Baku Mathematical Journal
2024, Vol. 3, No 1, PP. 84-109
https://doi.org/10.32010/j.bmj.2024.09

ON AN INVERSE SCATTERING PROBLEM FOR THE QUADRATIC PENCIL OF THE STURM-LIOUVILLE OPERATOR WITH A PIECE-WISE CONSTANT COEFFICIENT

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Received: 15.11.2023 / Revised: 15.01.2024 / Accepted: 26.01.2024

In memory of M. G. Gasymov on his 85th birthday

Abstract. *In this work, direct and inverse scattering problems on the real axis for the quadratic pencil of the Sturm-Liouville operator with piece-wise constant coefficient are studied. The new integral representations for solutions are given, the scattering data is defined, the main integral equations of the inverse scattering problem are obtained, the spectral characteristics of the scattering data are investigated and uniqueness theorem for the solution of inverse problem is proven, the necessary and sufficient conditions for recovering of the potentials are examined.*

Keywords: quadratic pencil of the Sturm-Liouville operator, scattering data, Jost solution, integral representation, inverse scattering problem, main integral equation

Mathematics Subject Classification (2020): 34A55, 34L25, 34L40

1. Introduction

Consider the differential equation

$$-y'' + q(x)y + 2\lambda p(x)y = \lambda^2 \rho(x)y,\ x \in (-\infty, +\infty), \tag{1}$$

where

$$\rho(x) = \begin{cases} 1, & x \geq 0, \\ \alpha^2, & x < 0, \end{cases}$$

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$\alpha \neq 1, \alpha > 0$, λ is a spectral parameter, $q(x)$ and $p(x)$ are real-valued functions such that $p(x)$ is absolutely continuous on every closed interval of the real axis and

$$\int_{-\infty}^{+\infty} |p(x)|\, dx < \infty, \quad \int_{-\infty}^{+\infty} (1+|x|)\left(|q(x)| + |p'(x)|\right) dx < +\infty. \tag{2}$$

Let us define

$$\sigma^{+}(x) = \left(\int_{x}^{+\infty} (s-x)\,|q(t)| + 2\,|p(t)| \right) dt$$

and

$$\sigma^{-}(x) = \left(\int_{-\infty}^{x} (x-s)\,|q(t)| + \frac{2}{\alpha}\,|p(t)| \right) dt.$$

We denote by $f_{\pm}(x,\lambda)$ the solution of (1) with the condition

$$\lim_{x\to\pm\infty} f_{\pm}(x,\lambda)\, e^{\mp i\lambda\mu(x)} = 1,$$

where $\mu(x) = x\sqrt{\rho(x)}$. The solutions $f_{+}(x,\lambda)$ and $f_{-}(x,\lambda)$ will be called the right and the left Jost solutions of (1) respectively.

Theorem 1. *(see [36]) For any λ from the closed upper half plane $Im\lambda \geq 0$, the discontinuous Sturm-Liouville equation (1) has the Jost solutions $f_{+}(x,\lambda)$ which is represented as*

$$f_{+}(x,\lambda) = R_{+}(x)\, e^{i\lambda\mu(x)} + R_{-}(x) e^{-i\lambda\mu(x)} + \int_{\mu(x)}^{+\infty} K^{+}(x,t)\, e^{i\lambda t} dt, \tag{3}$$

where

$$R_{+}(x) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right) e^{i\int_{x}^{+\infty} \frac{p(t)}{\sqrt{\rho(t)}} dt},$$

$$R_{-}(x) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{\rho(x)}}\right) e^{i\int_{x}^{+\infty} \frac{p(t)sgnt}{\sqrt{\rho(t)}} dt},$$

and the kernel $K^{+}(x,t)$ satisfies the inequality

$$\int_{\mu(x)}^{+\infty} \left|K^{+}(x,t)\right| dt \leq C\left\{ e^{\sigma^{+}(\mu(x))} - 1 \right\}$$

for some $C > 0$. Moreover, the following expressions are satisfied:

$$K^{+}(x,\mu(x)) = R_{+}(x)\left\{ \frac{1}{2}\int_{x}^{+\infty} \left(\frac{q(s)}{\sqrt{\rho(s)}} + \frac{p^{2}(s)}{\rho(s)\sqrt{\rho(s)}} \right) ds + \right.$$

$$\frac{i}{2}\int\limits_{x}^{+\infty}\left[\left(\frac{1}{\sqrt{\rho(x)}}\right)^2+\left(1-\frac{\sqrt{\rho(s)}}{\sqrt{\rho(x)}}\right)^2\right]p'(s)ds\Bigg\}, \tag{4}$$

$$K^{+}(x,-\mu(x)+0)-K^{+}(x,-\mu(x)-0)=$$

$$R_{-}(x)\left\{\frac{1}{2}\int\limits_{x}^{+\infty}\left(\frac{q(s)}{\sqrt{\rho(s)}}+\frac{p^2(s)}{\rho(s)\sqrt{\rho(s)}}\right)sgns ds+\right.$$

$$\left.+\frac{i}{2}\int\limits_{x}^{+\infty}\left[\left(\frac{1}{\sqrt{\rho(x)}}\right)^2+\left(1+\frac{sgns\sqrt{\rho(s)}}{\sqrt{\rho(x)}}\right)^2\right]p'(s)ds\right\}. \tag{5}$$

Theorem 2. *(see [36]) For any λ from the closed upper half plane $Im\lambda\geq 0$, the discontinuous Sturm-Liouville equation (1) has the Jost solutions $f_{-}(x,\lambda)$ which is represented as*

$$f_{-}(x,\lambda)=T_{+}(x)e^{i\lambda\mu(x)}+T_{-}(x)e^{-i\lambda\mu(x)}+\int\limits_{-\infty}^{\mu(x)}K^{-}(x,t)e^{-i\lambda t}dt, \tag{6}$$

where

$$T_{+}(x)=\frac{1}{2}\left(1-\frac{\alpha}{\sqrt{\rho(x)}}\right)e^{-i\int\limits_{-\infty}^{x}\frac{p(t)sgnt}{\sqrt{\rho(t)}}dt},$$

$$T_{-}(x)=\frac{1}{2}\left(1+\frac{\alpha}{\sqrt{\rho(x)}}\right)e^{i\int\limits_{-\infty}^{x}\frac{p(t)}{\sqrt{\rho(t)}}dt},$$

and the kernel $K^{-}(x,t)$ satisfies the inequality

$$\int\limits_{-\infty}^{\mu(x)}\left|K^{-}(x,t)\right|dt\leq C\left\{e^{\sigma^{-}(\mu(x))}-1\right\}$$

for some $C>0$. Moreover, the following expressions are satisfied:

$$K^{-}(x,\mu(x))=T_{-}(x)\left\{\frac{1}{2}\int\limits_{-\infty}^{x}\left(\frac{q(s)}{\sqrt{\rho(s)}}+\frac{p^2(s)}{\rho(s)\sqrt{\rho(s)}}\right)ds-\right.$$

$$\left.-\frac{i}{2}\int\limits_{-\infty}^{x}\left[\left(\frac{1}{\sqrt{\rho(x)}}\right)^2+\left(1-\frac{\sqrt{\rho(x)}}{\sqrt{\rho(s)}}\right)^2\right]p'(s)ds\right\}, \tag{7}$$

$$K^{-}(x,-\mu(x)+0)-K^{-}(x,-\mu(x)-0)$$

$$=T_{+}(x)\left\{\frac{1}{2}\int\limits_{-\infty}^{x}\left(\frac{q(s)}{\sqrt{\rho(s)}}+\frac{p^2(s)}{\rho(s)\sqrt{\rho(s)}}\right)sgns ds+\right.$$

$$+\frac{i}{2}\int\limits_{-\infty}^{x}\left[1+\left(1-\frac{sgns\sqrt{\rho\left(x\right)}}{\sqrt{\rho\left(s\right)}}\right)^{2}\right]p'(s)ds\right\}. \tag{8}$$

Theorem 3. *(see [36]) The kernel functions $K^{\pm}(x,t)$ of the integral representation (3), (6) have both partial derivatives of the first order. If $q(x)$ and $p'(x)$ are differentiable functions, then the kernel functions $K^{\pm}(x,t)$ satisfy (a. e.) the partial differential equations*

$$\frac{\partial^2 K^{\pm}(x,t)}{\partial x^2}-\rho\left(x\right)\frac{\partial^2 K^{\pm}(x,t)}{\partial t^2}=q(x)K^{\pm}(x,t)\pm 2ip(x)\frac{\partial K^{\pm}(x,t)}{\partial t},\ t\neq\mu(x),$$

with conditions (4), (5) and (7), (8) for $K^{+}(x,t)$ and $K^{-}(x,t)$ respectively.

We see that the equalities (4), (5) and (7), (8) can be written alternatively as

$$\frac{d}{dx}\left(K^{+}(x,\mu\left(x\right))e^{-i\omega^{+}(x)}\right)=$$

$$=-\frac{1}{4}\left(1+\frac{1}{\sqrt{\rho\left(x\right)}}\right)\left(\frac{q\left(x\right)}{\sqrt{\rho\left(x\right)}}+\frac{p^{2}(x)}{\rho\left(x\right)\sqrt{\rho\left(x\right)}}+i\frac{p'(x)}{\rho\left(x\right)}\right),$$

$$\frac{d}{dx}\left(K^{-}(x,\mu(x))e^{-i\omega^{-}(x)}\right)=$$

$$=\frac{1}{4}\left(1+\frac{\alpha}{\sqrt{\rho\left(x\right)}}\right)\left(\frac{q\left(x\right)}{\sqrt{\rho\left(x\right)}}+\frac{p^{2}(x)}{\rho\left(x\right)\sqrt{\rho\left(x\right)}}-i\frac{p'(x)}{\rho\left(x\right)}\right),$$

$$\frac{d}{dx}\left(\left(K^{+}(x,-\mu(x)+0)-K^{+}(x,-\mu(x)-0)\right)e^{i\widetilde{\omega}^{+}(x)dt}\right)$$

$$=\frac{1}{4}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\left(\frac{q(x)}{\sqrt{\rho(x)}}+\frac{p^{2}(x)}{\rho\left(x\right)\sqrt{\rho\left(x\right)}}-i\frac{p'(x)}{\rho\left(x\right)}\right),$$

$$\frac{d}{dx}\left(\left(K^{-}(x,-\mu(x)+0)-K^{-}(x,-\mu(x)-0)\right)e^{i\widetilde{\omega}^{-}(x)}\right)$$

$$=\frac{1}{4}\left(1-\frac{\alpha}{\sqrt{\rho\left(x\right)}}\right)\left(\frac{q(x)}{\sqrt{\rho(x)}}+\frac{p^{2}(x)}{\rho\left(x\right)\sqrt{\rho\left(x\right)}}+i\frac{p'(x)}{\rho\left(x\right)}\right),$$

where

$$\omega^{\pm}(x)=\pm\int\limits_{x}^{\pm\infty}\frac{p(t)}{\sqrt{\rho\left(t\right)}}dt,\ \widetilde{\omega}^{\pm}(x)=\pm\int\limits_{x}^{\pm\infty}\frac{p(t)sgnt}{\sqrt{\rho\left(t\right)}}dt.$$

Clearly,

$$\widetilde{\omega}^{+}(x)=2\omega^{+}(0)-\omega^{+}(x),\ \ \widetilde{\omega}^{-}(x)=\omega^{-}(x)-2\omega^{-}(0).$$

By setting $\mathrm{Re}K^{\pm}(x,t)=M^{\pm}(x,t)$ and $\mathrm{Im}K^{\pm}(x,t)=N^{\pm}(x,t)$ from (4), (5) and (7), (8) we have

$$\frac{1}{2\sqrt{\rho\,(x)}}\left(1+\frac{1}{\sqrt{\rho\,(x)}}\right)\left(q\,(x)+\frac{p^2(x)}{\rho\,(x)}\right)=$$

$$=-2\frac{d}{dx}\left[M^{+}(x,\mu\,(x))\cos\omega^{+}(x)+N^{+}(x,\mu\,(x))\sin\omega^{+}(x)\right],$$

$$\omega^{+}(x)=\int\limits_{x}^{+\infty}\left(1+\sqrt{\rho\,(t)}\right)\sqrt{\rho\,(t)}\left[M^{+}(t,\mu\,(t))\sin\omega^{+}(t)-N^{+}(t,\mu\,(t))\cos\omega^{+}(t)\right]dt-$$

$$-\int\limits_{x}^{+\infty}\left(1-\sqrt{\rho(t)}\right)\sqrt{\rho\,(t)}\left[PM^{+}(t,-\mu\,(t))\sin\widetilde{\omega}^{+}(t)-PN^{+}(t,-\mu\,(t))\cos\widetilde{\omega}^{+}(t)\right]dt,$$

$$\frac{1}{2\sqrt{\rho\,(x)}}\left(1+\frac{\alpha}{\sqrt{\rho\,(x)}}\right)\left(q\,(x)+\frac{p^2(x)}{\rho\,(x)}\right)=$$

$$2\frac{d}{dx}\left[M^{-}(x,\mu\,(x))\cos\omega^{-}(x)+N^{-}(x,\mu\,(x))\sin\omega^{-}(x)\right],$$

$$\omega^{-}(x)=\int\limits_{-\infty}^{x}\left(1+\frac{\sqrt{\rho\,(t)}}{\alpha}\right)\sqrt{\rho\,(t)}\left[M^{-}(t,\mu\,(t))\sin\omega^{-}(t)-N^{-}(t,\mu\,(t))\cos\omega^{-}(t)\right]dt+$$

$$\int\limits_{-\infty}^{x}\left(1-\frac{\sqrt{\rho\,(t)}}{\alpha}\right)\sqrt{\rho\,(t)}+\left[PM^{-}(t,-\mu\,(t)\sin\widetilde{\omega}^{-}(t)-PN^{-}(t,-\mu\,(t))\cos\widetilde{\omega}^{-}(t)\right]dt,$$

where $Pg(t,z)=g(t,z+0)-g(t,z-0)$ denotes the jumping of the function $g(t,z)$ with respect to z.

When the condition (2) is satisfied equation (1) has certain solutions $u^{-}(x,\lambda)$, $u^{+}(x,\lambda)$ which hold the asymptotic expressions

$$u^{-}(x,\lambda)=\frac{r^{-}(\lambda)}{\alpha}e^{-i\lambda\alpha x}+\frac{e^{i\lambda\alpha x}}{\alpha}+o(1),\quad x\to-\infty,$$

$$u^{-}(x,\lambda)=t(\lambda)e^{i\lambda x}+o(1),\quad x\to+\infty,$$

$$u^{+}(x,\lambda)=t(\lambda)e^{-i\lambda\alpha x}+o(1),\quad x\to-\infty,$$

$$u^{+}(x,\lambda)=r^{+}(\lambda)e^{i\lambda x}+e^{-i\lambda x}+o(1),\quad x\to+\infty,$$

where

$$r^{+}(\lambda)=-\frac{\overline{b(\lambda)}}{a(\lambda)},\quad r^{-}(\lambda)=\frac{b(\lambda)}{a(\lambda)},$$

$$a(\lambda)=\frac{1}{2\lambda i}W\left\{f_{+}(x,\lambda),f_{-}(x,\lambda))\right\},\quad\lambda\in\mathbb{R}^{*}=\mathbb{R}\backslash\{0\},\tag{9}$$

$$b(\lambda)=-\frac{1}{2\lambda i}W\left\{f_{+}(x,\lambda),\overline{f_{-}(x,\lambda)}\right\},\quad\lambda\in\mathbb{R}^{*}=\mathbb{R}\backslash\{0\},\ \text{where }\mathbb{R}=(-\infty,+\infty).$$

The solutions $u^-(x,\lambda)$ and $u^+(x,\lambda)$ are called the eigenfunctions of the left and the right scattering problems respectively. Here the coefficients $r^-(\lambda)$, $r^+(\lambda)$ and $t(\lambda)$ are called the left, the right reflection coefficients and the transmission coefficient respectively. Let $\lambda_k (k=1,2,...,n)$ be the zeros of the function $a(\lambda)$ in the upper halfplane $\mathrm{Im}\lambda > 0$. We denote by m_k the multiplicity of the root λ_k of the equation $a(\lambda)=0$.

The following properties of the roots λ_k can be obtained from the formula (9) (see also [28], [29]).

Lemma 1. *There exist the number chains* $\left(\varkappa^{\pm}_{k,0},...,\varkappa^{\pm}_{k,m_{k-1}}\right)$ *such that the identities*

$$\frac{1}{j!}\frac{d^j}{d\lambda^j}f^{\mp}(x,\lambda)\bigg|_{\lambda=\lambda_k}=\sum_{\nu=0}^{j}\varkappa^{\pm}_{k,j-\nu}\frac{1}{\nu!}\,\frac{d^{\nu}}{d\lambda^{\nu}}f^{\pm}(x,\lambda)\bigg|_{\lambda=\lambda_k}$$

are satisfied, where $j=0,...,m_k-1,\ k=1,..n,\ \varkappa^{\pm}_{k,0}\neq 0$.

It is easy to prove that the following relationship holds between the right and the left normalization chains:

$$\varkappa^{+}_{k,0}\varkappa^{-}_{k,0}=1,$$

$$\varkappa^{\pm}_{k,j}=\frac{(-1)^j}{\left(\varkappa^{\pm}_{k,0}\right)^{j+1}}\begin{vmatrix}\varkappa^{\pm}_{k,1} & \varkappa^{\pm}_{k,0} & 0 & ... & 0\\ \varkappa^{\pm}_{k,2} & \varkappa^{\pm}_{k,1} & \varkappa^{\pm}_{k,0} & ... & 0\\ ... & ... & ... & ... & ...\\ \varkappa^{\pm}_{k,j} & \varkappa^{\pm}_{k,j-1} & \varkappa^{\pm}_{k,j-2} & ... & \varkappa^{\pm}_{k,1}\end{vmatrix}.$$

Definition. *The collections of numbers* $\left(\varkappa^{+}_{k,0},...,\varkappa^{+}_{k,m_{k-1}}\right)$ *and* $\left(\varkappa^{-}_{k,0},...,\varkappa^{-}_{k,m_{k-1}}\right)$ *will be called the right and the left normalization chains, respectively, corresponding to the root* λ_k *of the equation* $a(\lambda)=0$ *for the equation (1).*

The collections

$$\left\{r^{+}(\lambda),\ \lambda_k,\ \varkappa^{+}_{k,j},\ j=0,...,m_k-1,\ k=1,..n\right\}$$

and

$$\left\{r^{-}(\lambda),\ \lambda_k,\ \varkappa^{-}_{k,j},\ j=0,...,m_k-1,\ k=1,..n\right\}$$

are called the right and the left scattering data, respectively, for the equation (1).

The construction of the potential functions in equation (1) with the help of the asymptotics of the wave functions is referred as the inverse scattering problem. In other words, the inverse scattering problem for the (1), (2) consists in recovering the coefficients $q(x)$ and $p(x)$ from the scattering data.

In this work the potentials $q(x)$ and $p(x)$ are constructed by slightly varying the method of Marchenko-Fadeev. We set

$$F^{\pm}(x)=\sum_{k=1}^{n}P^{\pm}_{k}(x)e^{\pm i\lambda_k x}+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(r^{\pm}(\lambda)\mp\frac{1-\alpha}{1+\alpha}e^{\mp i(\theta\pm\omega)}\right)e^{\pm i\lambda x}d\lambda$$

with

$$P_k^{\pm}(x) = -ie^{\mp i\lambda_k x} \sum_{j=0}^{m_k-1} \varkappa_{k,m_k-1-j}^{\pm} \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} \frac{(\lambda-\lambda_k)^{m_k} e^{\pm i\lambda x}}{a(\lambda)}) \right\}_{\lambda=\lambda_k}$$

and derive the integral equations

$$R_+(x)F^+(\mu(x)+y) + R_-(x)F^+(-\mu(x)+y) + \overline{K^+(x,y)} + \int_{\mu(x)}^{+\infty} F^+(y+t)K^+(x,t)dt+$$

$$+\frac{1-\sqrt{\rho(x)}}{1+\sqrt{\rho(x)}} e^{-i(\omega+\theta)} K^+(x,-y) = 0, \ y \geq \mu(x), \tag{10}$$

$$T_+(x)F^-(-\mu(x)+y) + T_-(x)F^-(\mu(x)+y) + \overline{K^-(x,y)}+$$

$$+\int_{-\infty}^{\mu(x)} F^-(y+t)K^-(x,t)dt + \frac{1-\sqrt{\rho(x)}}{1+\sqrt{\rho(x)}} e^{i(\theta-\omega)} K^-(x,-y) = 0, \ y \leq \mu(x) \tag{11}$$

for the unknown functions $K^{\pm}(x,t)$. These integral equations are called the main equations of the inverse problem of scattering theory for the equation (1). The main equations are different from the classical equations in Marchenko-Fadeev's method because the discontinuity of the function $\rho(x)$ which strongly influences the structure of the main equations.

Obviously, in order to write the integral equations (10) and (11) we need the functions $F^{\pm}(x)$ which are defined by the scattering data. If equation (10) and (11) which are constructed by the scattering data, have unique solutions $K^+(x,y)$ and $K^-(x,y)$, respectively, the functions $q(x)$ and $p(x)$ can be found from (4), (5) and (7), (8). In the paper it is examined the solvability of the equations (10), (11) and introduced the algorithm for recovering the potentials $q(x)$ and $p(x)$.

In the spectral theory of Sturm-Liouville and Dirac operators, especially in the examination of inverse spectral and inverse scattering problems on finite and infinite intervals, the method of transformation operators, introduced by V.A. Marchenko [32]-[34] and later developed by fundamental works of Faddeev, Levitan, Gasımov, Gelfand [5]-[7], [12], [15], [27], gave new impetus to researchers. After these important studies inverse problems in various statements, especially the full-line inverse scattering problem for one dimensional Schrödinger equation have been extensively improved and investigated by many authors. For details we refer to [4], [16], [24]-[26], [34].

The full-line inverse scattering problem for an energy dependent (or generalized) Schrödinger equation, as a generalization of the Marchenko method, was first investigated by Jaulent an Jean [19], [20] and by Kaup [23] in connection with a nonlinear evolution equation (see also [1], [2], [39]). Later for the equation, which was considered in [23], the full-line inverse scattering problem was investigated in [38] by reduction this problem to the inverse scattering problem for the matrix-valued energy dependent Schrödinger equation. More systematically, in the case $\rho(x) = 1$ the direct and inverse scattering problems for (1) without discrete spectrum was investigated by Maksudov and Guseinov [28], [29], where a procedure through which the potentials $q(x)$ and $p(x)$ are recovered from the scattering data are established under conditions (2). This problem and the inverse scattering problem on the half line for the equation (1) in the case $\rho(x) = 1$ recently has been investigated in [21], [22] where the differentiability assumptions on the function $p(x)$ is not required.

Some direct and inverse spectral problems for ordinary differential operator pencils have been also investigated in series works of M.G. Gasymov (see for instance [9]-[11], [13], [14]) who has made important contributions to the spectral theory of the ordinary differential operators.

In recent years, it has been proven that Sturm-Liouville operators with piecewise constant coefficients or discontinuity conditions have integral representations similar to transformation operators for their special solutions. Therefore, it became necessary to investigate different Sturm-Liouville inverse problems with piecewise constant coefficients or discontinuity conditions with the modified Marchenko method. On the half line the Sturm-Liouville inverse scattering problem with piecewise constant coefficient studied in [3], [8], [17], [30]. In the papers [3], [8] the solution of inverse problem on the half line $[0, +\infty)$ is reduced to solution of two inverse problems on the intervals $[0, a]$ and $[a, +\infty)$. In the case $p(x) = 0$ and $\rho(x) \neq 1$ the inverse scattering problem on the half line was also in [17], [30] by using the new (non-triangular) integral representation of the Jost solution for the equation (1). The direct and inverse scattering problems for the Sturm-Liouville equation with discontinuities in an interior point recently have been investigated in [18] where new integral representations for the Jost solutions of the Schrödinger equation, are obtained and applied to the investigation of the inverse problem.

In the study [31], on the real line authors consider direct and inverse scattering problems for the equation (1) with $p(x) = 0$ and $\rho(x) \neq 1$. It turns out that in this case, the discontinuity of the function $\rho(x)$ strongly affects to the structure of the representation of the Jost solutions (see [28]) and the main equations of the inverse problem.

In the study [37] in a finite interval for the equation (1) with initial conditions integral representations have been obtained for two linearly independent solutions. [36] has devoted to the construction of the integral representations (3) and (6) of the right and left Jost solutions respectively, for the equation (1).

Note also that the study [35] is devoted to both the direct scattering problem and the inverse scattering problem without discrete spectrum for the equation (1) with $\rho(x) = 1$ and discontinuities in an interior point.

In the present paper direct and inverse problems of the scattering theory for the quadratic pencil of the Sturm-Liouville equation with the discontinuous coefficient is investigated in a class of decreasing potential functions. First of all, using the known integral expressions of the solutions of the quadratic pencil of the discontiouous Sturm-Liouville equation, satisfying the Jost conditions for $x \to +\infty$ and $x \to -\infty$, the spectral properties of the problem are investigated. Since the kernel functions of the integral representations for the solutions have a jump discontinuity, the relationship between the potential functions of the equation and the kernel functions has a difficult structure, but by taking advantage of this relationship, it has become possible to reach new integral equations of the Faddeev-Marchenko type, which have a fundamental role in the solution of the inverse problem. The solution of the inverse problem is obtained from the unique solvability of the integral equations of the Faddeev-Marchenko type, based on the scattering data in the special case where there is no discrete spectrum. In section 3, under assumption that the discreet spectrum is absent, the direct problem is investigated, the main integral equations are derived and the unique solvability of these equations is proved. In section 4 the solution of the inverse scattering problem is given, the necessary and sufficient conditions for recovering of the potentials are obtained.

2. The Direct Scattering Problem

In the present section we consider the direct scattering problem and introduce some properties of the scattering data necessary for the solution of the inverse problem. Since the functions $q(x)$, $p(x)$ and the number α are real, the functions $\overline{f_+(x,\lambda)}$ and $\overline{f_-(x,\lambda)}$ are also solutions of the problem (1), (2) for real λ. Because of

$$f_+(x,\lambda) = e^{i\lambda x}\left(1+o(1)\right), \quad x\to+\infty,$$

$$f_+'(x,\lambda) = e^{i\lambda x}\left(i\lambda e^{i\omega_+(x)}+o(1)\right), \quad x\to+\infty,$$

$$f_-(x,\lambda) = e^{-i\alpha\lambda x}(1+o(1)), \quad x\to-\infty,$$

$$f_-'(x,\lambda) = e^{-i\alpha\lambda x}(-i\alpha\lambda e^{i\omega_-(x)}+o(1)), \quad x\to-\infty,$$

which follows from the representations (3), (6), we have the Wronskians

$$W\left\{f_+(x,\lambda),\overline{f_+(x,\lambda)}\right\} = f_+'(x,\lambda)\overline{f_+(x,\lambda)} - f_+(x,\lambda)\overline{f_+'(x,\lambda)} = 2i\lambda,$$

$$W\left\{f_-(x,\lambda),\overline{f_-(x,\lambda)}\right\} = f_-'(x,\lambda)\overline{f_-(x,\lambda)} - f_-(x,\lambda)\overline{f_-'(x,\lambda)} = -2i\lambda\alpha$$

for all real λ. Consequently, when $\lambda\neq 0$, the pairs $f_+(x,\lambda),\overline{f_+(x,\lambda)}$ and $f_-(x,\lambda),\overline{f_-(x,\lambda)}$ form two fundamental systems of solutions. Then for all real $\lambda\neq 0$, each solution of equation (1) can be expressed as a linear combination of the solutions $f_+(x,\lambda)$, $\overline{f_+(x,\lambda)}$ or $f_-(x,\lambda)$, $\overline{f_-(x,\lambda)}$. In particular, we have

$$f_+(x,\lambda) = \frac{b(\lambda)}{\alpha}f_-(x,\lambda) + \frac{a(\lambda)}{\alpha}\overline{f_-(x,\lambda)}, \tag{12}$$

$$f_-(x,\lambda) = -\overline{b(\lambda)}f_+(x,\lambda) + a(\lambda)\overline{f_+(x,\lambda)}, \tag{13}$$

where

$$a(\lambda) = \frac{1}{2\lambda i}W\left\{f_+(x,\lambda),f_-(x,\lambda))\right\}, \quad \lambda\in\mathbb{R}^*, \tag{14}$$

$$b(\lambda) = -\frac{1}{2\lambda i}W\left\{f_+(x,\lambda),\overline{f_-(x,\lambda)}\right\}, \quad \lambda\in\mathbb{R}^*. \tag{15}$$

Further, according to (12)and (13) we have

$$\left|a(\lambda)\right|^2 - \left|b(\lambda)\right|^2 = \alpha, \ \lambda\in\mathbb{R}^*. \tag{16}$$

Dividing both sides of (12)and (13) by $a(\lambda)$, for $\lambda\in\mathbb{R}^*$, and using the relations

$$t(\lambda) = \frac{1}{a(\lambda)},$$

$$r^{+}(\lambda)=-\frac{\overline{b(\lambda)}}{a(\lambda)},r^{-}(\lambda)=\frac{b(\lambda)}{a(\lambda)} \tag{17}$$

we obtain certain solutions of equation (1)

$$\begin{gathered}u^{-}(x,\lambda)\equiv t(\lambda)f_{+}(x,\lambda)=\tfrac{r^{-}(\lambda)}{\alpha}f_{-}(x,\lambda)+\tfrac{1}{\alpha}\overline{f_{-}(x,\lambda)},\\ u^{+}(x,\lambda)\equiv t(\lambda)f_{-}(x,\lambda)=r^{+}(\lambda)f_{+}(x,\lambda)+\overline{f_{+}(x,\lambda)}\end{gathered} \tag{18}$$

for which the asymptotic expressions

$$\begin{gathered}u^{-}(x,\lambda)=\frac{r^{-}(\lambda)}{\alpha}e^{-i\lambda\alpha x}+\frac{e^{i\lambda\alpha x}}{\alpha}+o(1),\quad x\to-\infty,\\ u^{-}(x,\lambda)=t(\lambda)e^{i\lambda x}+o(1),\quad x\to+\infty,\\ u^{+}(x,\lambda)=t(\lambda)e^{-i\lambda\alpha x}+o(1),\quad x\to-\infty,\\ u^{+}(x,\lambda)=r^{+}(\lambda)e^{i\lambda x}+e^{-i\lambda x}+o(1),\quad x\to+\infty,\end{gathered}$$

are satisfied. The solutions $u^{-}(x,\lambda)$ and $u^{+}(x,\lambda)$ are called the eigenfunctions of the left and the right scattering problems respectively, and the coefficients $r^{-}(\lambda)$, $r^{+}(\lambda)$ and $t(\lambda)$ are called the left, the right reflection coefficients and the transmission coefficient respectively.

Since $f_{+}(x,\lambda)$ and $f_{-}(x,\lambda)$ admit an analytic continuation to half-plane $Im\lambda>0$, formula (14) implies that the function $a(\lambda)$ also admits an analytic continuation to half-plane $Im\lambda>0$ with the same formula. From (16) we also have there $a(\lambda)\neq 0$ for $\lambda\in\mathbb{R}^{*}$.

Lemma 2. *The function $a(\lambda)$ may have a only finite number of zeros on the half plane* $\mathrm{Im}\lambda>0$.

Proof. Assuming the converse we suppose the function $a(\lambda)$ has a countable collection of mutually disjoint zeros $\lambda_k (k=1,2,...)$. Then from the equality $a(\lambda_k)=0$ by virtue of (14) we conclude that the solutions $f_{+}(x,\lambda_k)$ and $f_{-}(x,\lambda_k)$ and are linearly dependent, i. e. there is a constant c_k such that

$$f_{+}(x,\lambda_k)=c_k f_{-}(x,\lambda_k),\ x\neq 0,\ x\in\mathbb{R}.$$

Note that for $Im\lambda>0$ the solution $f_{+}(x,\lambda)$ is exponentially decreasing as $x\to+\infty$ and the solution $f_{-}(x,\lambda)$ behaves in a similar way when $x\to-\infty$. Thus, for complex values of λ from the upper half-plane the function $f_{+}(x,\lambda)$ $(f_{-}(x,\lambda))$ is a unique solution of equation (1) which belongs to the space $L_2(0,\infty)$ $(L_2(-\infty,0))$. Therefore equation (1) has a solution belonging to $L_2(-\infty,\infty)$ only for those values of λ $(Im\lambda>0)$ for which the functions $f_{+}(x,\lambda)$ and $f_{-}(x,\lambda)$ are linearly dependent. Then we have that $f_{+}(x,\lambda_k)$ is a nontrivial solution of the problem (1), (2) belonging to $L_2\left(-\infty,+\infty\right)$. Therefore, if L_0 is the minimal closed operator generated in the space $L_2\left(-\infty,+\infty\right)$ by the differential expression $-\frac{d^2}{dx^2}+q(x)$, then L_0 is a selfadjoint operator and for the solution $y_k(x)=f_{+}(x,\lambda_k)$ we have

$$\lambda_k^2\rho(x)y_k\left(x\right)-2\lambda_k p(x)y_k\left(x\right)-L_0 y_k\left(x\right)=0.$$

The last equation implies that

$$\lambda_k = \frac{(py_k, y_k) \pm \sqrt{(py_k, y_k)^2 + (L_0 y_k, y_k)(\rho y_k, y_k)}}{(\rho y_k, y_k)},$$

where $(.,.)$ denotes the standard inner product in the space $L_2(-\infty, +\infty)$. Since λ_k is not a real number we have $(L_0 y_k, y_k) < 0$ for all k. Since the numbers $\lambda_k (k = 1, 2, 3, ...)$ are mutually disjoint then from the asymptotic formula $f_+(x, \lambda_k) = e^{i\lambda_k x}(1 + o(1))$, $x \to +\infty$ we have that the system of functions $\{y_k\}$ is linearly independent. As is known, the number of negative points of the spectrum of a self-adjoint operator L_0 is equal to the maximum dimension of linear manifolds on which the inequality $(L_0 f, f) < 0$ is satisfied. This means that the selfadjoint operator L_0 has a countable collection of negative eigenvalues. But it contradicts the condition (2) on the function $q(x)$. ◀

The inverse scattering problem for the equation (1) with the conditions (2) consists of recovering the coefficients $q(x)$ and $p(x)$ of the equation (1) by the given right or left scattering data and proving some necessary and sufficient conditions for an arbitrary collection $\{r(\lambda), \lambda_k, \varkappa_{k,j}, j = 0, ..., m_k - 1, k = 1, ..n\}$ to be the right or the left scattering data for some problem of the form of (1), (2).

Lemma 3. *For $\lambda \in \mathbb{R}^*$ the function $a(\lambda)$, $b(\lambda)$ defined by formulas (14), (15) have the following representations*

$$a(\lambda) = \frac{1+\alpha}{2} e^{i\omega} - \frac{de^{i\omega}}{2i\lambda} + \frac{1}{2i\lambda} \int_0^{+\infty} \varphi(t) e^{i\lambda t} dt, \tag{19}$$

$$b(\lambda) = \frac{\alpha - 1}{2} e^{i\theta} - \frac{c}{2i\lambda} e^{i\theta} + \frac{1}{2i\lambda} \int_{-\infty}^{+\infty} \psi(t) e^{i\lambda t} dt, \tag{20}$$

where

$$\omega = \int_{-\infty}^{+\infty} \frac{p(x)}{\sqrt{\rho(x)}} dx, \ \theta = \int_{-\infty}^{+\infty} \frac{p(x) sgnx}{\sqrt{\rho(x)}} dx,$$

$$d = \frac{\alpha+1}{2} \int_{-\infty}^{+\infty} \left(\frac{p^2(x)}{\rho(x)\sqrt{\rho(x)}} + \frac{q(x)}{\sqrt{\rho(x)}} \right) dx + \frac{\alpha - 1}{2} \left(\frac{1}{\alpha^2} - 1 \right) ip(0),$$

$$c = \frac{\alpha-1}{2} \int_{-\infty}^{+\infty} \left(\frac{p^2(x)}{\rho(x)\sqrt{\rho(x)}} + \frac{q(x)}{\sqrt{\rho(x)}} \right) sgnx dx + \frac{\alpha + 1}{.2} \left(\frac{1}{\alpha^2} - 1 \right) ip(0)$$

and $\varphi(t) \in L_1(0, +\infty)$, $\psi(t) \in L_1(-\infty, +\infty)$.

Proof. Since the Wronskian of two solutions of the equation (1) doesn't depend on x we have

$$a(\lambda) = \frac{1}{2i\lambda} W\left[f_+(x,\lambda), f_-(x,\lambda)\right]$$

$$= \frac{1}{2i\lambda}\left[f_+(0^+,\lambda)' f_-(0^+,\lambda) - f_+(0^+,\lambda) f_-(0^+,\lambda)'\right].$$

Using the integral representation for $f_\pm(x,\lambda)$ we compute

$$a(\lambda) = \frac{1}{2i\lambda}\left[\left(i\lambda - -\frac{i}{2}p(0) - \frac{1}{2}\int\limits_0^{+\infty}\left(p^2(t)+q(t)\right)dt\right)e^{i\omega_+(0)} + \int\limits_0^{+\infty} K_x^+(0^+,t)e^{i\lambda t}dt\right] \times$$

$$\left[e^{i\omega_-(0)} - \frac{1}{2i\lambda}\left(\int\limits_{-\infty}^0 \left(\frac{p^2(t)}{\alpha^3} + \frac{q(t)}{\alpha}\right)dt - \frac{i}{\alpha^2}p(0)\right)e^{i\omega_-(0)} + \frac{1}{i\lambda}\int\limits_{-\infty}^0 K_t^-(0^+,t)e^{-i\lambda t}dt\right] -$$

$$\frac{1}{2i\lambda}\left[e^{i\omega_+(0)} - \frac{1}{2i\lambda}\left(\int\limits_0^{+\infty}\left(p^2(t)+q(t)\right)dt - ip(0)\right)e^{i\omega_+(0)} - \frac{1}{i\lambda}\int\limits_{-\infty}^0 K_t^+(0^+,t)e^{i\lambda t}dt\right] \times$$

$$\left(-i\alpha\lambda + \frac{i}{2\alpha}p(0) + \frac{1}{2}\int\limits_{-\infty}^0\left(\frac{p^2(t)}{\alpha^2} + q(t)\right)dt\right)e^{i\omega_-(0)} + \int\limits_{-\infty}^0 K_x^-(0^+,t)e^{-i\lambda t}dt$$

which implies (19) after some simple operations. Similarly the formula (20) is proved. ◄

From the Lemma 3 we have that

$$a(\lambda) = \frac{1+\alpha}{2}e^{i\omega} + O\left(\frac{1}{\lambda}\right), \quad |\lambda| \to \infty,\ \mathrm{Im}\lambda \geq 0, \tag{21}$$

and

$$b(\lambda) = \frac{\alpha-1}{2}e^{i\theta} + O\left(\frac{1}{\lambda}\right), \quad |\lambda| \to \infty, \quad \lambda \in \mathbb{R}.$$

The formula (16) implies

$$\frac{1}{|a(\lambda)|} \leq \frac{1}{\sqrt{\alpha}} \text{ for } \lambda \in \mathbb{R}^*.$$

The following lemma is proved by standard methods (see [28]).

Lemma 4. *The function* $\frac{1}{a(\lambda)}$ *is bounded in the region* $D_\delta = \{\lambda : \ |\lambda| \leq \delta,\ \mathrm{Im}\lambda \geq 0\}$ *fore some* $\delta > 0$.

Note that, from the formulas (16) and (17) we obtain

$$\left|r^\pm(\lambda)\right| < 1, \quad \lambda \in \mathbb{R}^*,$$

and the formulas (19) and (20) imply (for $\lambda \in \mathbb{R}$)

$$r^{-}(\lambda) = \frac{b(\lambda)}{a(\lambda)} = \frac{\alpha - 1}{\alpha + 1} e^{i(\theta - \omega)} + O\left(\frac{1}{\lambda}\right), \quad |\lambda| \to \infty,$$

$$r^{+}(\lambda) = -\frac{\overline{b(\lambda)}}{a(\lambda)} = \frac{1 - \alpha}{1 + \alpha} e^{-i(\theta + \omega)} + O\left(\frac{1}{\lambda}\right), \quad |\lambda| \to \infty.$$

Hence

$$r^{\pm}(\lambda) = \pm \frac{1 - \alpha}{1 + \alpha} e^{\mp i(\theta \pm \omega)} + O\left(\frac{1}{\lambda}\right), \quad |\lambda| \to \infty, \quad \lambda \in \mathbb{R}.$$

So $r^{\pm}(\lambda) \mp \frac{1-\alpha}{1+\alpha} e^{\mp i(\theta \pm \omega)} \in L_2(-\infty, +\infty)$ and consequently the function

$$F^{\pm}_{0R}(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \left(r^{\pm}(\lambda) \mp \frac{1 - \alpha}{1 + \alpha} e^{\mp i(\theta \pm \omega)} \right) e^{\pm i\lambda x} d\lambda$$

also belongs to $L_2(-\infty, +\infty)$.

Lemma 5. *(see [29]) The function $za(z)$is continuous on the closed upper half plane and there exists $C > 0$ such that*

$$1 - \left| r^{\pm}(\lambda) \right| \geq \frac{C\lambda^2}{1 + \lambda^2}, \quad \lambda \in \mathbb{R}^*.$$

Moreover

$$\lim_{\lambda \to 0} \lambda a(\lambda) \left[r^{\pm}(\lambda) + 1 \right] = 0.$$

Clear that the function $r^{\pm}(\lambda)$ is continuous for all real $\lambda \neq 0$. Moreover for all $\lambda \in \mathbb{R}^*$ we have

$$r(\lambda) = -\overline{r^{+}(\lambda)} \frac{\overline{a(\lambda)}}{a(\lambda)}. \tag{22}$$

It can be proved by analogous way as in [16], [29] that $r^{\pm}(\lambda)$ is continuous for all real λ and if $|r^{\pm}(0)| < 1$ then $r^{\pm}(0) = -1$.The formula (22) shows that the left reflection coefficient is uniquely defined by the right reflection coefficient. The following lemma gives the way to construct the function $a(z)$ by the right reflection coefficient.

Lemma 6. *The function $a(z)$ can be reconstructed by the right reflection coefficient as*

$$a(z) = \frac{1 + \alpha}{2} \exp\left\{ i\omega - \frac{1}{2\pi i\alpha} \int\limits_{-\infty}^{+\infty} \frac{\ln\left[\left(1 - |r^{+}(\lambda)|^2 \right) \right]}{\lambda - z} d\lambda \right\} \prod_{k=1}^{n} \frac{z - \lambda_k}{z - \overline{\lambda_k}}, \tag{23}$$

where

$$\omega = \int\limits_{-\infty}^{+\infty} \frac{p(x)}{\sqrt{\rho(x)}} dx.$$

Proof. According to Lemma 2 and Lemma 3 the function $g(z) = \frac{1+\alpha}{2} e^{i\omega} \frac{1}{a(z)} \prod_{k=1}^{n} \frac{z-\lambda_k}{z-\overline{\lambda_k}}$ is halomorphic on the upper halfplane, uniformly bounded, has no zeros and behaves as $1 + O(\frac{1}{z})$ when $|z| \to \infty$. Moreover, for real $\lambda \neq 0$ according to the formulas (16) and (17) we have

$$|g(\lambda)| = \frac{1}{\sqrt{\alpha}} \sqrt{1 - |r^{\pm}(\lambda)|^2}.$$

This allows us to reconstruct the function $\ln g(z)$ $(\mathrm{Im} z > 0)$ by its real part $\ln |g(\lambda)|$ $(-\infty < \lambda < \infty)$. For this we use the Poisson-Schwarz formula

$$\ln g(z) = \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{\ln |g(\lambda|}{\lambda - z} d\lambda$$

which yields (23). ◀

3. The Main Equations of the Inverse Problem

In order to derive the main integral equations for the case $x < 0$ we use identity (18). The second relation in (18) can be rewritten as

$$\left(\frac{1}{a(\lambda)} - \frac{2}{1+\alpha} e^{-i\omega} \right) f_-(x,\lambda) = \left(r^+(\lambda) + \frac{\alpha-1}{\alpha+1} e^{-i(\omega+\theta)} \right) f_+(x,\lambda)$$
$$-\frac{\alpha-1}{\alpha+1} e^{-i(\omega+\theta)} f_+(x,\lambda) - \frac{2}{1+\alpha} e^{-i\omega} f_-(x,\lambda) + \overline{f_+(x,\lambda)}.$$

Multiplying the both sides of the last equation $\frac{1}{2\pi} e^{i\lambda y}$, where $y > \mu(x)$, and integrating it with respect to λ over the interval $(-\infty, +\infty)$ we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{a(\lambda)} - \frac{2}{1+\alpha} e^{-i\omega} \right) f_-(x,\lambda) e^{i\lambda y} d\lambda =$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(r^+(\lambda) + \frac{\alpha-1}{\alpha+1} e^{-i(\omega+\theta)} \right) f_+(x,\lambda) e^{i\lambda y} d\lambda + \tag{24}$$
$$+\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\overline{f_+(x,\lambda)} - \frac{\alpha-1}{\alpha+1} e^{-i(\omega+\theta)} f_+(x,\lambda) - \frac{2e^{-i\omega}}{1+\alpha} f_-(x,\lambda) \right] e^{i\lambda y} d\lambda.$$

Further, using the representation (3), (6) of the solutions $f_{\pm}(x,\lambda)$ we find

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(r^+(\lambda) + \frac{\alpha-1}{\alpha+1} e^{-i(\omega+\theta)} \right) f_+(x,\lambda) e^{i\lambda y} d\lambda +$$
$$+\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\overline{f_+(x,\lambda)} - \frac{\alpha-1}{\alpha+1} e^{-i(\omega+\theta)} f_+(x,\lambda) - \frac{2e^{-i\omega}}{1+\alpha} f_-(x,\lambda) \right] e^{i\lambda y} d\lambda =$$
$$= F_R^+(x,y) + \int_{\mu(x)}^{\infty} K^+(x,t) F_{0R}^+(t+y) dt + \overline{K^+(x,y)} -$$

$$-\frac{\alpha-1}{\alpha+1}e^{-i(\omega+\theta)}K^{+}(x,-y)-\frac{2e^{-i\omega}}{1+\alpha}K^{-}(x,y),$$

where

$$F_R^{+}(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[r^{+}(\lambda)+\frac{\alpha-1}{\alpha+1}e^{-i(\omega+\theta)}\right]\left[R_{+}(x)\,e^{i\lambda\mu(x)}+R_{-}(x)e^{-i\lambda\mu(x)}\right]e^{i\lambda y}d\lambda=$$

$$=R_{+}(x)F_{0R}^{+}(\mu(x)+y)+R_{-}(x)F_{0R}^{+}(-\mu(x)+y)\,. \qquad (25)$$

Therefore, the right hand side of (24) takes the form of

$$F_R^{+}(x,y)+\int_{\mu(x)}^{\infty}K^{+}(x,t)F_{0R}^{+}(t+y)dt+\overline{K^{+}(x,y)}+$$

$$+\frac{1-\alpha}{1+\alpha}e^{-i(\omega+\theta)}K^{+}(x,-y)-\frac{2e^{-i\omega}}{1+\alpha}K^{-}(x,y).$$

Now, let's compute the left hand side of (24) by using the contour integration method. We have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{a(\lambda)}-\frac{2}{1+\alpha}e^{-i\omega}\right)f_{-}(x,\lambda)e^{i\lambda y}d\lambda=$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{a(\lambda)}-\frac{2}{1+\alpha}e^{-i\omega}\right)f_{-}(x,\lambda)e^{i\lambda\mu(x)}e^{i\lambda(y-\mu(x))}d\lambda.$$

Note that

i) the function

$$f_{-}(x,\lambda)e^{i\lambda\mu(x)}=T_{+}(x)e^{2i\lambda\mu(x)}+T_{-}(x)+\int_{-\infty}^{\mu(x)}K^{-}(x,t)e^{i\lambda(\mu(x)-t)}dt$$

is uniformly bounded on the half plane $Im\lambda\geq 0$;

ii) $\frac{1}{a(\lambda)}-\frac{2}{1+\alpha}e^{-i\omega}\to 0$, $|\lambda|\to\infty$ $(Im\lambda\geq 0)$ by virtue of (21);

iii) according to Lemma 2 the function $\frac{1}{a(\lambda)}$, consequently the function $\left(\frac{1}{a(\lambda)}-\frac{2}{1+\alpha}e^{-i\omega}\right)f_{-}(x,\lambda)e^{i\lambda y}$is bounded in some neighborhood of zero.

Hence, applying the Jordan's lemma we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{a(\lambda)}-\frac{2}{1+\alpha}e^{-i\omega}\right)f_{-}(x,\lambda)e^{i\lambda y}d\lambda=$$

$$=i\sum_{k=1}^{n}\underset{\lambda=\lambda_k}{res}\left(\frac{1}{a(\lambda)}-\frac{2}{1+\alpha}e^{-i\omega}\right)f_{-}(x,\lambda)e^{i\lambda y}.$$

Since

$$\underset{\lambda=\lambda_k}{res}\left(\frac{1}{a(\lambda)}-\frac{2}{1+\alpha}e^{-i\omega}\right)f_{-}(x,\lambda)e^{i\lambda y}=\underset{\lambda=\lambda_k}{res}\frac{1}{a(\lambda)}f_{-}(x,\lambda)e^{i\lambda y}=$$

$$=\frac{1}{(m_k-1)!}\left\{\frac{d^{m_k-1}}{d\lambda^{m_k-1}}\left[(\lambda-\lambda_k)^{m_k}\right]\frac{f_{-}(x,\lambda)e^{i\lambda y}}{a(\lambda)}\right\}_{\lambda=\lambda_k}=$$

$$\frac{1}{(m_k-1)!}\sum_{j=0}^{m_k-1}\frac{(m_k-1)!}{j!\,(m_k-1-j)!}\left\{\frac{d^{m_k-1-j}}{d\lambda^{m_k-1-j}}f_-(x,\lambda))\right\}_{\lambda=\lambda_k}\times$$

$$\times\left\{\frac{d^{m_k-1}}{d\lambda^{m_k-1}}\frac{(\lambda-\lambda_k)^{m_k}\,e^{i\lambda y}}{a(\lambda)}\right\}_{\lambda=\lambda_k}$$

$$=\sum_{j=0}^{m_k-1}\frac{1}{j!}\left\{\frac{d^j}{d\lambda^j}\frac{(\lambda-\lambda_k)^{m_k}\,e^{i\lambda y}}{a(\lambda)}\right\}_{\lambda=\lambda_k}\times$$

$$\times\sum_{\nu=0}^{m_k-1-j}\varkappa^{+}_{k,m_k-1-j-\nu}\frac{1}{\nu!}\left\{\frac{d^\nu}{d\lambda^\nu}f_+(x,\lambda)\right\}_{\lambda=\lambda_k}=$$

$$\sum_{j=0}^{m_k-1}\varkappa^{+}_{k,m_k-1-j}\frac{1}{j!}\left\{\frac{d^j}{d\lambda^j}\frac{(\lambda-\lambda_k)^{m_k}\,e^{i\lambda y}}{a(\lambda)}f_+(x,\lambda)\right\}_{\lambda=\lambda_k}=$$

$$=iR_+(x)P_k^+(\mu(x)+y)e^{i\lambda_k(\mu(x)+y)}+$$

$$+iR_-(x)P_k^+(y-\mu(x))e^{i\lambda_k(y-\mu(x))}+i\int\limits_{\mu(x)}^{+\infty}K^+(x,t)P_k(t+y)e^{i\lambda_k(t+y)},$$

where

$$P_k^+(x)=-ie^{-i\lambda_k x}\sum_{j=0}^{m_k-1}\varkappa^{+}_{k,m_k-1-j}\frac{1}{j!}\left\{\frac{d^j}{d\lambda^j}\frac{(\lambda-\lambda_k)^{m_k}\,e^{i\lambda x}}{a(\lambda)})\right\}_{\lambda=\lambda_k},$$

we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{a(\lambda)}-\frac{2}{1+\alpha}e^{-i\omega}\right)f_-(x,\lambda)e^{i\lambda y}d\lambda=$$

$$-R_+(x)\sum_{k=1}^{n}P_k^+(\mu(x)+y)e^{i\lambda_k(\mu(x)+y)}-$$

$$-R_-(x)\sum_{k=1}^{n}P_k^+(y-\mu(x))-\int\limits_{\mu(x)}^{+\infty}K^+(x,t)\sum_{k=1}^{n}P_k(t+y)e^{i\lambda_k(t+y)}.$$

Taking into our account the formula (25) we obtain the following integral equation for the kernel $K^+(x,y)$ $(y>\mu(x))$:

$$F_R^+(x,y)+\int_{\mu(x)}^{\infty}K^+(x,t)F_{0R}^+(t+y)dt+$$

$$+\overline{K^+(x,y)}-\frac{\alpha-1}{\alpha+1}e^{-i(\omega+\theta)}K^+(x,-y)-\frac{2e^{-i\omega}}{1+\alpha}K^-(x,y)=$$

$$-R_+(x)\sum_{k=1}^{n}P_k^+(\mu(x)+y)e^{i\lambda_k(\mu(x)+y)}-$$

$$-R_-(x)\sum_{k=1}^{n}P_k^+(y-\mu(x)e^{i\lambda_k(y-\mu(x))})-\int\limits_{\mu(x)}^{+\infty}K^+(x,t)\sum_{k=1}^{n}P_k(t+y)e^{i\lambda_k(t+y)},$$

$$R_+(x)[F_{0R}^+(\mu(x)+y)\sum_{k=1}^{n}P_k^+(\mu(x)+y)e^{i\lambda_k(\mu(x)+y)}]$$

$$+R_-(x)[F_{0R}^+(-\mu(x)+y)+\sum_{k=1}^{n}P_k^+(y-\mu(x)e^{i\lambda_k(y-\mu(x))}]+$$

$$+\overline{K^+(x,y)}-\frac{\alpha-1}{\alpha+1}e^{-i(\omega+\theta)}K^+(x,-y)-\frac{2e^{-i\omega}}{1+\alpha}K^-(x,y)+$$

$$+\int_{\mu(x)}^{\infty}K^+(x,t)[F_{0R}^+(t+y)+\sum_{k=1}^{n}P_k(t+y)e^{i\lambda_k(t+y)}]dt=0,y\geq\mu(x).$$

The last equation is simplified to the equation

$$R_+(x)F^+(\mu(x)+y)+R_-(x)[F^+(-\mu(x)+y)+\overline{K^+(x,y)}-\frac{1-\alpha}{1+\alpha}e^{-i(\omega+\theta)}K^+(x,-y)$$

$$-\frac{2e^{-i\omega}}{1+\alpha}K^-(x,y)+\int\limits_{\mu(x)}^{+\infty}K^+(x,t)F^+(t+y)dt=0,\ \ y\geq\mu(x),$$

where

$$F^+(x)=\sum_{k=1}^{n}P_k^+(x)e^{i\lambda_k x}+F_{0R}^+(x).$$

Since $K^-(x,y)=0$ for $y>\mu(x)$ we conclude that

$$R_+(x)F^+(\mu(x)+y)+R_-(x)F^+(-\mu(x)+y)+\overline{K^+(x,y)}+\int_{\mu(x)}^{\infty}F^+(y+t)K^+(x,t)dt+$$

$$+\frac{1-\sqrt{\rho(x)}}{1+\sqrt{\rho(x)}}e^{-i(\omega+\theta)}K^+(x,-y)=0,y\geq\mu(x),\tag{26}$$

where the function

$$F^+(x)=\sum_{k=1}^{n}P_k^+(x)e^{i\lambda_k x}+\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left(r^+(\lambda)-\frac{1-\alpha}{1+\alpha}e^{-i(\theta+\omega)}\right)e^{i\lambda x}d\lambda$$

are defined by the right scattering data.

By the similar way, using the first relation in (18), we can derive

$$T_+(x)F^-(-\mu(x)+y)+T_-(x)F^-(\mu(x)+y)+\overline{K^-(x,y)}+\int_{-\infty}^{\mu(x)}F^-(y+t)K^-(x,t)dt$$

$$-\frac{1-\sqrt{\rho(x)}}{1+\sqrt{\rho(x)}}e^{i(\theta-\omega)}K^{-}(x,-y)=0 \tag{27}$$

for $y \leq \mu(x)$, where

$$F^{-}(x)=\sum_{k=1}^{n}P_k^{-}(x)e^{-i\lambda_k x}+\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left(r^{-}(\lambda)+\frac{1-\alpha}{1+\alpha}e^{i(\theta-\omega)}\right)e^{-i\lambda x}d\lambda$$

and

$$P_k^{-}(x)=-ie^{i\lambda_k x}\sum_{j=0}^{m_k-1}\varkappa_{k,m_k-1-j}^{-}\frac{1}{j!}\left\{\frac{d^j}{d\lambda^j}\frac{(\lambda-\lambda_k)^{m_k}e^{-i\lambda x}}{a(\lambda)})\right\}_{\lambda=\lambda_k}$$

are defined by the left scattering data.

Hence, we have proved the following theorem.

Theorem 4. *The kernels $K^{\pm}(x,y)$ of representation (3) and (6) satisfy the main equations (26) and (27).*

4. Uniqueness

Theorem 5. *The main equations (26), (27) has a unique solution $K^{+}(x,.)\in L_1(\mu(x),+\infty)$, and $K^{-}(x,.)\in L_1(-\infty,\mu(x))$ for each fixed $x>-\infty$ and $x<\infty$, respectively.*

Proof. It is easy to show that the considered integral equations (26), (27) are generated by completely continuous operators, and to prove their unique solvability, it is enough to show that the corresponding homogeneous equations only have zero solutions. Consider the equation

$$g(y)+\overline{h_{+}(y)}-ch_{+}(-y)+\int_{\mu(x)}^{\infty}F_{+}(y+t)h_{+}(t)dt=0, \quad y\geq\mu(x), \tag{28}$$

where $h(y)=K^{+}(x,y), c=e^{i(\theta-\omega)}$, x plays a role of a parameter and $g(y)\in L_1(\mu(x),+\infty)$. Then (28) implies that

$$\begin{aligned}0=\int_{\mu(x)}^{\infty}|h_{+}(y)|^2dy-c\int_{\mu(x)}^{\infty}h_{+}(-y)\overline{h_{+}(y)}dy+\\+\int_{\mu(x)}^{\infty}\int_{\mu(x)}^{\infty}F_{+}(y+t)h_{+}(t)\overline{h_{+}(y)}dtdy=\\=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widetilde{h_{+}(\lambda)}\right|^2d\lambda+\frac{1}{2\pi}\int_{-\infty}^{\infty}r^{+}(\lambda)\widetilde{h^2(\lambda)}d\lambda,\end{aligned}$$

where $\widetilde{f(\lambda)}=\int_{\mu(x)}^{\infty}f(y)e^{-i\lambda y}d\lambda$. Since

$$\left|\widetilde{r^{+}(\lambda)}\right|=\left|r^{+}(\lambda)\right|,$$

$$\int_{-\infty}^{\infty} \left|r^{+}(\lambda)\widetilde{h^{2}(\lambda)}d\lambda\right| d\lambda \leq \int_{-\infty}^{\infty} \left|r^{+}(\lambda)\right| \left|\widetilde{h_{+}(\lambda)}\right|^{2} d\lambda$$

we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\widetilde{h_{+}(\lambda)}\right|^{2} d\lambda =\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|r^{+}(\lambda)\right| \left|\widetilde{f(\lambda)}\right|^{2} d\lambda,$$

i. e.

$$\int_{-\infty}^{\infty} \left(1-\left|r^{+}(\lambda)\right|\right) \left|\widetilde{f(\lambda)}\right|^{2} d\lambda \leq 0$$

which implies $\widetilde{h_{+}(\lambda)} \equiv 0$ because of $1-|r^{+}(\lambda)| > 0$ for all $\lambda \neq 0$. Thus, homogeneous equation (28) only have the zero solution. The uniquely solvability of equation (27) is proved analogously. ◀

Theorem 6. *Given function $r^{+}(\lambda)(-\infty < \lambda < \infty)$ is the right scattering data of the problem (1), (2) without discrete spectrum and real potential functions $q(x)$ and $p(x)$ if and only if the following conditions are satisfied:*

I) $r^{+}(\lambda)(-\infty < \lambda < \infty)$ is continuous, $|r^{+}(\lambda)| < 1$ for $\lambda \neq 0$, and $r^{+}(0) = -1$ if $|r^{+}(0)| = 1$. Moreover, there exists $C > 0$ such that

$$1-\left|r^{+}\left(\lambda\right)\right| \geq \frac{C\lambda^{2}}{1+\lambda^{2}}, \quad -\infty < \lambda < \infty,$$

and

$$r^{+}(\lambda) = \frac{1-\alpha}{1+\alpha}e^{-i(\theta+\omega)} + O\left(\frac{1}{\lambda}\right), \ |\lambda| \to \infty,$$

for some real θ and ω.

II) The function $za_{1}(z)$, where

$$a_{1}(z) = \frac{1+\alpha}{2}\exp\left\{-\frac{1}{2\pi i\alpha}\int_{-\infty}^{+\infty}\frac{\ln\left[\left(1-\left|r^{+}\left(\lambda\right)\right|^{2}\right)\right]}{\lambda - z}d\lambda\right\},$$

is continuous on the closed upper halfplane $Imz \geq 0$.

III) The functions

$$F_{0R}^{+}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(r^{+}\left(\lambda\right) - \frac{1-\alpha}{1+\alpha}e^{-i(\theta+\omega)}\right)e^{i\lambda x}d\lambda$$

and

$$F_{0R}^{-}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(-\overline{r^{+}\left(\lambda\right)}\frac{\overline{a_{1}(\lambda)}}{a_{1}(\lambda)} + \frac{1-\alpha}{1+\alpha}e^{i(\theta-\omega)}\right)e^{-i\lambda x}d\lambda$$

are absolutely continuous and their derivatives $\frac{d}{dx}F_{0R}^{+}(x)$, $\frac{d}{dx}F_{0R}^{-}(x)$ satisfy

$$\int_{x_{1}}^{+\infty}\frac{d}{dx}F_{0R}^{+}(\mu(x))dx < \infty, \int_{-\infty}^{x_{2}}\frac{d}{dx}F_{0R}^{-}(\mu(x))dx < \infty,$$

for all $x_{1} > -\infty$ and $x_{2} < \infty$ respectively.

Proof. In the direct problem we have proven that the right reflection coefficient of the scattering problem (1), (2) without discrete spectrum and real potential functions $q(x)$ and $p(x)$ satisfies the conditions I)-III). Let the conditions I)-III) are satisfied. Then based on the given function $r^{+}(\lambda)$ we construct the second function $r^{-}(\lambda)$ by the formula

$$a(z) = e^{i\omega} a_1(z) \ (\mathrm{Im} z \geq 0, z \neq 0),$$

$$r^{-}(\lambda) = -\overline{r^{+}(\lambda)} \frac{\overline{a(\lambda)}}{a(\lambda)},$$

$$F_{0R}^{-}(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \left(r^{-}(\lambda) + \frac{1-\alpha}{1+\alpha} e^{i(\theta-\omega)} \right) e^{-i\lambda x} d\lambda,$$

where numbers ω and θ will defined later. Then according to the Theorem 14 we can write the main integral equations (26) and (27) from which we have

$$K^{\pm}(x,t) = K_0^{\pm}(x,t) \cos \omega^{\pm}(x) + K_1^{\pm}(x,t) \cos \omega^{\pm}(x), \tag{29}$$

where $\omega^{\pm}(x)$ is defined below and $K_0^{\pm}(x,t), K_1^{\pm}(x,t)$ are solutions of the integral equations

$$\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right) F_{0R}^{+}(\mu(x)+y) + \frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)) e^{i(\omega+\theta)} F_{0R}^{+}(-\mu(x)+y) +$$

$$+\overline{K_0^{+}(x,y)} + \int_{\mu(x)}^{\infty} F_{0R}^{+}(y+t) K_0^{+}(x,t) dt +$$

$$+\frac{1-\sqrt{\rho(x)}}{1+\sqrt{\rho(x)}} e^{-i(\omega+\theta)} K_0^{+}(x,-y) = 0, \ y \geq \mu(x), \tag{30}$$

$$\frac{i}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right) F_{0R}^{+}(\mu(x)+y) - \frac{i}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)) e^{i(\omega+\theta)} F_{0R}^{+}(-\mu(x)+y)$$

$$+\overline{K_1^{+}(x,y)} + \int_{\mu(x)}^{\infty} F_{0R}^{+}(y+t) K_1^{+}(x,t) dt +$$

$$+\frac{1-\sqrt{\rho(x)}}{1+\sqrt{\rho(x)}} e^{-i(\omega+\theta)} K_1^{+}(x,-y) = 0 \ , y \geq \mu(x), \tag{31}$$

$$\frac{1}{2}\left(1+\frac{\alpha}{\sqrt{\rho(x)}}\right) F_{0R}^{-}(\mu(x)+y) + \frac{1}{2}\left(1-\frac{\alpha}{\sqrt{\rho(x)}}\right)) e^{i(\theta-\omega)} F_{0R}^{-}(-\mu(x)+y)$$

$$+\overline{K_0^{-}(x,y)} + \int_{-\infty}^{\mu(x)} F_{0R}^{-}(y+t) K_0^{-}(x,t) dt -$$

$$-\frac{1-\sqrt{\rho(x)}}{1+\sqrt{\rho(x)}} e^{i(\theta-\omega)} K_0^{-}(x,-y) = 0, \ y \leq \mu(x), \tag{32}$$

$$\frac{i}{2}\left(1+\frac{\alpha}{\sqrt{\rho\left(x\right)}}\right)F_{0R}^{-}(\mu(x)+y)-\frac{i}{2}\left(1-\frac{\alpha}{\sqrt{\rho\left(x\right)}}\right))e^{i(\theta-\omega)}F_{0R}^{-}\left(-\mu(x)+y\right)$$

$$+\overline{K_{1}^{-}\left(x,y\right)}+\int_{-\infty}^{\mu(x)}F_{0R}^{-}(y+t)K_{1}^{-}\left(x,t\right)dt-$$

$$-\frac{1-\sqrt{\rho(x)}}{1+\sqrt{\rho(x)}}e^{i(\theta-\omega)}K_{1}^{-}\left(x,-y\right)=0,\ \ y\leq\mu(x). \tag{33}$$

At this stage we define functions $\omega^{\pm}(x)$ as the solutions of the nonlinear integral equations

$$\omega^{\pm}(x)=\pm\int_{x}^{\pm\infty}\varPhi_{\pm}(t,\omega^{\pm}(t))dt, \tag{34}$$

where

$$\varPhi_{\pm}(t,z)=\left(1+\frac{\sqrt{\rho(t)}}{\beta}\right)\sqrt{\rho(t)}\left[\left(\mathrm{Re}K_{0}^{\pm}(t,\mu(t))-\mathrm{Im}K_{1}^{\pm}(t,\mu(t)\right)\sin 2z\right.$$

$$+2\mathrm{Re}K_{1}^{\pm}(t,\mu(t))\sin^{2}z-2\mathrm{Im}K_{0}^{\pm}(t,\mu(t))\cos^{2}z\big]$$

$$+\left(1-\frac{\sqrt{\rho(t)}}{\beta}\right)\sqrt{\rho(t)}\left[\left(\mathrm{Re}PK_{0}^{\pm}(t,-\mu(t))-\mathrm{Im}PK_{1}^{\pm}(t,-\mu(t)\right)\sin 2(\theta\pm\omega\mp z)\right.$$

$$\left.+2\mathrm{Re}PK_{1}^{\pm}(t,-\mu(t))\sin^{2}(\theta\pm\omega\mp z)-2\mathrm{Re}PK_{0}^{\pm}(t,-\mu(t))\cos^{2}(\theta\pm\omega\mp z)\right], \tag{35}$$

$$\beta=\frac{1+\alpha\pm(1-\alpha)}{2} \tag{36}$$

and $Pg(t,z)=g(t,z+0)-g(t,z-0)$ denotes the jumping of the function $g(t,z)$ with respect z. Therefore from equations (29)-(36) we can define functions $\omega^{\pm}(x),\widetilde{\omega}^{\pm}(x),K^{\pm}(x,t)$ uniquely and by these we can construct the solutions

$$f_{+}\left(x,\lambda\right)=\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho\left(x\right)}}\right)e^{i\lambda\mu(x)+i\omega^{+}(x)}$$

$$+\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho\left(x\right)}}\right)e^{-i\lambda\mu(x)+\widetilde{\omega}^{+}(x)}+\int\limits_{\mu(x)}^{+\infty}K^{+}\left(x,t\right)e^{i\lambda t}dt \tag{37}$$

and

$$f_{-}\left(x,\lambda\right)=\frac{1}{2}\left(1-\frac{\alpha}{\sqrt{\rho\left(x\right)}}\right)e^{i\lambda\mu(x)+\widetilde{i\omega}^{-}(x)}$$

$$+\frac{1}{2}\left(1+\frac{\alpha}{\sqrt{\rho\left(x\right)}}\right)e^{-i\lambda\mu(x)+i\omega^{-}(x)}+\int\limits_{-\infty}^{\mu(x)}K^{-}\left(x,t\right)e^{-i\lambda t}dt \tag{38}$$

of the equations

$$f''_{+}(x,\lambda)+\left[\lambda^2\rho(x)-2\lambda p_{+}(x)-q_{+}(x)\right]f_{+}(x,\lambda)=0$$

and

$$f''_{-}(x,\lambda)+\left[\lambda^2\rho(x)-2\lambda p_{-}(x)-q_{-}(x)\right]f_{-}(x,\lambda)=0, \tag{39}$$

where

$$p_{\pm}(x)=\sqrt{\rho(x)}\frac{d\omega^{\pm}(x)}{dx},$$

$$q_{\pm}(x)=-\frac{p_{\pm}^2(x)}{\rho(x)}-\frac{4\rho(x)}{\beta+\sqrt{\rho(x)}}\frac{d}{dx}\times$$

$$\times\left[\mathrm{Re}K^{\pm}(x,\mu(x))\cos\omega^{\pm}(x)+\mathrm{Im}K^{\pm}(x,\mu(x))\sin\omega^{\pm}(x)\right]$$

and

$$\int\limits_{x_1}^{+\infty}|p_{+}(x)|\,dx<\infty,\quad \int\limits_{x_1}^{+\infty}(1+|x|)\left(|q_{+}(x)|+\left|p'_{+}(x)\right|\right)dx<+\infty, \tag{40}$$

$$\int\limits_{-\infty}^{x_2}|p_{+}(x)|\,dx<\infty,\quad \int\limits_{-\infty}^{x_2}(1+|x|)\left(|q_{-}(x)|+\left|p'_{-}(x)\right|\right)dx<+\infty$$

are satisfied for all $x_1>-\infty$ and $x_2<\infty$ respectively.

Further using the main integral equations (29)-(33)and the integral representations (37), (38) of the solutions it can be shown (see [29], [33]) that for all real values of the solutions $f_{+}(x,\lambda)$ and $f_{-}(x,\lambda)$ have connections

$$\begin{array}{c} r^{+}(\lambda)f_{+}(x,\lambda)+\overline{f_{+}(x,\lambda)}=\frac{1}{a(\lambda)}f_{-}(x,\lambda),\\ \frac{r^{-}(\lambda)}{\alpha}f_{-}(x,\lambda)+\frac{1}{\alpha}\overline{f_{-}(x,\lambda)}=\frac{1}{a(\lambda)}f_{+}(x,\lambda). \end{array} \tag{41}$$

Now we obtain from (41)

$$f''_{-}(x,\lambda)+\left[\lambda^2\rho(x)-2\lambda p_{+}(x)-q_{+}(x)\right]f_{-}(x,\lambda)=0. \tag{42}$$

Subtracting equations (39) and (42) we conclude that

$$p_{-}(x)=p_{+}(x),q_{-}(x)=q_{+}(x)\ (-\infty<x<\infty).$$

Consequently we can define

$$p(x)=p_{+}(x),\ q(x)=q_{+}(x)\ (-\infty<x<\infty) \tag{43}$$

uniquely and from the inequalities (40) we have

$$\int\limits_{-\infty}^{+\infty}|p(x)|\,dx<\infty,\quad \int\limits_{-\infty}^{+\infty}(1+|x|)\left(|q(x)|+|p'(x)|\right)dx<+\infty.$$

It remains to show that $r^+(\lambda)$ and $r^-(\lambda)$ are the right and left reflection coefficients of the constructed equation

$$y'' + \left[\lambda^2\rho(x) - 2\lambda p(x) - q(x)\right] y = 0, -\infty < x < \infty. \tag{44}$$

We define the right and left reflection coefficients of the constructed equation (44) by $\widetilde{r}^+(\lambda)$ and $\widetilde{r}^-(\lambda)$ respectively. By the formula (43) the functions $f_+(x,\lambda)$ and $f_-(x,\lambda)$ are the Jost solutions of the equation (44). Therefore by the results of the direct scattering problem we can write

$$\begin{array}{c} \widetilde{r}^+(\lambda) f_+(x,\lambda) + \overline{f_+(x,\lambda)} = \frac{1}{\widetilde{a}(\lambda)} f_-(x,\lambda), \\ \frac{\widetilde{r}^-(\lambda)}{\alpha} f_-(x,\lambda) + \frac{1}{\alpha}\overline{f_-(x,\lambda)} = \frac{1}{\widetilde{a}(\lambda)} f_+(x,\lambda). \end{array} \tag{45}$$

Now from (41) and (45) we have

$$\begin{array}{c} a(\lambda)\, r^+(\lambda) f_+(x,\lambda) + a(\lambda)\, \overline{f_+(x,\lambda)} = f_-(x,\lambda), \\ \widetilde{a}(\lambda)\, \widetilde{r}^+(\lambda) f_+(x,\lambda) + \widetilde{a}(\lambda)\, \overline{f_+(x,\lambda)} = f_-(x,\lambda) \end{array}$$

which imply

$$\left(a(\lambda)\, r^+(\lambda) - \widetilde{a}(\lambda)\, \widetilde{r}^+(\lambda)\right) f_+(x,\lambda) + \left(a(\lambda) - \widetilde{a}(\lambda)\right) \overline{f_+(x,\lambda)} = 0.$$

Since $W\left\{f_+(x,\lambda), \overline{f_+(x,\lambda)}\right\} = 2i\lambda \neq 0$ for $\lambda \neq 0$ we obtain $a(\lambda)\, r^+(\lambda) - \widetilde{a}(\lambda)\, \widetilde{r}^+(\lambda) = 0$, $a(\lambda) - \widetilde{a}(\lambda) = 0$. Consequently $a(\lambda) = \widetilde{a}(\lambda)$, $r^+(\lambda) = \widetilde{r}^+(\lambda)$. Similarly from (45) we have $r^-(\lambda) = \widetilde{r}^-(\lambda)$. Since $a(z) = e^{i\omega} a_1(z)$ $(\text{Im} z \geq 0, z \neq 0)$ from the condition II) we have that the function $a(z)$ has no zeros in the upper halfplane. Then the equation (44) has no discrete spectrum. Until now, the numbers ω and θ were arbitrary real numbers. Since $\omega + \theta = 2\omega^+(0)$ it os sufficient to define ω. From the conditions I) and II) of the theorem we find that

$$\lim_{\lambda \to 0} \lambda a(\lambda) \left(r^+(\lambda) + 1\right) = 0.$$

Then using the formula

$$r^-(\lambda) = -\overline{r^+(\lambda)} \frac{\overline{a(\lambda)}}{a(\lambda)}$$

we have

$$\lambda a(\lambda) \left(r^-(\lambda) + 1\right) = \lambda a(\lambda) \left(r^+(\lambda) + 1\right) - \lambda \left(a(\lambda) r^+(\lambda) + \overline{a(\lambda)}\right) \overline{r^+(\lambda)}.$$

Since $r^+(\lambda)$ is continuous and $\lim_{\lambda \to 0} \lambda a(\lambda)$ is finite there exist a finite limit

$$\lim_{\lambda \to 0} \lambda a(\lambda) r^+(\lambda) = \gamma.$$

Consequently we have

$$\lim_{\lambda \to 0} \lambda a(\lambda) \left(r^-(\lambda) + 1\right) = -2\text{Re}\gamma.$$

If we require that $\text{Re}\gamma = 0$ we can define $e^{i\omega}$ uniquely. ◀

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