence, R.I., 1971.

8. Lions J.L., Magenes E. *Non-Homogeneous Boundary Value Problems and Applications.* Dunod, Paris, 1968; Mir, Moscow, 1971; Springer, Berlin, 1972.
9. Mirzoev S.S. Conditions for the well-defined solvability of boundary-value problems for operator differential equations. *Dokl. Akad. Nauk SSSR*, 1983, **273** (2), pp. 292-295 (in Russian).
10. Plamenevskii B.A. On the existence and asymptotics of solutions of differential equations with unbounded operator coefficients in a Banach space. *Math. USSR-Izv.*, 1972, **6** (6), pp. 1327-1379.
11. Shakhmurov V.B. Coercive boundary value problems for strongly degenerate operator-differential equations. *Dokl. Akad. Nauk SSSR*, 1986, **290** (3), pp. 553-556 (in Russian).
12. Shakhmurov V.B., Babaev Azad A. Coercive problems for an equation with parameters. *Dokl. Akad. Nauk SSSR*, 1990, **315** (1), pp. 37-40 (in Russian).
13. Shkalikov A.A. Elliptic equations in Hilbert space and associated spectral problems. *Trudy Sem. Petrovsk.*, 1989, (14), pp. 140-224 (in Russian).
14. Yakubov S.Ya. *Linear Differential-Operator Equations and Their Applications.* Elm, Baku, 1985 (in Russian).