

# ON THE BOUNDEDNESS OF THE RESOLVENT OF THE OPERATOR GENERATED BY PARTIAL OPERATOR-DIFFERENTIAL EXPRESSIONS OF HIGHER ORDER IN HILBERT SPACE

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*In memory of M. G. Gasymov on his 85th birthday*

**Abstract.** *In the paper we consider the boundedness of the resolvent of a differential operator generated by partial operator-differential expression of higher even order in Hilbert space. The main theorem on the boundedness of the operator  $(L - \lambda E)^{-1}$  for rather large values of the parameter lying on some ray  $\lambda \in l$ ,  $|\lambda| \geq \lambda_0$  was proved.*

**Keywords:** Hilbert space, operator-differential equation, resolvent, Fourier transformation, self-adjoint operator, completely continuous operator, compactness, positive-definite operator

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## 1. Introduction

Let  $H_0, H_1, \dots, H_{2m}$  be Hilbert spaces,  $H_{i+1} \subset H_i, i = 0, 1, 2, \dots, 2m-1$ , where all embeddings are compact.

Let us consider the differential expression

$$L(x, D)u = \sum_{|\alpha| \leq 2m} A_\alpha(x) D^\alpha u, \quad x \in \mathbb{R}^n,$$

where  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ ,  $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ ,  $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$ . The function  $u(x) \in H_{2m}$  is such that  $D^\alpha u \in H_{2m-|\alpha|}$ . It is assumed that for each  $x \in \mathbb{R}^n$ ,  $A_\alpha(x) \neq 0 : H_{2m-|\alpha|} \rightarrow H_0$  are bounded operators.  $A_0(x) = A_0 + \gamma(x)$ , where  $A_0 : H_0 \rightarrow H_0$  is such a positive definite self-adjoint operator that  $A_0^{-1}$  is completely continuous. The complex-valued function  $\gamma(x)$  is assumed to be measurable and locally bounded.

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## 2. Estimation of the Resolvent of the Operator $L(x, D)$

Before finding our main goal, the estimation of the resolvent of the operator  $L(x, D)$  we formulate the conditions to which the coefficients of the operator  $L(x, D)$  must satisfy in future.

We denote

$$R_0(x, \xi) = \left[ \sum_{|\alpha|=2m} A_\alpha(x) (i\xi)^\alpha \right]^{-1}, \quad \xi = (\xi_1, \xi_2, \dots, \xi_n), \quad \xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \dots \xi_n^{\alpha_n}.$$

Assume that

I.  $R_0(x, \xi)$  for all  $\xi \in \mathbb{R}^n \setminus \{0\}$  is a bounded operator  $H_0 \rightarrow H_{2m}$ , moreover

$$\sum_{i=0}^{2m} |\xi|^i \|R_0\|_{H_0 \rightarrow H_{2m-i}} \leq \delta_1,$$

where  $\delta_1 = \text{const}$  is independent of  $x, \xi$ .

II. There exists such a ray  $l = \{\lambda : \arg \lambda = \beta\}$  of a complex plane  $\lambda$  that the operator

$$R(x, \xi, \lambda) = \left[ \sum_{|\alpha| \leq 2m} A_\alpha(x) (i\xi)^\alpha - \lambda E \right]^{-1},$$

is a bounded operator  $H_0 \rightarrow H_{2m}$  for  $\lambda \in l$ ,  $\xi \in \mathbb{R}^n$ ,  $|x| > c$  and

$$(|\gamma(x)| + |\lambda|) \|R_0\|_{H_0 \rightarrow H_{2m}} + \sum_{i < 2m} |\xi|^{2m-i} \|R_0\|_{H_0 \rightarrow H_{2m-i}} \leq \delta_2.$$

III. The quantities  $\sup_{\|x-x_0\| \leq h} \|A_\alpha(x) - A_\alpha(x_0)\|$ ,  $\sup \left| \frac{\gamma(x) - \gamma(x_0)}{\gamma(x_0)} \right|$  tend to zero as  $h \rightarrow 0$  uniformly with respect to  $n, x_0$ ,  $|\alpha| = 2m$ .

IV.  $\sum_{0 < |\alpha| \leq 2m} \|A_\alpha(x)\| < \delta_3$ .

Denote by  $\tilde{H}_i$  a space with a scalar product

$$(f, g)_{\tilde{H}_i} = \int_{\mathbb{R}^n} (f(x), g(x))_{H_i} dx, \quad f, g \in H_i, \quad i = 0, 1, \dots, 2m.$$

We have the following theorem.

**Theorem 1.** *Let conditions I-IV be fulfilled, and*

$$\sum_{|\alpha| \leq 2m} \int_{\mathbb{R}^n} \|D^\alpha u\|_{H_0}^2 dx + \int_{\mathbb{R}^n} \gamma^2(x) \|u\|_{H_0}^2 dx < \infty$$

for  $\lambda \in l$ ,  $|\lambda| \geq \lambda_0$ . Then

$$\sum_{|\alpha| \leq 2m} \|D^\alpha u\|_{H_0}^2 dx \leq c_1 \|(L - \lambda E) u\|_{H_0}^2,$$

where  $c_1 = \text{const}$ , is independent of  $\lambda$ .

*Proof.* Denote  $(L - \lambda E)u = f$ . Divide  $\mathbb{R}^n$  into the system of cubes with such ribs  $h$  that combination of their interiors coincide with  $\mathbb{R}^n$  and each point is overlapped by finitely many cubes. Let  $S_i$  be any of the cubes of this system,  $P_i$  be the center of  $S_i$ . Let us consider partition of unity

$$\sum_{i=1}^{\infty} \theta_i(x) \equiv 1, \quad \theta_i(x) \in \mathring{C}^\infty(S_i).$$

It is easy to see that

$$(L - \lambda E) \theta_i(x) u = f_i, \quad (1)$$

where

$$f_i = \sum_{\substack{0 < |\alpha| \leq 2m \\ |\alpha'| > 0, \alpha' + \beta' = \alpha}} A_\alpha(x) C_{\alpha' \beta'} D^{\alpha'} \theta_i(x) D^{\beta'} u + f \theta_i(x), \quad C_{\alpha' \beta'} = \text{const.}$$

Hence it follows

$$\begin{aligned} A_0(P_i) \theta_i u + \sum_{|\alpha|=2m} A_\alpha(P_i) D^{\alpha'} \theta_i u - \lambda \theta_i u &= f_i - \sum_{|\alpha| < 2m} A_\alpha(x) D^{\alpha'} \theta_i u - \\ &- \sum_{|\alpha|=2m} (A_\alpha(x) - A_\alpha(P_i)) D^{\alpha'} \theta_i u = F_i(x). \end{aligned} \quad (2)$$

Denote  $\theta_i u = v_i$ . To the both sides of equality (2) we apply Fourier transform with respect to  $x$ . It can be done since  $v_i(x)$  has a compact support. As a result we obtain:

$$\left[ A_0(P_i) + \sum_{|\alpha|=m} A_\alpha(P_i) (i\xi)^\alpha - \lambda \right] \tilde{v}_i(\lambda) = \tilde{F}_i(\lambda).$$

From assumption II for  $\lambda \in l$  we have:

$$\begin{aligned} (|\gamma(P_i)| + |\lambda|) \|\tilde{v}_i(\lambda)\|_{H_0} + \|\tilde{v}_i(\lambda)\|_{H_{2m}} &\leq C_2 \left\| \tilde{F}_i(\lambda) \right\|_{H_0} \leq \\ &\leq C_3 \left\| \tilde{f}_i(\lambda) \right\|_{H_0} + C_3 \|\tilde{v}_i(\lambda)\|_{H_{2m-1}} + \varepsilon(h) \|\tilde{v}_i(\lambda)\|_{H_{2m}}, \end{aligned}$$

where  $\varepsilon(h) \rightarrow 0$  as  $h \rightarrow 0$ .

Since the embedding's  $H_i \subset H_{i-1}$  are compact we have the following estimate:

$$\|w\|_{H_{2m-1}} \leq \varepsilon \|w\|_{H_{2m}} + C(\varepsilon) \|W\|_{H_0} \quad \text{for all } \varepsilon > 0. \quad (3)$$

Having chosen  $\varepsilon = \frac{1}{2}\varepsilon_3$ , for rather small  $h$  from (3) we obtain

$$(|\gamma(P_i)| + |\lambda|) \|\tilde{v}_i(\lambda)\|_{H_0} + \|\tilde{v}_i(\lambda)\|_{H_{2m}} \leq C_4 \left\| \tilde{f}_i(\lambda) \right\|_{H_0},$$

if is  $|\lambda|$  rather large.

Hence and from the Parseval equality we have:

$$\left(|\gamma^2(P_i)| + |\lambda|^2\right) \int_{S_i} \|v_i\|_{H_0}^2 dx + \int_{S_i} \|v_i\|_{H_{2m}}^2 dx \leq C \int_{S_i} \|f_i\|_{H_0}^2 dx.$$

This inequality yields:

$$\begin{aligned} & \left(|\gamma(x)|^2 + |\lambda|^2\right) \|v_i(\lambda)\|_{H_0}^2 dx + \int_{S_i} \|v_i(\lambda)\|_{H_{2m}}^2 dx \leq \\ & \leq C(h) \int_{S_i} \|u\|_{H_{2m-1}}^2 dx + c \int_{S_i} \|f\|_{H_0}^2 dx. \end{aligned} \quad (4)$$

We sum inequality (4) over all  $i = 1, 2, \dots$ .

As a result we obtain:

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(|\gamma(x)|^2 + |\lambda|^2\right) \|u\|_{H_0}^2 dx + \int_{\mathbb{R}^n} \|u(x)\|_{H_{2m}}^2 dx \leq \\ & \leq c(h) \int_{\mathbb{R}^n} \|u\|_{H_{2m-1}}^2 dx + c \int_{\mathbb{R}^n} \|f\|_{H_0}^2 dx. \end{aligned} \quad (5)$$

To estimate the first addend in the right hand side of inequality (5), we use inequality (3)

As a result, having taken in (3)  $\varepsilon > 0$  so that  $\varepsilon \cdot c(h) = \frac{1}{2}$ , we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} |\gamma(x)|^2 + |\lambda|^2 \|u\|_{H_0}^2 dx + \int_{\mathbb{R}^n} \|u(x)\|_{H_{2m}}^2 dx \leq \\ & \leq c \int_{\mathbb{R}^n} \|f\|_{H_0}^2 dx + c(\varepsilon, h) \int_{\mathbb{R}^n} \|u\|_{H_0}^2 dx. \end{aligned} \quad (6)$$

If  $|\lambda|^2 > 2c(\varepsilon, h)$ , then the required inequality (1) follows from (6).  $\blacktriangleleft$

**Theorem 2.** *If the conditions of Theorem 1 are fulfilled, then the operator  $(L - \lambda E)^{-1} : H_0 \rightarrow H_0$  is a bounded operator for each  $\lambda \in l$ ,  $|\lambda| \geq \lambda_0$ .*

*Proof.* To prove the theorem, it suffices to prove the existence of the solution to the equation

$$L(x, D)u - \lambda u = f(x) \text{ for any } f(x) \in H_0.$$

Let  $\theta_i(x)$  be the same partition of unity that was used when proving Theorem 1,  $f_i = \theta_i f$ .

Let us consider the equation

$$\sum_{|\alpha|=2m} A_\alpha(P_i) D^\alpha w_i + A_0(P_i) w_i = f_i(x).$$

Its solution exists and can be determined by applying Fourier transform by the following formula:

$$\tilde{w}_i = \left[ \sum_{|\alpha|=2m} A_\alpha(P_i)(i\xi)^\alpha + A_0(P_i) - \lambda \right]^{-1} \tilde{f}_i(\lambda).$$

Let  $\sigma_i(x)$  be such a partition of unity that  $\sigma_i(x)\psi_i(x) \equiv \psi_i$ . We determine the function

$$w(x) = \sum_{i=1}^{\infty} \sigma_i(x)w_i(x) = L(f).$$

Then we have:

$$\begin{aligned} (L - \lambda E)w &= \sum_{i=1}^{\infty} \left[ \sum_{|\alpha|=2m} \sigma_i(x)A_\alpha(x)D^\alpha w_i + \sum_{\substack{0 < |\alpha| \leq 2m \\ |\alpha'| > 0, \alpha' + \beta' = \alpha}} d_{\alpha'\beta'} A_\alpha(x)D^{\beta'} w_i + \right. \\ &\quad \left. + A_0(x)\sigma_i w_i - \lambda \sigma_i(x)w_i \right] = \\ &= \sum_{i=1}^{\infty} \left[ \sigma_i f_i + \sum_{|\alpha|=2m} \sigma_i [A_\alpha(x) - A_\alpha(x_i)] D^\alpha w_i + \right. \\ &\quad \left. + \sum_{\alpha' + \beta' = \alpha} A_\alpha(x) d_{\alpha'\beta'}(x) D^{\beta'} w_i - (A_0(x) - A_0(x_i)) \sigma_i w_i \right] = \\ &= f + \sum_{i=1}^{\infty} \left[ \sum_{|\alpha|=2m} \sigma_i(x) [A_\alpha(x) - A_\alpha(x_0)] D^\alpha w_i + \right. \\ &\quad \left. + \sum_{\alpha' + \beta' = \alpha} A_\alpha(x) d_{\alpha'\beta'}(x) D^{\beta'} w_i + (A_0(x) - A_0(x_0)) \sigma_i w_i \right] = f + Tf. \end{aligned} \quad (7)$$

Here  $d_{\alpha'\beta'}(x)$  are infinitely differentiable functions with a support in  $S_i$ ,  $T$  is an operator  $H_0 \rightarrow H_0$ . Estimate  $\|T\|$ . For  $|\alpha| = 2m$  we have

$$\|\sigma_i(x) (A_\alpha(x) - A_\alpha(x_0)) D^\alpha w_i\|_{H_0} \leq \varepsilon \int_{S_i} \|D^\alpha w_i\|_{H_0}^2 dx.$$

If  $h$  is rather small, then condition III yields

$$(|\gamma(x)| + |\lambda|) \|w\|_{H_0} + \|w\|_{H_{2m}} \leq c \|f\|_{H_0}.$$

This means that

$$\|\sigma_i(x) [A_\alpha(x) - A_\alpha(x_i)] D^\alpha w_i\|_{H_0} \leq c\varepsilon \|f\|_{H_0}. \quad (8)$$

Furthermore,

$$\begin{aligned} & \left\| A_\alpha(x) d_{\alpha'\beta'}(x) D^{\beta'} w_i \right\|_{H_0} \leq c \|w\|_{H_{2m-1}} \leq \\ & \leq \varepsilon \|w\|_{H_{2m}} + \|w\|_{H_0} \leq \varepsilon \|f\|_{H_0} + (1 + |\lambda|)^{-1} c_\varepsilon \|f_i\|_{H_0}. \end{aligned} \quad (9)$$

Finally we obtain

$$\|A_0(x) - A_0(x_i) \sigma_i w_i\|_{H_0} \leq c\varepsilon \|f_i\|_{H_0}. \quad (10)$$

From (8)-(10) it follows that  $\|T\| \leq \frac{1}{2}$  if  $h$  is rather small.

So,  $(L - \lambda E)w = f + Tf$ .

By the Banach theorem there exists such  $\varphi$  that  $T\varphi + \varphi = f$ .

Hence we obtain that  $(L - \lambda E)\varphi = T\varphi + \varphi = f$ .

The, the solvability of equation (7) is proved.  $\blacktriangleleft$

We observe that Cauchy problem and existence of solutions of boundary value problems and asymptotic properties of solutions for ordinary operator-differential equations was studying by M.G. Gasymov [6], Yu.A. Dubinskii [5], B.A. Plamenevskii [10], S.S. Mirzoev [9], A.A. Shkalikov [13] and others. In comporve with the ordinary operator-differential equations the partial operator-differential equations was small investigated. In this direction we can refer to the works of S. Agmon, A. Douglis, L. Nirenberg [1], G.I. Aslanov [2]-[4], A.A. Shkalikov [13], V.B. Shakhmurov [11], V.B. Shakhmurov and Azad A. Babaev [12] and others. In general the studying of the solutions of operator-differential equations we refer to detail to fundamental monographies S.G. Krein [7], J.L. Lions and E. Magenes [8] and S.Ya. Yakubov [14].

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