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EULER-TYPE SYSTEM OF EQUATIONS IN VARIATIONAL PROBLEMS WITH DELAYED ARGUMENT

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In memory of M. G. Gasymov on his 85th birthday

Abstract. *The article studies a closed-end variational problem with a delayed arguments. As a result, a system of equations with a delayed argument of Euler-type is obtained using the first variation of the functional for the extremum.*

Keywords: variational problem with a delayed argument, necessary condition, system of Euler-type equations.

Mathematics Subject Classification (2020): 42B20, 42B25, 42B35

1. Introduction

Control systems described by dynamic systems with aftereffects quite fully reflect many real processes [6]. Therefore, even today there is a need for a deep study of the issues of minimizing the functional, for example in variational problems with delay and in problems of optimal control with delay.

In 1961 G.L. Kharatishvili [5] obtained an analogue of Pontryagin's maximum principle for the problem of optimal processes with delay. In this direction, continuing research, a number of important results were obtained (see, for example [1], [2], [8]-[10], [13]-[15]).

For the first time in 1970 G.A. Kamenskii [4] studied variational problems with delay and obtained an analogue of the Euler equation.

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Works [3], [16] were published in this direction in which analogues of the Euler equation, Weierstrass conditions, Legendre conditions, as well as other necessary minimum conditions for more natural boundary conditions compared to the work [4] were obtained. Note that in the works [3], [4] analogues of the Euler equation were obtained, which are a system of neutral type equations of second order with delayed and advanced arguments. To solve the resulting boundary value problems, as a rule, a method similar to the sweep method is used. It is clear that finding a solution to such a system of equations presents difficulties. Therefore, it is relevant to obtain a system of Euler-type equations, which is a system of neutral-type equations with only a delayed argument. This idea, as the main goal, is implemented in this paper.

2. Problem Statement

We consider a variational problem for a functional depending on one constant delay $h > 0$. Namely, the problems of the following form are studied:

$$J(x(\cdot)) = \int_{t_0}^{t_1} L\left(t,\, x\left(t\right), \dot{x}\left(t\right), \, x\left(t-h\right), \, \dot{x}\left(t-h\right)\right)\, dt \to \min, \tag{1}$$

$$x\left(t\right) = \varphi\left(t\right), \quad t \in \left[t_0 - h,\ t_0\right], \quad x\left(t_1\right) = x_1 \in R^n\,, \tag{2}$$

where R^n is the n–dimensional Euclidean space, t_0, t_1, and x_1 are given points, and $t_1 - t_0 > h$; further, the given function $\varphi\left(t\right) : \left[t_0\,,\ t_1\right] \to R^n$ is twice continuously differentiable. The function $L\left(t,\, x, \tilde{x}, y, \tilde{y}\right) : [t_0,\, t_1] \times R^n \times R^n \times R^n \times R^n \to R = (-\infty,\, +\infty)$ called the integrant, is assumed to be twice continuously differentiable with respect to a set of variables.

The desired function $x(\cdot)$ is continuous and the derivatives $\dot{x}(\cdot)$ and $\ddot{x}(\cdot)$ are piecewise smooth. We call such functions $x(\cdot)$, satisfying the boundary condition (2) admissible functions.

3. Necessary Conditions for a Minimum

Theorem. *Let an admissible function $\bar{x}(\cdot)$ be a solution to problem (1), (2) and be twice continuously differentiable at the points of the set $T \subset [t_0,\, t_1]$, where $[t_0,\, t_1] \setminus T$ be a finite set. Then:*

(i) if $t_1 \in [t_0,\ t_0 + 2h]$, then an admissible function $\bar{x}(\cdot)$ taking into account (2) satisfies a system of equations with a delayed argument of the form:

$$\begin{cases} \bar{L}_x\left(t\right) - \frac{d}{dt}\bar{L}_{\dot{x}}\left(t\right) = 0, & t \in [t_0,\ t_1] \cap T, \\ \bar{L}_y\left(t\right) - \frac{d}{dt}\bar{L}_{\dot{y}}\left(t\right) = 0, & t \in [t_0 + h,\ t_1] \cap T, \end{cases} \tag{3}$$

where the functions $\bar{L}_x\left(t\right)$, $\bar{L}_{\dot{x}}\left(t\right)$, $\bar{L}_y\left(t\right)$ and $\bar{L}_{\dot{y}}\left(t\right)$, $t \in [t_0,\ t_1] \cap T$ are calculated along the function $\bar{x}(\cdot)$, taking into account (2);

(ii) if $t_1 > t_0 + 2h$, then the admissible function $\bar{x}(\cdot)$ taking into account (2) satisfies the system of equations with a delayed argument of the form

$$\begin{cases} \bar{L}_x(t) - \frac{d}{dt}\bar{L}_{\dot{x}}(t) = 0, & t \in [t_1 - h,\ t_1] \cap T, \\ \bar{L}_x(t-h) + \bar{L}_y(t) - \frac{d}{dt}\left[\bar{L}_{\dot{x}}(t-h) + \bar{L}_{\dot{y}}(t)\right] = 0, & t \in [t_0 + 2h,\ t_1] \cap T, \\ \bar{L}_y(t) - \frac{d}{dt}\bar{L}_{\dot{y}}(t) = 0, & t \in [t_0 + h,\ t_0 + 2h] \cap T, \\ \bar{L}_x(t) - \frac{d}{dt}\bar{L}_{\dot{x}}(t) = 0, & t \in [t_0,\ t_0 + h] \cap T. \end{cases} \tag{4}$$

Proof. At first we prove part (i) of Theorem. Since an admissible function $\bar{x}(\cdot)$ is a solution to the problem (1), (2), then, by [3], the first variation $\delta J(\delta x(\cdot);\bar{x}(\cdot))$ of the problem (1), (2) has the form:

$$\delta J(\delta x(\cdot);\bar{x}(\cdot)) = \int_{t_0}^{t_1} \left[\bar{L}_x^T(t)\delta x(t) + \bar{L}_{\dot{x}}^T(t)\delta\dot{x}(t) + \bar{L}_y^T(t)\delta y(t) + \bar{L}_{\dot{y}}^T(t)\delta\dot{y}(t)\right] dt = 0,$$

$$\forall \delta x(t) \in C^2([t_0 - h,\ t_1],\ R^n), \delta x(t) = 0, t \in [t_0 - h,\ t_0] \cup \{t_1\}. \tag{5}$$

We choose the function $\delta x(\cdot)$ so that it is equal to zero only in $[t_0 - h,\ t_1 - h] \cup \{t_1\}$. In this case, using the method of integration by parts, equality (5) takes the form.

$$\delta J(\delta x(\cdot);\bar{x}(\cdot)) = \int_{t_1-h}^{t_1} \left[\bar{L}_x^T(t)\delta x(t) + \bar{L}_{\dot{x}}^T(t)\delta\dot{x}(t)\right] dt =$$

$$= \int_{t_1-h}^{t_1} \left[\bar{L}_x(t) - \frac{d}{dt}\bar{L}_{\dot{x}}(t)\right]\delta x(t)dt = 0. \tag{6}$$

If the function $\delta x(\cdot)$ is equal to zero only at $[t_0 - h,\ t_0] \cup [t_0 + h,\ t_1]$, then taking into account that $t_1 \in [t_0 + h,\ t_o + 2h]$ and $\delta y(t_0 + h) = \delta y(t_1) = 0$ equality (5) takes the form:

$$\delta J(\delta x(\cdot);\bar{x}(\cdot)) = \int_{t_0}^{t_0+h} \left[\bar{L}_x^T(t)\delta x(t) + \bar{L}_{\dot{x}}^T(t)\delta\dot{x}(t)\right] dt +$$

$$+ \int_{t_0+h}^{t_1} \left[\bar{L}_y^T(t)\delta y(t) + \bar{L}_{\dot{y}}^T(t)\delta\dot{y}(t)\right] dt = \int_{t_0}^{t_0+h} \left[\bar{L}_x^T(t) - \frac{d}{dt}\bar{L}_{\dot{x}}^T(t)\right]\delta x(t)dt +$$

$$+ \int_{t_0+h}^{t_1} \left[\bar{L}_y^T(t) - \frac{d}{dt}\bar{L}_{\dot{y}}^T(t)\right]\delta y(t)dt = 0. \tag{7}$$

From (6) and (7) taking into account $t_1 - h \le t_0 + h$ the proof of (3) easily follows. Part (i) of Theorem is proved.

Let us prove part (ii) of Theorem. Equality (5) can be written as:

$$\delta J\left(\delta x\left(\cdot\right);\bar{x}\left(\cdot\right)\right)=\int\limits_{t_0}^{t_0+h}\left[\bar{L}_x^T(t)\delta x(t)+\bar{L}_{\dot{x}}^T(t)\delta\dot{x}(t)\right]dt+$$

$$+\int\limits_{t_1-h}^{t_1}\left[\bar{L}_x^T(t)\delta x(t)+\bar{L}_{\dot{x}}^T(t)\delta\dot{x}(t)\right]dt+\int\limits_{t_0+h}^{t_1-h}\left[\bar{L}_x^T(t)\delta x(t)+\bar{L}_{\dot{x}}^T(t)\delta\dot{x}(t)\right]dt+$$

$$+\int\limits_{t_0+h}^{t_1}\left[\bar{L}_y^T(t)\delta y(t)+\bar{L}_{\dot{y}}^T(t)\delta\dot{y}(t)\right]dt=\int\limits_{t_0}^{t_0+h}\left[\bar{L}_x^T(t)\delta x(t)+\bar{L}_{\dot{x}}^T(t)\delta\dot{x}(t)\right]dt+$$

$$+\int\limits_{t_0+2h}^{t_1}\left\{\left[\bar{L}_x^T(t-h)+\bar{L}_y^T(t)\right]\delta y(t)+\left[\bar{L}_{\dot{x}}^T(t-h)+\bar{L}_{\dot{y}}^T(t)\right]\delta\dot{y}(t)\right\}dt+$$

$$+\int\limits_{t_0+h}^{t_0+2h}\left[\bar{L}_y^T(t)\delta y(t)+\bar{L}_{\dot{y}}^T(t)\delta\dot{y}(t)\right]dt+\int\limits_{t_1-h}^{t_1}\left[\bar{L}_x^T(t)\delta x(t)+\bar{L}_{\dot{x}}^T(t)\delta\dot{x}(t)\right]dt=0,$$

$$\forall\delta x(t)\in C^2\left(\left[t_0-h,\ t_1\right],\ R^n\right),\delta x(t)=0,t\in[t_0-h,\,t_0]\cup\{t_1\}. \tag{8}$$

As above, we select functions $\delta x(\cdot)$ and apply the method of integration by parts. Then we have:

(a) let the function $\delta x(\cdot)$ be equal to zero in $[t_0-h,\ t_1-h]$, then equality (8), taking into account $\delta x(t_1-h)=\delta x(t_1)=0$ takes the form:

$$\delta J\left(\delta x\left(\cdot\right);\bar{x}\left(\cdot\right)\right)=\int\limits_{t_1-h}^{t_1}\left[\bar{L}_x^T(t)\delta x(t)+\bar{L}_{\dot{x}}^T(t)\delta\dot{x}(t)\right]dt=$$

$$=\int\limits_{t_1-h}^{t_1}\left[\bar{L}_x^T(t)-\frac{d}{dt}\bar{L}_{\dot{x}}^T(t)\right]\delta x(t)dt=0; \tag{9}$$

(b) let the function $\delta x(\cdot)$ be equal to zero in $[t_0-h,\ t_0]\cup[t_0+h,\ t_1]$, then for (8) taking into account $\delta x(t_0)=\delta x(t_0+h)=0$ we have the equality of the form

$$\delta J\left(\delta x\left(\cdot\right);\bar{x}\left(\cdot\right)\right)=\int\limits_{t_0}^{t_0+h}\left[\bar{L}_x^T(t)\delta x(t)+\bar{L}_{\dot{x}}^T(t)\delta\dot{x}(t)\right]dt+$$

$$+\int_{t_0+h}^{t_0+2h}\left[\bar{L}_y^T(t)\delta y(t)+\bar{L}_{\dot{y}}^T(t)\delta\dot{y}(t)\right]dt=\int_{t_0}^{t_0+h}\left[\bar{L}_x^T(t)-\frac{d}{dt}\bar{L}_{\dot{x}}^T(t)\right]\delta x(t)dt+$$

$$+\int_{t_0+h}^{t_0+2h}\left[\bar{L}_y^T(t)-\frac{d}{dt}\bar{L}_{\dot{y}}^T(t)\right]\delta y(t)=0; \quad (10)$$

(c) let the function $\delta x(\cdot)$ be equal to zero in $[t_0-h,\ t_0+h]\cup[t_1-h,\ t_1]$, then for (8), taking into account $\delta x(t_0+h)=\delta x(t_1-h)=0$, i.e. $\delta y(t_0+2h)=\delta y(t_1)=0$, we have

$$\delta J\left(\delta x\left(\cdot\right);\bar{x}\left(\cdot\right)\right)=\int_{t_0+2h}^{t_1}\left\{\bar{L}_x^T(t-h)+\bar{L}_y^T(t)-\frac{d}{dt}\left[\bar{L}_{\dot{x}}^T(t-h)+\bar{L}_{\dot{y}}^T(t)\right]\right\}\delta y(t)dt=0. \quad (11)$$

Now we assume $\bar{L}_y^T(t)-\frac{d}{dt}\bar{L}_{\dot{y}}^T(t)=0,\ t\in[t_0+h,t_0+2h]$, then from (9)-(11), by virtue of the Lagrange lemma [7], the proof of (4) follows, i.e. part (ii) of Theorem is proven.

Therefore, Theorem is completely proven. ◄

Note that the statement of Theorems, i.e. systems of equations (3) and (4), can be called an Euler-type equation in the problem (1), (2). It is clear that each of these systems of equations is a system with delay. Therefore, the statements of Theorem are more constructive than the corresponding statements of the works [3], [4].

In addition, due to the method of obtaining systems of equations (3), as well as (4), we assert that the solutions of each of these systems may not satisfy the Euler equation of problem (1), (2) [3]. At the same time, we introduce that this solution cannot be a minimum in problem (1), (2).

4. Summary

The result obtained in this work could be obtained by a similar method for more complex variational problems, for example, with a finite number of delays, as well as with both free ends.

Note also that obtaining analogues of the results of [11], [12] for problem (1), (2) is promising.

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