Baku Mathematical Journal
2024, Vol. 3, No 1, PP. 75-83
https://doi.org/10.32010/j.bmj.2024.08

THE REGULARIZED TRACE FORMULA FOR A DISCONTINUOUS STURM-LIOUVILLE PROBLEM WITH A SPECTRAL PARAMETER DEPENDENT JUMP CONDITION

L.N. ATAMOVA, A.A. HUSEYNLI*

Received: 15.11.2023 / Revised: 12.01.2024 / Accepted: 24.01.2024

In memory of M. G. Gasymov on his 85th birthday

Abstract. *The first regularized trace formula for a discontinuous Sturm-Liouville problem with a spectral parameter dependent jump condition is obtained.*

Keywords: boundary value problem, eigenvalue, asymptotics, regularized trace

Mathematics Subject Classification (2020): 34B05, 34B10, 34B24, 34E05

1. Introduction

We consider the boundary value problem

$$-y'' + q(x)y = \lambda y, \quad x\epsilon(-1,0)\cup(0,1), \tag{1}$$

$$\left.\begin{aligned} y(-1) = y(1) = 0, \\ y(-0) = y(+0), \\ y'(-0) - y'(+0) = \lambda m y(0), \end{aligned}\right\} \tag{2}$$

where $q(x) \in W_2^1(-1,1)$ is a complex-valued function, $m \neq 0$ is a complex number. The aim of the paper is to calculate the first regularized trace for the problem (1), (2).

The theory of regularized traces of first and higher orders of ordinary differential operators has a long history. The regularized trace formula for the regular Sturm-Liouville operator with Dirichlet boundary condition and smooth potential was first established

* Corresponding author.

Lala N. Atamova
Khazar University, Baku, Azerbaijan
E-mail: ljafarova@khazar.org

Ali A. Huseynli
Khazar University, Baku, Azerbaijan
E-mail: ahuseynli@khazar.org

by I.M. Gelfand and B.M. Levitan in [5]. In [1], another proof of the Gelfand-Levitan's regularized trace formula was given. In [7] and [11], the regularized trace formulas for the Sturm-Liouville operator with periodic and antiperiodic boundary conditions were obtained. In [6], regularized trace formula for the regular Sturm-Liouville operator with spectral parameter dependent boundary conditions was established. M.G. Gasymov was the first, who considered in [2] singular differential operators on the whole and half-axis with discrete spectrum and established trace formulas for these operators. In [3], M.G. Gasymov and B.M. Levitan obtained trace formulas for singular differential operators on the half-axis with different type of boundary conditions.

For the review and relatively recent results of the theory, see [10].

2. Main Results

Denote $\lambda := \rho^2$, $Im\rho := \tau$. Let $y_1(x,\lambda)$ be the solution of equation (1) on $(-1,1)$ satisfying the initial conditions

$$y_1(-1) = 0, \quad y_1'(-1) = \rho \tag{3}$$

and $y_2(x,\lambda)$ be the solution of equation (1) on $(-1,1)$ satisfying the initial conditions

$$y_2(1) = 0, \quad y_2'(1) = \rho. \tag{4}$$

Using the fact that $q(x) \in W_2^1(-1,1)$ one can obtain the following asymptotic representations:

$$y_1(x,\lambda) = \sin\rho(1+x) - q_1(x)\frac{\cos\rho(x+1)}{\rho} + q_{20}(x)\frac{\sin\rho(x+1)}{\rho^2} -$$

$$-\frac{1}{2\rho^2}\int_{-1}^{x} \sin\rho(x-1-2t)q(t)q_1(t)dt + \int_{-1}^{x} q'(t)\frac{\sin\rho(x-1-2t)}{4\rho^2}dt + O\left(\frac{e^{|\tau|(1+x)}}{\rho^3}\right),$$

$$y_1'(x,\lambda) = \rho\ cos\rho(1+x) - q_{21}(x)\frac{\cos\rho(x+1)}{\rho} + \sin\rho(x+1)q_1(x) -$$

$$-\frac{1}{2\rho}\int_{-1}^{x} \cos\rho(x-1-2t)q(t)q_1(t)dt + \frac{1}{4\rho}\int_{-1}^{x} q'(t)\cos\rho(x-1-2t)dt + O\left(\frac{e^{|\tau|(1+x)}}{\rho^2}\right),$$

$$y_2(x,\lambda) = \sin\rho(1-x) - q_2(x)\frac{\cos\rho(1-x)}{\rho} + q_{30}(x)\frac{\sin\rho(1-x)}{\rho^2} -$$

$$-\frac{1}{2\rho^2}\int_{x}^{1} \sin\rho(2t-x-1)q(t)q_2(t)dt - \int_{x}^{1} q'(t)\frac{\sin\rho(2t-x-1)}{4\rho^2}dt + O\left(\frac{e^{|\tau|(1-x)}}{\rho^3}\right),$$

$$y_2'(x,\lambda) = \rho\ cos\rho(1-x) + q_{31}(x)\frac{\cos\rho(1-x)}{\rho} - \sin\rho(1-x)q_2(x) -$$

$$-\frac{1}{2\rho}\int_{x}^{1} \cos\rho(2t-x-1)q(t)q_2(t)dt - \frac{1}{4\rho}\int_{x}^{1} q'(t)\cos\rho(2t-x-1)dt + O\left(\frac{e^{|\tau|(1-x)}}{\rho^2}\right),$$

where

$$q_{2j}(x)=\frac{1}{4}\left(q(x)+(-1)^{j}q(-1)\right)+\frac{(-1)^{j+1}}{2}\int_{-1}^{x}q(t)q_1(t)dt,$$

$$q_{3j}(x)=\frac{1}{4}\left(q(x)+(-1)^{j}q(1)\right)+\frac{(-1)^{j+1}}{2}\int_{x}^{1}q(t)q_2(t)dt,\quad j=0,1,$$

$$q_1(x)=\frac{1}{2}\int_{-1}^{x}q(t)dt,\quad q_2(x)=\frac{1}{2}\int_{x}^{1}q(t)dt\ .$$

In that case one can prove the following asymptotic expansion of the characteristic function $\Delta(\lambda)$ of problem (1), (2), which is sharper than the expansion of the same $\Delta(\lambda)$ obtained in [4], where $q(x)$ is a summable function on $(-1,1)$:

$$\Delta\left(\rho^2\right)=[-m\rho^2+2q_2(0)-mq_{30}(0)+2q_1(0)+\frac{1}{\rho^2}q_1(0)q_{30}(0)-$$

$$-mq_{20}(0)+\frac{1}{\rho^2}q_2(0)q_{20}(0)-\frac{m}{\rho^2}q_{20}(0)q_{30}(0)+mq_1(0)q_2(0)-\frac{1}{\rho^2}q_{31}(0)q_1(0)-$$

$$-\frac{1}{\rho^2}q_2(0)q_{21}(0)]\sin^2\rho+[2\rho+\rho mq_2(0)-\frac{1}{\rho}q_{31}(0)+\frac{1}{\rho}q_{30}(0)+\rho mq_1(0)-$$

$$-\frac{2}{\rho}q_1(0)q_2(0)+\frac{m}{\rho}q_{30}(0)q_1(0)+\frac{1}{\rho}q_{20}(0)-\frac{1}{\rho}q_{21}(0)+\frac{m}{\rho}q_{20}(0)q_2(0)]sin\rho\cos\rho+$$

$$+[-\frac{1}{4\rho}\int_0^1 q'(t)\cos\rho(2t-1)dt+\frac{m}{4}\int_0^1 q'(t)\sin\rho(2t-1)dt-$$

$$-\frac{1}{2\rho}\int_0^1 q(t)q_2(t)\cos\rho(2t-1)dt-\frac{1}{2\rho}\int_{-1}^0 q(t)q_1(t)\cos\rho(2t+1)dt-$$

$$-\frac{m}{2}\int_{-1}^0 q(t)q_1(t)\sin\rho(2t+1)dt+\frac{1}{4\rho}\int_{-1}^0 q'(t)\cos\rho(2t+1)dt]\sin\rho+$$

$$+[-\frac{1}{4\rho}\int_0^1 q'(t)\sin\rho(2t-1)dt+q_1(0)\frac{1}{4\rho}\int_0^1 q'(t)\cos\rho(2t-1)dt-$$

$$-\frac{m}{4\rho}q_1(0)\int_0^1 q'(t)\sin\rho(2t-1)dt+\frac{1}{4\rho}\int_{-1}^0 q'(t)\sin\rho(2t+1)dt-$$

$$-\frac{1}{2\rho}\int_0^1 q_2(t)q(t)\sin\rho(2t-1)dt+\frac{1}{2\rho}\int_{-1}^0 q_1(t)q(t)\sin\rho(2t+1)dt+$$

$$+\frac{m}{2\rho}q_2(0)\int_{-1}^0 q_1(t)q(t)\sin\rho(2t+1)dt]cos\rho+(-q_2(0)-q_1(0)-mq_1(0)q_2(0))+$$

$$+O\left(\frac{e^{|\tau|}}{\rho^2}\right). \tag{5}$$

The following theorem was proved in [4].

Theorem 1. *Let $q(x)$ be a complex-valued function, summable on $[-1,1]$ and $d := 4 + (mq_2(0))^2 + (mq_1(0))^2 - 2m^2 q_2(0) q_1(0)$* [1]*.*

Then, the spectrum of problem (1), (2) consists of two sequences of eigenvalues $\lambda_{1,n} = \rho_{1,n}^2$, $n = 0,1,\ldots$, and $\lambda_{2,n} = \rho_{2,n}^2$, $n = 1,2,\ldots$, counted with their algebraic multiplicities, for which the following asymptotic equalities hold:

$$\rho_{1,n} = \pi n + \frac{\alpha_1}{n} + o\left(\frac{1}{n}\right) \tag{6}$$

and

$$\rho_{2,n} = \pi n + \frac{\alpha_2}{n} + o\left(\frac{1}{n}\right), \tag{7}$$

where α_1 and α_2 are defined by the following formulas:

$$\alpha_1 = \frac{-\left(2 + mq_2(0) + mq_1(0)\right) + \sqrt{d}}{-2m\pi},$$

$$\alpha_2 = \frac{-\left(2 + mq_2(0) + mq_1(0)\right) - \sqrt{d}}{-2m\pi},$$

$$0 \le arg\sqrt{d} < \pi.$$

If $d \neq 0$, then the eigenvalues of the problem are asymptotically simple.

From (5) it follows that if $q(x) \in W_2^1(-1,1)$, then the following asymptotics, that are more precise than the asymptotics (6) and (7), are true:

$$\rho_{1,n} = \pi n + \frac{\alpha_1}{n} + \frac{\xi_n^{(1)}}{n^2}, \quad \left\{\xi_n^{(1)}\right\} \in l_2,$$

$$\rho_{2,n} = \pi n + \frac{\alpha_2}{n} + \frac{\xi_n^{(2)}}{n^2}, \quad \left\{\xi_n^{(2)}\right\} \in l_2.$$

Then

$$\lambda_{1,n} = (\rho_{1,n})^2 = \pi^2 n^2 + 2\pi\alpha_1 + \frac{\eta_n^{(1)}}{n}, \quad \left\{\eta_n^{(1)}\right\} \in l_2, \tag{8}$$

$$\lambda_{2,n} = (\rho_{2,n})^2 = \pi^2 n^2 + 2\pi\alpha_2 + \frac{\eta_n^{(2)}}{n}, \quad \left\{\eta_n^{(2)}\right\} \in l_2.$$

Therefore, the sum

$$S_\lambda := \sum_{n=0}^{\infty}\left(\lambda_{1,n} - \pi^2 n^2 - 2\pi\alpha_1\right) + \sum_{n=1}^{\infty}\left(\lambda_{2,n} - \pi^2 n^2 - 2\pi\alpha_2\right) \tag{9}$$

is convergent (more precisely, both of the above series are convergent). The sum (9) is called *the first regularized trace* for the problem (1), (2).

[1] In [4], the formula for d contains a typo.

Since $\Delta(\lambda)$ is an entire function of order $\frac{1}{2}$ and by conditions (3) and (4) $\lambda = 0$ is zero of $\Delta(\lambda)$, by Hadamard's factorization theorem (see [8, Section 4.2]) we have

$$\Delta(\lambda) = A\lambda\Phi(\lambda) = A\lambda\left(\lambda - \lambda_{1,0}\right)\prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_{1,n}}\right)\prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_{2,n}}\right) =$$

$$= A\lambda\left(\lambda - \lambda_{1,0}\right)\Phi_1(\lambda)\Phi_2(\lambda), \tag{10}$$

where $\Phi_1(\lambda) = \prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_{1,n}}\right)$ and $\Phi_2(\lambda) = \prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_{2,n}}\right)$.

Now we study the behavior of $\Phi_1(\lambda)$ and $\Phi_2(\lambda)$ for negative values of λ with sufficiently large obsolute values. Take $\lambda = -\mu^2, \quad \mu > 0$. We can write:

$$\phi_1(-\mu^2) = \prod_{n=1}^{\infty}\left(1 + \frac{\mu^2}{\lambda_{1,n}}\right) = \frac{\prod\limits_{n=1}^{\infty}\left(1 + \frac{\mu^2}{\lambda_{1,n}}\right)}{\prod\limits_{n=1}^{\infty}\left(1 + \frac{\mu^2}{\pi^2 n^2}\right)}\ \frac{sh\ \mu}{\mu} =$$

$$= \prod_{n=1}^{\infty}\frac{\pi^2 n^2}{\lambda_{1,n}}\prod_{n=1}^{\infty}\frac{\lambda_{1,n} + \mu^2}{\pi^2 n^2 + \mu^2}\ \frac{sh\ \mu}{\mu} = \prod_{n=1}^{\infty}\frac{\pi^2 n^2}{\lambda_{1,n}}\ \prod_{n=1}^{\infty}\left(1 - \frac{\pi^2 n^2 - \lambda_{1,n}}{\pi^2 n^2 + \mu^2}\right)\ \frac{sh\ \mu}{\mu} =$$

$$= C_1\prod_{n=1}^{\infty}\left(1 - \frac{\pi^2 n^2 - \lambda_{1,n}}{\pi^2 n^2 + \mu^2}\right)\ \frac{sh\ \mu}{\mu},$$

where $C_1 = \prod\limits_{n=1}^{\infty}\frac{\pi^2 n^2}{\lambda_{1,n}}$.

Denote

$$\varphi_1(\mu) = \prod_{n=1}^{\infty}\left(1 - \frac{\pi^2 n^2 - \lambda_{1,n}}{\pi^2 n^2 + \mu^2}\right).$$

It is clear that for the function

$$\psi_1(\mu) = \sum_{n=1}^{\infty} Log\left(1 - \frac{\pi^2 n^2 - \lambda_{1,n}}{\pi^2 n^2 + \mu^2}\right),$$

where $Log\ z$ denotes the principial branch of the logarithm , it holds $e^{\psi_1(\mu)} = \varphi_1(\mu)$.

We have

$$\psi_1(\mu) = \sum_{n=1}^{\infty} Log\left(1 - \frac{\pi^2 n^2 - \lambda_{1,n}}{\pi^2 n^2 + \mu^2}\right) = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\ \left(\frac{\pi^2 n^2 - \lambda_{1,n}}{\pi^2 n^2 + \mu^2}\right)^k =$$

$$= -\sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\left(\frac{\pi^2 n^2 - \lambda_{1,n}}{\pi^2 n^2 + \mu^2}\right)^k.$$

We use the formulas (see [9])

$$\sum_{k=2}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\frac{\left|\pi^2 n^2 - \lambda_{1,n}\right|^k}{\left(\pi^2 n^2 + \mu^2\right)^k} = O(\mu^{-3}) \tag{11}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{\mu^2+\pi^2 n^2} = \frac{1}{2\mu} - \frac{1}{2\mu^2} + O\left(e^{-2\mu}\right). \tag{12}$$

Taking into account (8) we get

$$-\sum_{n=1}^{\infty} \frac{\pi^2 n^2 - \lambda_{1,n}}{\pi^2 n^2 + \mu^2} = \sum_{n=1}^{\infty} \frac{\lambda_{1,n} - \pi^2 n^2 - 2\pi\alpha_1}{\pi^2 n^2 + \mu^2} + \sum_{n=1}^{\infty} \frac{2\pi\alpha_1}{\mu^2+\pi^2 n^2} =$$

$$= \sum_{n=1}^{\infty} \frac{2\pi\alpha_1}{\mu^2+\pi^2 n^2} + \frac{1}{\mu^2}\sum_{n=1}^{\infty}\left(\lambda_{1,n} - \pi^2 n^2 - 2\pi\alpha_1\right) - \frac{1}{\mu^2}\sum_{n=1}^{\infty} \frac{(\lambda_{1,n} - \pi^2 n^2 - 2\pi\alpha_1)\pi^2 n^2}{\mu^2+\pi^2 n^2} =$$

$$= \sum_{n=1}^{\infty} \frac{2\pi\alpha_1}{\mu^2+\pi^2 n^2} + \frac{1}{\mu^2}\sum_{n=1}^{\infty}\left(\lambda_{1,n} - \pi^2 n^2 - 2\pi\alpha_1\right) - \frac{\pi^2}{\mu^2}\sum_{n=1}^{\infty} \frac{\eta_n^{(1)} n}{\mu^2+\pi^2 n^2}.$$

Now we estimate the last sum in the above equality. We have

$$\left|\frac{\pi^2}{\mu^2}\sum_{n=1}^{\infty} \frac{\eta_n^{(1)} n}{\mu^2+\pi^2 n^2}\right| \le \frac{\pi^2}{\mu^2}\sum_{n=1}^{\infty} \frac{\left|\eta_n^{(1)}\right| n}{\mu^2+\pi^2 n^2} \le$$

$$\le \frac{\pi^2}{\mu^2}\left(\sum_{n=1}^{\infty}\left|\eta_n^{(1)}\right|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\frac{n^2}{\left(\mu^2+\pi^2 n^2\right)^2}\right)^{\frac{1}{2}} \le$$

$$\le \frac{\pi}{\mu^2}\left(\sum_{n=1}^{\infty}\left|\eta_n^{(1)}\right|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\frac{1}{\mu^2+\pi^2 n^2}\right)^{\frac{1}{2}} = o\left(\frac{1}{\mu^2}\right). \tag{13}$$

From (11), (12) and (13) we calculate:

$$\psi_1(\mu) = \sum_{n=1}^{\infty} \frac{2\pi\alpha_1}{\mu^2+\pi^2 n^2} + \frac{1}{\mu^2}\sum_{n=1}^{\infty}\left(\lambda_{1,n} - \pi^2 n^2 - 2\pi\alpha_1\right) + o\left(\frac{1}{\mu^2}\right) =$$

$$= \frac{\pi\alpha_1}{\mu} - \frac{\pi\alpha_1}{\mu^2} + o\left(\frac{1}{\mu^2}\right) + \frac{1}{\mu^2}\sum_{n=1}^{\infty}\left(\lambda_{1,n} - \pi^2 n^2 - 2\pi\alpha_1\right) + o\left(\frac{1}{\mu^2}\right) =$$

$$= \frac{\pi\alpha_1}{\mu} + \frac{1}{\mu^2}\left(\sum_{n=1}^{\infty}\left(\lambda_{1,n} - \pi^2 n^2 - 2\pi\alpha_1\right) - \pi\alpha_1\right) + o\left(\frac{1}{\mu^2}\right) =$$

$$= \frac{\pi\alpha_1}{\mu} + \frac{1}{\mu^2}\left(\sum_{n=0}^{\infty}\left(\lambda_{1,n} - \pi^2 n^2 - 2\pi\alpha_1\right) - \lambda_{1,0} + \pi\alpha_1\right) + o\left(\frac{1}{\mu^2}\right) =$$

$$= \frac{\pi\alpha_1}{\mu} + \frac{1}{\mu^2}\left(S_\lambda^{(1)} - \lambda_{1,0} + \pi\alpha_1\right) + o\left(\frac{1}{\mu^2}\right),$$

where

$$S_\lambda^{(1)} = \sum_{n=0}^{\infty}\left(\lambda_{1,n} - \pi^2 n^2 - 2\pi\alpha_1\right).$$

We obtain

$$\varphi_1(\mu) = 1 + \frac{\pi\alpha_1}{\mu} + \frac{1}{\mu^2}\left(S_\lambda^{(1)} - \lambda_{1,0} + \pi\alpha_1 + \pi^2\alpha_1^2\right) + o\left(\frac{1}{\mu^2}\right).$$

Therefore,

$$\Phi_1\left(-\mu^2\right) = C_1\left(1 + \frac{\pi\alpha_1}{\mu} + \frac{1}{\mu^2}\left(S_\lambda^{(1)} - \lambda_{1,0} + \pi\alpha_1 + \pi^2\alpha_1^2\right) + o\left(\frac{1}{\mu^2}\right)\right)\frac{sh\ \mu}{\mu}. \quad (14)$$

By the same way for $\Phi_2\left(-\mu^2\right)$ we have

$$\Phi_2\left(-\mu^2\right) =$$

$$= C_2\left(1 + \frac{\pi\alpha_2}{\mu} + \frac{1}{\mu^2}\left(\sum_{n=1}^{\infty}\left(\lambda_{2,n} - \pi^2 n^2 - 2\pi\alpha_2\right) - \pi\alpha_2 + \pi^2\alpha_2^2\right) + o\left(\frac{1}{\mu^2}\right)\right)\frac{sh\ \mu}{\mu} =$$

$$= C_2\left(1 + \frac{\pi\alpha_2}{\mu} + \frac{1}{\mu^2}\left(S_\lambda^{(2)} + \pi\alpha_2 + \pi^2\alpha_2^2\right) + o\left(\frac{1}{\mu^2}\right)\right)\frac{sh\ \mu}{\mu}, \quad (15)$$

where $C_2 = \prod_{n=1}^{\infty}\frac{\pi^2 n^2}{\lambda_{2,n}}$ and

$$S_\lambda^{(2)} = \sum_{n=1}^{\infty}\left(\lambda_{2,n} - \pi^2 n^2 - 2\pi\alpha_2\right).$$

From (10), (14) and (15) we get

$$\Delta\left(-\mu^2\right) = A\left(-\mu^2\right)\left(-\mu^2 - \lambda_{1,0}\right)\times$$

$$\times C_1\left(1 + \frac{\pi\alpha_1}{\mu} + \frac{1}{\mu^2}\left(S_\lambda^{(1)} - \lambda_{1,0} + \pi\alpha_1 + \pi^2\alpha_1^2\right) + o\left(\frac{1}{\mu^2}\right)\right)\times$$

$$\times C_2\left(1 + \frac{\pi\alpha_2}{\mu} + \frac{1}{\mu^2}\left(S_\lambda^{(2)} + \pi\alpha_2 + \pi^2\alpha_2^2\right) + o\left(\frac{1}{\mu^2}\right)\right)\frac{sh\ ^2\mu}{\mu^2} =$$

$$= AC_1C_2\left(\mu^2 + \lambda_{1,0}\right)\Big[1 + \frac{\pi\left(\alpha_1 + \alpha_2\right)}{\mu} + (S_\lambda^{(1)} + S_\lambda^{(2)} - \lambda_{1,0} + \pi\left(\alpha_1 + \alpha_2\right) + \pi^2\left(\alpha_1^2 + \alpha_2^2\right) +$$

$$+\pi^2\alpha_1\alpha_2)\frac{1}{\mu^2} + o\Big(\frac{1}{\mu^2}\Big)\Big]sh^2\ \mu. \quad (16)$$

By comparing the coefficients of μ^2 in formulas (5) and (16) we get that $AC_1C_2 = -m$. Now comparing the coefficients of other terms of (5) and (16) for sufficiently large positive values of μ we obtain the following formula:

$$-m\left(S_\lambda^{(1)} + S_\lambda^{(2)} - \lambda_{1,0} + \pi\left(\alpha_1 + \alpha_2\right) + \pi^2\left(\alpha_1^2 + \alpha_2^2\right) + \pi^2\alpha_1\alpha_2 + \lambda_{1,0}\right) =$$

$$= 2\left(q_1(0) + q_2(0)\right) + mq_1(0)q_2(0) - mq_{30}(0) - mq_{20}(0) =$$

$$= \int_{-1}^{1} q(t)dt + \frac{1}{4}m\int_0^1 q(t)dt\int_{-1}^0 q(t)dt -$$

$$-m\left(\frac{1}{4}\left(2q(0)+q(-1)+q(1)\right)-\frac{1}{8}\left(\left(\int_0^1 q(t)dt\right)^2+\left(\int_{-1}^0 q(t)dt\right)^2\right)\right).$$

We state the above result as a theorem.

Theorem 2. *Let $q(x)$ be a complex-valued function in $W_2^1(-1,1)$ and $\{\lambda_{1,n}\}_{n=0}^{\infty}\cup\{\lambda_{2,n}\}_{n=1}^{\infty}$ be its eigenvalues, counted with their algebraic multiplicities. The following first regularized trace formula holds:*

$$S_\lambda=\sum_{n=0}^{\infty}\left(\lambda_{1,n}-\pi^2n^2-2\pi\alpha_1\right)+\sum_{n=1}^{\infty}\left(\lambda_{2,n}-\pi^2n^2-2\pi\alpha_2\right)=$$

$$=-\pi\left(\alpha_1+\alpha_2\right)-\pi^2\left(\alpha_1^2+\alpha_2^2\right)-\pi^2\alpha_1\alpha_2-$$

$$-\frac{1}{m}\int_{-1}^{1}q(t)dt-\frac{1}{4}\int_0^1 q(t)dt\cdot\int_{-1}^0 q(t)dt+\frac{1}{4}\left(2q(0)+q(-1)+q(1)\right)-$$

$$-\frac{1}{8}\left(\left(\int_0^1 q(t)dt\right)^2+\left(\int_{-1}^0 q(t)dt\right)^2\right).$$

References

1. Dikii L.A. On a formula of Gel'fand-Levitan. *Uspehi Math.Nauk*, 1953, **8** (2(54)), pp. 119-123 (in Russian).
2. Gasymov M.G. On the sum of the differences of the eigenvalues of two self-adjoint operators. *Dokl. Akad. Nauk SSSR*, 1963, **150** (6), pp. 1202-1205 (in Russian).
3. Gasymov M.G., Levitan B.M. On the sum of the differences of the eigenvalues of two singular Sturm-Liouville operators. *Dokl. Akad. Nauk SSSR*, 1963, **151** (5), pp. 1014-1017 (in Russian).
4. Gasymov T.B., Huseynli A.A. Asymptotics of eigenvalues and eigenfunctions of a discontinuous boundary value problem. *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.*, 2011, **35**, pp. 21-32.
5. Gel'fand I.M., Levitan B.M. On a simple identity for the characteristic values of a differential operator of the second order. *Dokl. Akad. Nauk SSSR*, 1953, **88**, pp. 593-596 (in Russian).
6. Guliyev N.J. The regularized trace formula for the Sturm-Liouville equation with spectral parameter in the boundary conditions. *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.*, 2005, **22**, pp. 99-102.
7. Lax P.D. Trace formulas for the Schroedinger operator. *Comm. Pure Appl. Math.*, 1994, **47** (4), pp. 503-512.
8. Levin B.Ya. *Lectures on Entire Functions.* Amer. Math. Soc., Providence, RI, 1996.
9. Levitan B.M. Calculation of the regularized trace for the Sturm-Liouville operator. *Uspehi Mat. Nauk*, 1964, **19** (1(115)), pp. 161-165 (in Russian).
10. Sadovnichii V.A., Podol'skii V.E. Traces of operators. *Russ. Math. Surv.*, 2006, **61** (5), pp. 885-953.

11. Sansuc J.-J., Tkachenko V. Characterization of the periodic and anti-periodic spectra of nonselfadjoint Hill's operators. *New Results in Operator Theory and Its Applications, Oper. Theory Adv. Appl.*, **98**, Birkhäuser, Basel, 1997, pp. 216-224.