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ON THE MINIMALITY OF A PART OF EIGEN- AND ASSOCIATED VECTORS FOR A CLASS OF THIRD-ORDER QUASI-ELLIPTIC OPERATOR PENCILS

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In memory of M. G. Gasymov on his 85th birthday

Abstract. *In the paper, we derive sufficient conditions on the coefficients of a third-order quasi-elliptic operator pencil, ensuring the minimality of its system of eigen- and associated vectors corresponding to eigenvalues from the left half-plane. Additionally, we prove a theorem on the minimality of decreasing elementary solutions to a homogeneous equation in a Sobolev-type space.*

Keywords: operator pencil, eigen- and associated vectors, decreasing elementary solutions, regular solution, internal compactness, minimal system

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1. Some Definitions and Facts from the Theory of Linear Operators

Let H be a separable Hilbert space and let A be a self-adjoint positive-definite operator on H with the domain of definition $Dom(A)$. When $\gamma \geq 0$, let's denote the Hilbert space H_γ by introducing the scalar product $(x, y)_\gamma = (A^\gamma x, A^\gamma y)$ within the domain of definition of the operator A^γ, i.e., in $Dom(A^\gamma)$. When $\gamma = 0$, we assume that $(x, y)_0 = (x, y)$ and $H_0 = H$.

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The notation $\sigma(\cdot)$ will be understood as the spectrum of the operator $(\cdot)$.

Let's consider in the Hilbert space H a third-order quasi-elliptic operator pencil [4]:

$$P(\lambda) = (\lambda E - A)^2(\lambda + A) + \lambda^2 A_1 + \lambda A_2, \tag{1}$$

where λ is the spectral parameter, E is the identity operator, and the remaining coefficients of the operator pencil $P(\lambda)$ satisfy the conditions:

1) A is a self-adjoint positive-definite operator with a completely continuous inverse operator A^{-1};

2) the operators $B_1 = A_1 A^{-1}$ and $B_2 = A_2 A^{-1}$ are bounded in H.

Let's assume that

$$P_0(\lambda) = (\lambda E - A)^2(\lambda + A), \quad P_1(\lambda) = \lambda^2 A_1 + \lambda A_2.$$

Then

$$P(\lambda) = P_0(\lambda) + P_0(\lambda).$$

The operator pencil (1) can be represented in the form

$$P(\lambda) = (E + L(\lambda))A^3, \tag{2}$$

where

$$L(\lambda) = \lambda(B_2 - E)A^{-1} + \lambda^2(B_1 - E)A^{-2} + \lambda^3 A^{-3}.$$

Since A^{-1} is a completely continuous operator, and the operators $B_1 - E$ and $B_2 - E$ are bounded operators, then $L(\lambda)$ is an operator function with completely continuous values. And since $L(0) = 0$, then $E + L(\lambda)$ is invertible at the point $\lambda = 0$, therefore, the operator pencil $E + L(\lambda)$, according to M.V. Keldysh's theorem [7], has a discrete spectrum with a unique limit point at infinity. From the representation (2), it follows that the operator pencil $P(\lambda)$ also possesses this property. Note that all points of the spectrum of the pencil $P(\lambda)$ are poles of the resolvent $P^{-1}(\lambda)$.

Definition 1. *Let $x_0 \neq 0$, $x_0, x_1, x_2, ..., x_m \in H_{5/2}$ satisfy the conditions*

$$P(\lambda_0)x_0 = 0,$$
$$P(\lambda_0)x_1 + \frac{1}{1!}\left.\frac{dP(\lambda)}{d\lambda}\right|_{\lambda=\lambda_0} x_0 = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$P(\lambda_0)x_m + \frac{1}{1!}\left.\frac{dP(\lambda)}{d\lambda}\right|_{\lambda=\lambda_0} x_{m-1} + \frac{1}{2!}\left.\frac{d^2P(\lambda)}{d\lambda^2}\right|_{\lambda=\lambda_0} x_{m-2} + \frac{1}{3!}\left.\frac{d^3P(\lambda)}{d\lambda^3}\right|_{\lambda=\lambda_0} x_{m-3} = 0.$$

Then λ_0 is called an eigenvalue of the operator pencil $P(\lambda)$, and $x_0, x_1, ..., x_m$ are eigen- and associated vectors of the operator pencil $P(\lambda)$ corresponding to the eigenvalue λ_0.

It is evident that a single eigenvalue λ_0 may correspond to multiple sets of eigen- and associated vectors. In the future, we will assume that all sets of eigen- and associated vectors are canonical [7].

Let $K(\Pi_-)$ denote the canonical system of all eigen- and associated vectors corresponding to eigenvalues from the left half-plane $\Pi_- = \{\lambda : Re\lambda < 0\}$.

Definition 2. *Let λ_0 be an eigenvalue of the operator pencil* (1)*, where $Re\lambda_0 < 0$. Then, if $x_0, x_1, ..., x_m$ is a system of eigen- and associated vectors of the operator pencil $P(\lambda)$ corresponding to the eigenvalue λ_0, then*

$$u_n(t) = e^{\lambda_0 t}\left(\frac{t^n}{n!}x_0 + \frac{t^{n-1}}{(n-1)!}x_1 + ... + x_n\right), \quad n = 0, 1, ..., m,$$

are called decreasing elementary solutions to the homogeneous equation $P(d/dt)u(t) = 0$.

Denote by $L_2((a,b);H)$ a Hilbert space of functions defined almost everywhere on the interval $(a,\ b)$ with values in H and the norm

$$\|f\|_{L_2((a,b);H)} = \left(\int_a^b \|f\|^2 dt\right)^{1/2} < \infty.$$

Following the monograph [10], let's denote by

$$W_2^n((a,b);H) = \{u : u^{(n)} \in L_2((a,b);H), \quad A^n u \in L_2((a,b);H)\}$$

a Hilbert space with norm

$$\|u\|_{W_2^n((a,b);H)} = \left(\left\|u^{(n)}\right\|^2_{L_2((a,b);H)} + \|A^n u\|^2_{L_2((a,b);H)}\right)^{1/2}.$$

For $(a,b) = \mathbb{R}$ and $(a,b) = \mathbb{R}_+$, we use the notations $W_2^n(\mathbb{R};H)$ and $W_2^n(\mathbb{R}_+;H)$, respectively.

Note that if $\omega \in W_2^3(\mathbb{R}_+;H)$ (see [10]), then it can be extended to the left half-plane as the zero function over the entire real line, where

$$\omega_1(t) = \begin{cases} \omega(t), & t \geq 0, \\ 0, & t \leq 0, \end{cases} \tag{3}$$

belongs to the space $W_2^3(\mathbb{R};H)$.

Furthermore, we observe that if $u \in W_2^n((a,b);H)$, then

a) $A^{n-j}u^{(j)} \in L_2((a,b);H)$ and $\left\|A^{n-j}u^{(j)}\right\|_{L_2((a,b);H)} \leq const\|u\|_{W_2^n((a,b);H)}$, $j = \overline{1, n-1}$ (the theorem on intermediate derivatives);

b) for $t_0 \in [a,b]$ $u^{(j)}(t_0) \in H_{n-j-1/2}(j = \overline{0, n-1})$ and $\|u(t_0)\|_{n-j-1/2} \leq$ $\leq const\|u\|_{W_2^n((a,b);H)}$, $j = \overline{0, n-1}$ (the theorem on traces).

The paper proves the minimality of the system $K(\Pi_-)$ in the space $H_{5/2}$ and the system of decreasing elementary solutions in the space $W_2^2(\mathbb{R}_+;H)$ subject to additional conditions imposed on the coefficients of the operator pencil $P(\lambda)$.

It should be noted that M.V. Keldysh, in a well-known work [7], studied the issues of completeness and minimality of all eigen- and associated vectors of a certain class of polynomial operator pencils. In the work by M.G. Gasimov [6], an original method was proposed, connecting the solvability of a boundary-value problem on the semi-axis for a certain class of operator-differential equations with spectral problems for a part of the eigen- and associated vectors of the corresponding polynomial operator pencil, corresponding to eigenvalues from the left half-plane. This work was further developed in papers [11]-[18]. Here, it is worth mentioning the works [1]-[3], [5], where they studied both the spectral properties of a quasi-elliptic operator pencil of the third order, different from the one considered in this paper, and the issues of solvability of boundary-value problems on the semi-axis for the operator-differential equation related to this pencil.

2. One Theorem for Functions from the Space $W_2^3(\mathbb{R}_+; H)$

As we noted, the function $\omega_1(t)$ defined by equality (3), belongs to the space $W_2^3(\mathbb{R}; H)$. The following theorem holds.

Theorem 1. *Let conditions 1) and 2) be satisfied. Then, under the fulfillment of condition*

$$\|B_1\| + \|B_2\| < \left(\frac{27}{4}\right)^{1/2}$$

for every $\omega \in W_2^3(\mathbb{R}_+; H)$, the following inequality holds

$$\|P(d/dt)\omega\|_{L_2(\mathbb{R}_+;H)} \geq const\|\omega\|_{W_2^3(\mathbb{R}_+;H)}. \tag{4}$$

Proof. Let $\omega(t)$ be any function from $W_2^3(\mathbb{R}_+; H)$. Then let's denote

$$\psi(t) = P(d/dt)\omega(t), t \in \mathbb{R}_+.$$

We can write that for $t \in \mathbb{R}$, the equality holds

$$\psi_1(t) = P(d/dt)\omega_1(t), t \in \mathbb{R}.$$

Since $\omega_1(t) = 0$ when $t \leq 0$, then $\psi_1(t) = 0$ for $t \leq 0$. After the Fourier transformation, we have

$$\hat{\psi}_1(\xi) = P(i\xi)\hat{\omega}_1(\xi).$$

Let's demonstrate that under the conditions of the theorem, the resolvent $P^{-1}(i\xi)$ exists. Indeed, since $P_0^{-1}(i\xi)$ exists for $\xi \in R$, then from the representation

$$P(i\xi) = (E + P_1(i\xi)P_0^{-1}(i\xi))P_0(i\xi)$$

we obtain that $P(i\xi)$ is invertible when the operator $E + P_1(i\xi)P_0^{-1}(i\xi)$ is invertible. We have

$$\left\|P_1(i\xi)P_0^{-1}(i\xi)\right\| \leq \|B_1\| \cdot \left\|(i\xi)^2 AP_0^{-1}(i\xi)\right\| + \|B_2\| \cdot \left\|i\xi \cdot A^2 P_0^{-1}(i\xi)\right\|. \tag{5}$$

On the other hand, from the spectral decomposition of the operator A, it follows that

$$\left\|(i\xi)^2 AP_0^{-1}(i\xi)\right\| \leq \sup_{\mu\in\sigma(A)} \left\|\xi^2\mu(\xi^2+\mu^2)^{-3/2}\right\| \leq \left(\frac{4}{27}\right)^{1/2}. \tag{6}$$

Similarly, we have

$$\left\|i\xi A^2 P_0^{-1}(i\xi)\right\| \leq \sup_{\mu\in\sigma(A)} \left\|\xi\mu^2(\xi^2+\mu^2)^{-3/2}\right\| \leq \left(\frac{4}{27}\right)^{1/2}. \quad (7)$$

Taking into account the inequalities (6) and (7) in the inequality (5), we obtain

$$\left\|P_1(i\xi)P_0^{-1}(i\xi)\right\| \leq \left(\frac{4}{27}\right)^{1/2} (\|B_1\| + \|B_2\|).$$

According to the theorem's conditions, $\left\|P_1(i\xi)P_0^{-1}(i\xi)\right\| < 1$ for $\xi \in \mathbb{R}$.

So, the resolvent $P^{-1}(i\xi)$ exists for $\xi \in \mathbb{R}$ and

$$\left\|P^{-1}(i\xi)\right\| \leq const \left\|P_0^{-1}(i\xi)\right\| \leq const \sup_{\mu\in\sigma(A)} (\xi^2+\mu^2)^{-3/2} \leq const\mu_0^{-3/2} = const.$$

Therefore,

$$\hat{\omega}_1(\xi) = P^{-1}(i\xi)\hat{\psi}_1(\xi).$$

Then it is obvious that

$$\|\omega\|_{W_2^3(\mathbb{R}_+;H)} = \|\omega_1\|_{W_2^3(\mathbb{R};H)} = \|\hat{\omega}_1(\xi)\|_{W_2^3(\mathbb{R};H)} = \left\|P^{-1}\hat{\psi}_1(\xi)\right\|_{W_2^3(\mathbb{R};H)} \leq$$

$$\leq const\left\|\hat{\psi}_1(\xi)\right\|_{L_2(\mathbb{R};H)} = const\|\psi\|_{L_2(\mathbb{R}_+;H)} = const\|P(d/dt)\omega\|_{L_2(\mathbb{R}_+;H)}.$$

Thus, inequality (4) has been proven. ◀

3. On the Internal Compactness of the Space of Regular Solutions to the Homogeneous Equation

Definition 3. *If the function $u(t) \in W_2^3(\mathbb{R}_+;H)$ satisfies the equation $P(d/dt)\,u(t) = 0$ almost everywhere in $\mathbb{R}_+$, then it is called a regular solution to the homogeneous equation.*

The set of regular solutions to the homogeneous equation is denoted by

$$L(P) = \left\{u : u \in W_2^3(\mathbb{R}_+;H), P(d/dt)\,u(t) = 0\right\}.$$

The space $L(P)$ is a complete subspace of $W_2^3(\mathbb{R}_+;H)$ according to conditions 1) and 2), and the theorem on intermediate derivatives.

Definition 4. *Let $0 \leq a < a' < b' < b$ and $M > 0$ are real numbers. If the set*

$$L_M = \left\{u : u \in L(P), \|u\|_{W_2^2((a,b);H)} \leq M\right\}$$

is compact with respect to the norm $W_2^2((a',b');H)$, then we say that the space of regular solutions to the homogeneous equation is internally compact.

Notice that the definition of internal compactness was first provided by P.D. Lax [8], who applied the obtained results to elliptic equations in an infinite domain.

Let's augment the space $L(P)$ with the norm $\|u\|_{W_2^2(\mathbb{R}_+;H)}$ and denote the resulting space after augmentation by $\hat{L}_M$. Let's show that $\hat{L}_M$ is a compact set with respect to the norm $\|u\|_{W_2^2((a',b');H)}$.

The following theorem holds.

Theorem 2. *Assuming all the conditions of Theorem 1 are satisfied, then $\hat{L}_M$ is a compact set with respect to the norm $\|u\|_{W_2^2((a',b');H)}$.*

Proof. From the conditions of the theorem, it follows that for any $w \in W_2^3(\mathbb{R}_+;H)$ the inequality holds

$$\|P\,(d/dt)\,w(t)\|_{L_2(\mathbb{R}_+;H)} \geq const\|w\|_{W_2^3(\mathbb{R}_+;H)}. \tag{8}$$

Let $\varphi(t)$ be an infinitely differentiable scalar function defined on $\mathbb{R}$, such that

$$\varphi(t) = \begin{cases} 1, t \in (a',b'), \\ 0, t \in \mathbb{R}\backslash(a',b'), \end{cases}$$

and $u(t)$ be a regular solution to the equation $P\,(d/dt)\,u(t) = 0$. Then $\varphi(t)u(t) \in W_2^3(\mathbb{R};H)$ and $\left|\varphi^{(k)}(t)\right| \leq const, t \in \mathbb{R}, k = 1,2,3.$

From the inequality (8), it follows that

$$\|P\,(d/dt)\,\varphi(t)u(t)\|_{L_2(\mathbb{R}_+;H)} \geq const\|\varphi(t)u(t)\|_{W_2^3(\mathbb{R}_+;H)} \geq const\|\varphi(t)u(t)\|_{W_2^3((a,b);H)} \geq$$

$$\geq const\|\varphi(t)u(t)\|_{W_2^3((a',b');H)}. \tag{9}$$

On the other hand, it is easy to see that

$$P\,(d/dt)\,\varphi(t)u(t) = \varphi(t)P\,(d/dt)\,u(t) + Q(\varphi,u),$$

where $Q(\varphi,u)$ is some operator function depending on $\varphi(t)$ and $u\,(t)$.

Since $P\,(d/dt)\,u(t) = 0$, then

$$\|P\,(d/dt)\,\varphi(t)u(t)\|_{L_2(\mathbb{R}_+;H)} = \|Q(\varphi,u)\|_{L_2(\mathbb{R}_+;H)}.$$

Using the theorem on intermediate derivatives, it is easy to obtain that

$$\|Q(\varphi,u)\|_{L_2(\mathbb{R}_+;H)} \leq const\|u\|_{W_2^2((a,b);H)}. \tag{10}$$

Using (9) and (10), we have

$$\|u\|_{W_2^3((a',b');H)} \leq const\|u\|_{W_2^2((a,b);H)}.$$

Since $u \in \hat{L}_M$, then $\|u\|_{W_2^3((a',b');H)} \leq const$. Because A^{-1} is a completely continuous operator, then the embedding $W_2^3((a',b');H) \subset W_2^2((a',b');H)$ is compact [6], i.e., $\hat{L}_M$ is compact with respect to the norm $\|u\|_{W_2^2((a',b');H)}$. ◄

4. The Minimality of Decreasing Elementary Solutions and the System $K(\Pi_-)$

The following theorem holds.

Theorem 3. *Let the conditions of Theorem 1 be satisfied. Then the system of decreasing elementary solutions is minimal in the space* $W_2^3(\mathbb{R}_+;H)$.

Proof. Recall that $K(\Pi_-)$ is the canonical system of eigen- and associated vectors corresponding to eigenvalues from the left half-plane. We have just proven that the space of regular solutions to the homogeneous equation is internally compact. Then, according to [8], the elementary decreasing solutions to the homogeneous equation will constitute a system of eigen- and associated vectors for a completely continuous operator (see also [16])

$$(Tu)(t) = u(t+1),$$

and its eigenvalues will be $e^{\lambda_0}(Re\lambda_0 < 0)$, and the eigen- and associated vectors will be decreasing elementary solutions corresponding to the eigenvalue $e^{\lambda_0}(Re\lambda_0 < 0)$. According to the results of [9], these vectors are minimal in $W_2^2(\mathbb{R}_+;H)$. Thus, the operator T, acting in the space $\hat{L}_M$, is completely continuous. But, on the other hand, $W_2^3(\mathbb{R}_+;H) \subset W_2^2(\mathbb{R}_+;H)$ and elementary decreasing solutions belong to $W_2^3(\mathbb{R}_+;H)$. Since the system of decreasing elementary solutions is minimal in $W_2^2(\mathbb{R}_+;H)$ and the norm $\|u\|_{W_2^3(\mathbb{R}_+;H)}$ is stronger than the norm $\|u\|_{W_2^2(\mathbb{R}_+;H)}$, this system will be minimal in $W_2^3(\mathbb{R}_+;H)$. Indeed, if there exists $\varepsilon_1 > 0$ and for all coefficients c_n

$$\left\| u_n - \sum_{k\neq n} c_k u_k \right\|_{W_2^2(\mathbb{R}_+;H)} > \varepsilon_1,$$

then

$$\left\| u_n - \sum_{k\neq n} c_k u_k \right\|_{W_2^3(\mathbb{R}_+;H)} \geq const \left\| u_n - \sum_{k\neq n} c_k u_k \right\|_{W_2^2(\mathbb{R}_+;H)} \geq \varepsilon_1 \cdot const = \varepsilon_2.$$

◄

Theorem 4. *Let conditions 1) and 2) hold, and suppose the inequality is satisfied*

$$N_2(\mathbb{R}_+)\,\|B_1\| + N_1(\mathbb{R}_+)\,\|B_2\| < 1,$$

where $N_1(\mathbb{R}_+) = (\frac{\sqrt{5}-1}{8})^{1/2}$, $N_2(\mathbb{R}_+) = \beta_0^{-1}$, β_0 *is a positive root of the equation* $4\beta^3 - 11\beta^2 - 20\beta - 1 = 0$. *Then the system* $K(\Pi_-)$ *is minimal in* $H_{5/2}$.

Proof. From the conditions of the Theorem, it follows that the conditions of Theorem 3 are satisfied. Indeed,

$$N_1(\mathbb{R}_+) > \left(\frac{4}{27}\right)^{1/2}, \quad N_2(\mathbb{R}_+) > \left(\frac{4}{27}\right)^{1/2}.$$

Then, according to Theorem 3, the system of decreasing elementary solutions is minimal in $W_2^3(\mathbb{R}_+; H)$.

On the other hand, it follows from the results of [12] that when the conditions of Theorem are satisfied, the boundary-value problem

$$P\,(d/dt)\,u(t) = 0, \quad t \in \mathbb{R}_+,$$

$$u(0) = \zeta, \quad \zeta \in H_{5/2},$$

has a unique regular solution, and

$$\|u\|_{W_2^3(\mathbb{R}_+;H)} \leq const\|\zeta\|_{5/2}. \tag{11}$$

If we consider the operator $\Gamma u(0) = u(t)$, acting from the space $H_{5/2}$ to the space $\hat{L}_M$, then we will see that, from inequality (11) and the theorem on traces, it follows that the operator $\Gamma : H_{5/2} \to \hat{L}_M$ satisfies the condition

$$const\|u(0)\|_{5/2} \leq \|\Gamma u(0)\|_{W_2^3(\mathbb{R}_+;H)} \leq const\|u(0)\|_{5/2}.$$

From this, it follows that Γ^{-1} exists, is bounded, and therefore, it transforms the minimal system into a minimal system. Consequently, $K(\Pi_-)$ is minimal in $H_{5/2}$. ◄

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