Baku Mathematical Journal
2024, Vol. 3, No 1, PP. 64-74
https://doi.org/10.32010/j.bmj.2024.07

CONVERGENCE OF ITERATES OF PROBABILITY MEASURES ON LOCALLY COMPACT GROUPS

H. MUSTAFAYEV

Received: 07.11.2023 / Revised: 08.01.2024 / Accepted: 23.01.2024

In memory of M. G. Gasymov on his 85th birthday

Abstract. *Let G be a locally compact group with the left Haar measure m_G. For a subset S of G, by $[S]$ we denote the closed subgroup of G generated by S. Let μ be a probability measure on G and $H := [\mathrm{supp}\,(\widetilde{\mu} * \mu)]$, where $d\widetilde{\mu}(g) := d\mu\left(g^{-1}\right)$. We show that:*

a) If G is a compact group, then

$$w^* - \lim_{n\to\infty} (\widetilde{\mu} * \mu)^n = \overline{m}_H,$$

where $\overline{m}_H(E) = m_H(E \cap H)$ for every Borel subset E of G.

b) If H is not compact, then

$$w^* - \lim_{n\to\infty} (\widetilde{\mu} * \mu)^n = 0.$$

Some related problems are also discussed.

Keywords: locally compact group, probability measure, weak* convergence

Mathematics Subject Classification (2020): 46HXX, 43A10, 43A20, 43A25

1. Introduction

Let G be a locally compact group with the left Haar measure m_G (in the case when G is compact, m_G will denote normalized Haar measure on G) and let $M(G)$ be the convolution measure algebra of G. As usual, $C_0(G)$ will denote the space of all complex valued continuous functions on G vanishing at infinity. Since $C_0(G)^* = M(G)$, the

Heybetkulu Mustafayev
Azerbaijan State Oil and Industry University, Baku, Azerbaijan;
Center for Mathematics and Its Applications at Khazar University, Baku, Azerbaijan
E-mail: mustafayev.heybatqulu@asoiu.edu.az, hsmustafayev@yahoo.com

space $M(G)$ carries the weak* topology $\sigma(M(G), C_0(G))$. In the following, the w*-topology on $M(G)$ always means this topology. Thus, a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $M(G)$ weak* converges to $\mu \in M(G)$ or w*-$\lim\limits_{n\to\infty} \mu_n = \mu$ if:

$$\lim_{n\to\infty} \int_G f d\mu_n = \int_G f d\mu, \quad \forall f \in C_0(G).$$

For a subset S of G, by $[S]$ we will denote the closed subgroup of G generated by S. A probability measure μ on G is said to be *adapted* if $[\mathrm{supp}\mu] = G$. Also, a probability measure μ on G is said to be *strictly aperiodic* if the support of μ is not contained in a proper closed left cosets gH $(H \neq G,\ g \in G\backslash H)$ of G.

Recall that the convolution product $\mu * \nu$ of two measures μ, $\nu \in M(G)$ is defined by

$$(\mu * \nu)(B) = \int\limits_G \mu\left(Bg^{-1}\right) d\nu(g) \quad \text{for every Borel subset } B \text{ of } G.$$

For an arbitrary $n \in \mathbb{N}$, by μ^n we will denote n-th convolution power of $\mu \in M(G)$, where $\mu^0 := \delta_e$ is the Dirac measure concentrated at the unit element e of G. A classical Kawada-Itô theorem [6, Theorem 7] asserts that if μ is an adapted measure on a compact metrisable group G, then the sequence of probability measures $\left\{\frac{1}{n}\sum\limits_{i=0}^{n-1}\mu^i\right\}_{n\in\mathbb{N}}$ weak* converges to the Haar measure on G (see also, [5, Theorem 3.2.4]). If μ is an adapted and strictly aperiodic measure on a compact metrisable group G, then w*-$\lim\limits_{n\to\infty} \mu^n = m_G$ [6, Theorem 8]. If μ is an adapted measure on a second countable non-compact locally compact group G, then w*-$\lim\limits_{n\to\infty} \mu^n = 0$ [9, Theorem 2]. In [2, Théorème 8], it was proved that if μ is a strictly aperiodic measure on a non-compact locally compact group G, then w*-$\lim\limits_{n\to\infty} \mu^n = 0$ (for related results see also, [1], [5], [10], [11]).

In this note, we present some results of Kawada-Itô type.

2. The Sequence $\{(\widetilde{\mu} * \mu)^n\}_{n\in\mathbb{N}}$

In this section, we study weak* convergence of the sequence $\{(\widetilde{\mu} * \mu)^n\}_{n\in\mathbb{N}}$ for the probability measure μ on a locally compact group G.

As is well known, equipped with the involution given by $d\widetilde{\mu}(g) = \overline{d\mu(g^{-1})}$, the algebra $M(G)$ becomes a Banach $*$-algebra. If μ is a probability measure on a locally compact group G, then as $\mathrm{supp}\widetilde{\mu} = (\mathrm{supp}\mu)^{-1}$, we have

$$\mathrm{supp}(\widetilde{\mu} * \mu) = \overline{\left\{(\mathrm{supp}\mu)^{-1} \cdot (\mathrm{supp}\mu)\right\}}.$$

If H is a closed subgroup of the locally compact group G, then $\overline{m}_H$ may be regarded as a measure on G by putting $\overline{m}_H(E) = m_H(E \cap H)$ for every Borel subset E of G.

The following two theorems are the main results of this note.

Theorem 1. *For an arbitrary probability measure μ on a compact (not necessarily metrisable) group G, we have*

$$w^* - \lim_{n\to\infty} (\widetilde{\mu} * \mu)^n = m_H,$$

where $H = [\operatorname{supp}(\widetilde{\mu} * \mu)]$.

Theorem 2. *Let μ be a probability measure on a locally compact group G. If* $[\operatorname{supp}(\widetilde{\mu} * \mu)]$ *is not compact, then*

$$w^* - \lim_{n\to\infty} (\widetilde{\mu} * \mu)^n = 0.$$

For the proof of Theorems 1 and 2, we need some preliminary results.

Let $\mathcal{H}$ be a complex Hilbert space and let $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. If T is a contraction on $\mathcal{H}$, then by the Mean Ergodic Theorem [7, Chapter 2],

$$P_T x := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k x \quad \text{in norm, for every } x \in \mathcal{H},$$

where P_T is a orthogonal projection onto $\ker(T - I)$. The operator P_T will be called *mean ergodic projection associated with T*. Moreover, we have

$$\mathcal{H} = \ker(T - I) \oplus \overline{(T - I)\,\mathcal{H}} \tag{1}$$

and $TP_T = P_T T = P_T$.

The following result is an immediate consequence of the identity (1).

Proposition 1. *Let T be a contraction on a Hilbert space $\mathcal{H}$ and assume that $\left\|T^{n+1}x - T^n x\right\| \to 0$ for all $x \in \mathcal{H}$. Then,*

$$\lim_{n\to\infty} T^n x = P_T x \quad \textit{in norm for every } x \in \mathcal{H},$$

where P_T is the mean ergodic projection associated with T.

Recall that an operator $T \in B(\mathcal{H})$ is said to be *positive* if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. For example, T^*T is a positive operator for any $T \in B(\mathcal{H})$. Now, let T be a positive contraction. Since $\lim_{n\to\infty} \left|\lambda^{n+1} - \lambda^n\right| = 0$ for all $0 \leq \lambda \leq 1$, by the Spectral Theorem, we have $\lim_{n\to\infty} \left\|T^{n+1} - T^n\right\| = 0$.

Even more can be deduced.

Proposition 2. *If T is a positive contraction on a Hilbert space, then*

$$\overline{\lim_{n\to\infty}}\, n \left\|T^{n+1} - T^n\right\| \leq \frac{1}{e}.$$

Proof. Notice that the spectrum of T is in $[0,1]$. Since

$$\max_{\lambda\in[0,1]}\left|\lambda^{n+1}-\lambda^{n}\right|=\frac{n^{n}}{(n+1)^{n+1}},$$

by the Spectral Theorem,

$$\left\|T^{n+1}-T^{n}\right\|\leq\max_{\lambda\in[0,1]}\left|\lambda^{n+1}-\lambda^{n}\right|=\frac{n^{n}}{(n+1)^{n+1}}=\frac{1}{n}\frac{n}{n+1}\frac{1}{\left(1+\frac{1}{n}\right)^{n}}.$$

It follows that

$$\overline{\lim_{n\to\infty}}\, n\left\|T^{n+1}-T^{n}\right\|\leq\frac{1}{e}.$$

◀

As a consequence of Propositions 1, we have the following.

Corollary 1. *If T is a contraction on a Hilbert space, then*

$$\lim_{n\to\infty}(T^{*}T)^{n}=P \quad \textit{in the strong operator topology,}$$

where P is an orthogonal projection onto $\ker(T^{*}T-I)$.

Let G be a locally compact group and let π be a strongly continuous unitary representations of G on a Hilbert space $\mathcal{H}_{\pi}$. For an arbitrary $\mu\in M(G)$, we can define $\widehat{\mu}(\pi)\in B(\mathcal{H}_{\pi})$, by

$$\langle\widehat{\mu}(\pi)x,y\rangle=\int_{G}\langle\pi(g)x,y\rangle d\mu(g) \quad (x,y\in\mathcal{H}_{\pi}).$$

The map $\mu\to\widehat{\mu}(\pi)$ is multiplicative, $*$-linear; $\widehat{\mu}(\pi)^{*}=\widehat{\widetilde{\mu}}(\pi)$, and contractive; $\|\widehat{\mu}(\pi)\|\leq\|\mu\|_{1}$, where $\|\mu\|_{1}$ is the total variation norm of μ. By $\widehat{G}$ we will denote the set of all equivalence classes of irreducible strongly continuous unitary representations of G. The function $\pi\to\widehat{\mu}(\pi)$ $\left(\pi\in\widehat{G}\right)$ is called *Fourier-Stieltjes transform* of the measure μ. If $\widehat{\mu}(\pi)=0$ for all $\pi\in\widehat{G}$, then $\mu=0$ (for instance, see [4, § 18]).

We will assume that the dual object $\widehat{G}$ of G is equipped with the Fell topology. Recall that a point $\pi_{0}\in\widehat{G}$ is a *limit point* of $M\subset\widehat{G}$ in the Fell topology, if the matrix function $g\to\langle\pi_{0}(g)x_{0},x_{0}\rangle$ $(x_{0}\in\mathcal{H}_{\pi_{0}})$ can be uniformly approximated on every compact $K\subset G$ by the matrix functions $g\to\langle\pi(g)x,x\rangle$ $(\pi\in M,\ x\in\mathcal{H}_{\pi})$. The set $M\subset\widehat{G}$ is said to be *closed* if it contains all of its limit points. It is well known that if G is compact, then every $\pi\in\widehat{G}$ is finite dimensional. Also, we know that if G is compact (resp. compact and metrisable) then $\widehat{G}$ is discrete (resp. countable). These facts are consequences of the Peter-Weyl theory [8, Chapter 4]. Also, recall that σ-compact locally compact group G is metrisable if and only if $\widehat{G}$ is separable (for instance see, [4]).

The following result was proved in [3, Proposition 2.1].

Lemma. *Let μ be a probability measure on a locally compact group G and let π be a strongly continuous unitary representations of G on a Hilbert space $\mathcal{H}_\pi$. Then, we have*

$$\ker\left[\widehat{\mu}(\pi) - I_\pi\right] = \{x \in \mathcal{H}_\pi : \pi(g)\,x = x,\ \forall g \in [\mathrm{supp}\mu]\},$$

where I_π is the identity operator on $\mathcal{H}_\pi$.

Now, we are in a position to prove Theorem 1.

Proof. (Proof of Theorem 1) Let $\pi \in \widehat{G}$ and let $\mathcal{H}_\pi$ be the representation space of π. Since G is a compact group, $\mathcal{H}_\pi$ is finite dimensional. Let $\dim \mathcal{H}_\pi := n_\pi$ and let $\left\{e_\pi^{(1)}, ..., e_\pi^{(n_\pi)}\right\}$ be the basic vectors in $\mathcal{H}_\pi$. Denote by $f_{i,j}^\pi$ the matrix functions of π, where

$$f_{i,j}^\pi(g) = \langle \pi(g)\, e_\pi^{(i)}, e_\pi^{(j)} \rangle \quad (i, j = 1, ..., n_\pi).$$

Then, we can write

$$\langle (\widetilde{\mu} * \mu)^n, f_{i,j}^\pi \rangle = \int_G \langle \pi(g)\, e_\pi^{(i)}, e_\pi^{(j)} \rangle d(\widetilde{\mu} * \mu)^n = \langle \left[\widehat{\mu}(\pi)^* \widehat{\mu}(\pi)\right]^n e_\pi^{(i)}, e_\pi^{(j)} \rangle \quad (\forall n \in \mathbb{N}).$$

By Corollary 1,

$$\langle \left[\widehat{\mu}(\pi)^* \widehat{\mu}(\pi)\right]^n e_\pi^{(i)}, e_\pi^{(j)} \rangle \to \langle P_\mu^\pi e_\pi^{(i)}, e_\pi^{(j)} \rangle \quad (n \to \infty),$$

where P_μ^π is an orthogonal projection onto $\ker\left[\widehat{\mu}(\pi)^* \widehat{\mu}(\pi) - I_\pi\right]$. So we have

$$\langle (\widetilde{\mu} * \mu)^n, f_{i,j}^\pi \rangle \to \langle P_\mu^\pi e_\pi^{(i)}, e_\pi^{(j)} \rangle \quad (n \to \infty).$$

By the Peter-Weyl C-Theorem [8, Chapter 4], the system of matrix functions

$$\left\{f_{i,j}^\pi : \pi \in \widehat{G},\ i, j = 1, ..., n_\pi\right\}$$

is linearly dense in $C(G)$. Consequently, the limit

$$\lim_{n\to\infty} \langle (\widetilde{\mu} * \mu)^n, f \rangle \ \text{ exists for all } f \in C(G).$$

Since

$$f \to \lim_{n\to\infty} \langle (\widetilde{\mu} * \mu)^n, f \rangle$$

is a bounded linear functional on $C(G)$, there exists a measure $\theta_\mu \in M(G)$ such that

$$\lim_{n\to\infty} \langle (\widetilde{\mu} * \mu)^n, f \rangle = \langle \theta_\mu, f \rangle, \quad \forall f \in C(G).$$

So we have

$$\text{w}^*\text{-}\lim_{n\to\infty} (\widetilde{\mu} * \mu)^n = \theta_\mu.$$

Now, let $H := [\text{supp}\,(\widetilde{\mu} * \mu)]$. It remains to show that $\theta_\mu = m_H$. Let us see first that θ_μ is an idempotent measure. Since the left (or right) multiplication on $M(G)$ is separately continuous, we have $(\widetilde{\mu} * \mu) * \theta_\mu = \theta_\mu$, which implies

$$(\widetilde{\mu} * \mu)^n * \theta_\mu = \theta_\mu, \quad \forall n \in \mathbb{N}.$$

As $(\widetilde{\mu} * \mu)^n \to \theta_\mu$ in the w*-topology, we have $\theta_\mu^2 = \theta_\mu$. Hence, θ_μ is an idempotent measure. Notice also that

$$\widehat{\theta_\mu}(\pi) = P_\mu^\pi, \quad \forall \pi \in \widehat{G}.$$

Further, since $\widehat{m_H}(\pi)$ is an orthogonal projection, by Lemma, we can write

$$\widehat{m_H}(\pi)\,\mathcal{H}_\pi = \ker[\widehat{m_H}(\pi) - I_\pi] = \{x \in \mathcal{H}_\pi : \pi(g)\,x = x, \ \forall g \in H\}.$$

For the same reasons,

$$\widehat{\theta_\mu}(\pi)\,\mathcal{H}_\pi = P_\mu^\pi \mathcal{H}_\pi = \ker\left[\widehat{\mu}(\pi)^* \,\widehat{\mu}(\pi) - I_\pi\right] = \{x \in \mathcal{H}_\pi : \pi(g)\,x = x, \ \forall g \in H\}.$$

Thus we have $\widehat{\theta_\mu}(\pi) = \widehat{m_H}(\pi)$ for all $\pi \in \widehat{G}$. It follows that $\theta_\mu = m_H$. ◀

Let μ be a probability measure on a compact metrisable group G. A Borel subset E of G is said to be a *continuity set* of μ if $\mu(\partial E) = 0$, where ∂E denotes topological boundary of E. By the well known Portmanteau theorem, the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of the probability measures on G, weak* converges to the measure μ if and only if $\mu_n(E) \to \mu(E)$ for any continuity set E of μ.

Corollary 2. *Let μ be a probability measure on a compact metrisable group G with $[\text{supp}\,(\widetilde{\mu} * \mu)] = G$. The following assertions hold:*

(a) For an arbitrary continuity set E of m_G,

$$\lim_{n\to\infty} (\widetilde{\mu} * \mu)^n (E) = m_G(E).$$

(b) Let ν be a probability measure on G and assume that for an arbitrary continuity set E of ν,

$$\lim_{n\to\infty} (\widetilde{\mu} * \mu)^n (E) = \nu(E).$$

Then, $\nu = m_G$.

Let G be a locally compact group. For an arbitrary $f \in L^p(G)$ $(1 \le p < \infty)$, we put

$$f^\vee(g) := f\left(g^{-1}\right) \quad \text{and} \quad \widetilde{f}(g) := \overline{f\left(g^{-1}\right)}.$$

Notice that for every $u, v \in L^2(G)$, the function $u * \widetilde{v}$ is in $C_0(G)$ and

$$\langle \mu, \ u * \widetilde{v}\rangle = \langle \mu * \overline{v}, \ \overline{u}\rangle, \quad \forall \mu \in M(G).$$

It follows that the set $\left\{u * \widetilde{v} : u, v \in L^2(G)\right\}$ is linearly dense in $C_0(G)$. Notice also that if $f \in L^p(G)$ $(1 < p < \infty,\, p \ne 2)$ and $h \in L^q(G)$ $(1/p + 1/q = 1)$, then $h * f^\vee \in C_0(G)$ and

$$\langle \mu, \ h * f^\vee\rangle = \langle \mu * f, \ h\rangle, \quad \forall \mu \in M(G).$$

It follows that $\{h * f^{\vee} : h \in L^q(G),\ f \in L^p(G)\}$ is linearly dense in $C_0(G)$.

Let π be the left regular representation of G on $L^p(G)$ $(1 \le p < \infty)$, where

$$\pi(g) f(s) = f\left(g^{-1}s\right) := f_g(s).$$

Then, π is continuous and $\widehat{\mu}(\pi) f = \mu * f$ for every $\mu \in M(G)$. We will denote this operator by $\lambda_p(\mu)$, the left convolution operator. The left convolution operator $\lambda_p(\mu) f := \mu * f$ is a bounded linear operator on $L^p(G)$, that is,

$$\|\lambda_p(\mu) f\|_p \le \|\mu\|_1 \|f\|_p \quad \text{and} \quad \|\lambda_1(\mu)\|_1 = \|\mu\|_1.$$

Proof. (Proof of Theorem 2) It suffices to show that

$$\langle (\widetilde{\mu} * \mu)^n,\ u * \widetilde{v}\rangle \to 0 \ \text{ for all } u, v \in L^2(G).$$

For this, we must show that

$$\langle (\widetilde{\mu} * \mu)^n * \overline{v}, \overline{u}\rangle \to 0.$$

Since $\lambda_2(\widetilde{\mu}) = \lambda_2(\mu)^*$, by Lemma,

$$\begin{aligned}\left\{f \in L^2(G) : \lambda_2(\mu)^* \lambda_2(\mu) f = f\right\} &= \left\{f \in L^2(G) : \lambda_2(\widetilde{\mu} * \mu) f = f\right\} \\ &= \left\{f \in L^2(G) : f_s = f,\ \forall s \in [\operatorname{supp}(\widetilde{\mu} * \mu)]\right\}.\end{aligned}$$

Since $[\operatorname{supp}(\widetilde{\mu} * \mu)]$ is not compact, from the identity $f_s = f$ for all $s \in [\operatorname{supp}\mu]$, we have $f = 0$ (a.e.). Hence,

$$\ker\left[\lambda_2(\mu)^* \lambda_2(\mu) - I\right] = 0.$$

By Corollary 1,

$$\left[\lambda_2(\mu)^* \lambda_2(\mu)\right]^n \to 0 \text{ in the strong operator topology.}$$

Now if $u, v \in L^2(G)$, then we get

$$\langle (\widetilde{\mu} * \mu)^n * \overline{v}, \overline{u}\rangle = \langle \left[\lambda_2(\mu)^* \lambda_2(\mu)\right]^n \overline{v}, \overline{u}\rangle \to 0.$$

◀

3. Norm Convergence

In this section, we present some results concerning norm convergence of the sequence $\{(\widetilde{\mu} * \mu)^n * f\}_{n \in \mathbb{N}}$ in $L^p(G)$ spaces.

Proposition 3. *Let μ be a probability measure on a locally compact group G and let $f \in L^p(G)$ $(1 < p < \infty)$. The following assertions hold:*

(a) If G is compact and $[\operatorname{supp}(\widetilde{\mu} * \mu)] = G$, then

$$(\widetilde{\mu} * \mu)^n * f \to \left(\int_G f dm_G\right) \mathbf{1} \quad \textit{in } L^p\textit{-norm},$$

where $\mathbf{1}$ is the identity one function on G.

(b) If $[\operatorname{supp}(\widetilde{\mu} * \mu)]$ is not compact, then

$$(\widetilde{\mu} * \mu)^n * f \to 0 \quad \textit{in } L^p\textit{-norm}.$$

Proof. If $\theta := \widetilde{\mu} * \mu$, then by Corollary 1, $\theta^n * f \to u$ in the L^p-norm for some $u \in L^p(G)$. On the other hand, by Theorem 1,

$$\text{w}^*\text{-} \lim_{n\to\infty} \theta^n = m_G.$$

If $v \in L^q(G)$ $(1/p + 1/q = 1)$, then as $v^\vee * f \in C(G)$, we can write

$$\langle u, v\rangle = \lim_{n\to\infty} \langle \theta^n * f, v\rangle = \langle \theta^n, v^\vee * f\rangle = \langle m_G, v^\vee * f\rangle = \langle m_G * f, v\rangle.$$

So we have

$$u = m_G * f = \int_G f dm_G.$$

If $[\text{supp}(\widetilde{\mu} * \mu)]$ is not compact, then by Theorem 2,

$$\text{w}^*\text{-} \lim_{n\to\infty} \theta^n = 0.$$

For an arbitrary $v \in L^q(G)$ $(1/p + 1/q = 1)$, since $v * f^\vee \in C_0(G)$, we get

$$\langle u, v\rangle = \lim_{n\to\infty} \langle \theta^n * f, v\rangle = \lim_{n\to\infty} \langle \theta^n, v * f^\vee\rangle = 0.$$

Hence, $u = 0$. ◄

Next, we have the following.

Proposition 4. *Let μ be a probability measure on a compact group G. If $[\text{supp}(\widetilde{\mu} * \mu)] = G$, then for an arbitrary $f \in C(G)$,*

$$(\widetilde{\mu} * \mu)^n * f \to \left(\int_G f dm_G\right) \mathbf{1} \quad \textit{uniformly on } G.$$

For the proof, we need some preliminary results. Let χ_π denote the character of $\pi \in \widehat{G}$;

$$\chi_\pi(g) = \sum_{i=1}^{n_\pi} \langle \pi(g) e_\pi^{(i)}, e_\pi^{(i)}\rangle.$$

For a given $f \in C(G)$, let $c_\pi(f) = n_\pi \chi_\pi * f$, where $n_\pi = \dim \pi$. It follows from the Peter-Weyl L^2-theorem [8, Chapter 4] that the Parseval identity

$$\|f\|_2^2 = \sum_{\pi\in\widehat{G}} \|c_\pi(f)\|_2^2$$

holds. It follows that there is at most countable set of values $\pi \in \widehat{G}$ for which $c_\pi(f) \neq 0$. This set is called the *spectrum* of f and is denoted by $\text{sp}(f)$. Any continuous function f on the compact group G can be uniformly approximated by linear combinations of matrix functions of those representations $\pi \in \widehat{G}$ for which $\pi \in \text{sp}(f)$ [8, Chapter 4].

Proof. (Proof of Proposition 4) Let $\theta := \widetilde{\mu} * \mu$. If $\pi \in \widehat{G}$ and

$$f_{x,y}^{\pi}(g) := \langle \pi(g) x, y \rangle \quad (x, y \in \mathcal{H}_{\pi}),$$

then we can write

$$\left(\theta^n * f_{x,y}^{\pi}\right)(g) = \int_G f_{x,y}^{\pi}\left(s^{-1}g\right) d\theta^n(s) = \int_G \langle \pi\left(s^{-1}g\right) x, y \rangle d\theta^n(s) =$$

$$= \int_G \langle \pi\left(s^{-1}\right) \pi(g) x, y \rangle d\theta^n(s) = \int_G \langle \pi(g) x, \pi(s) y \rangle d\theta^n(s) =$$

$$= \langle \pi(g) x, \widehat{\theta}(\pi)^n y \rangle \quad (\forall n \in \mathbb{N}).$$

Let $f \in C(G)$ and assume that $\int f dm_G = 0$. Let us show that $\theta^n * f \to 0$ uniformly on G. Let π_0 be the trivial representation of G. Since $\pi_0 \notin \mathrm{sp}(f)$, for an arbitrary $\varepsilon > 0$, there exist complex numbers $\lambda_1, ..., \lambda_k$ and $\pi_1, ..., \pi_k \in \widehat{G} \setminus \{\pi_0\}$ such that

$$|f(g) - \lambda_1 \langle \pi_1(g) x_1, y_1 \rangle - ... - \lambda_k \langle \pi_k(g) x_k, y_k \rangle| < \varepsilon, \quad \forall g \in G,$$

where $x_i, y_i \in \mathcal{H}_{\pi_i}$ $(i = 1, ..., k)$. Consequently, we have

$$\left| (\theta^n * f)(g) - \lambda_1 \langle \pi_1(g) x_1, \widehat{\theta}(\pi_1)^n y_1 \rangle - ... - \lambda_k \langle \pi_k(g) x_k, \widehat{\theta}(\pi_k)^n y_k \rangle \right| < \varepsilon,$$

which implies

$$|(\theta^n * f)(g)| \leq |\lambda_1| \left\| \widehat{\theta}(\pi_1)^n y_1 \right\| \|x_1\| + ... + |\lambda_k| \left\| \widehat{\theta}(\pi_k)^n y_k \right\| \|x_k\| + \varepsilon, \quad \forall g \in G.$$

It remains to show that $\left\| \widehat{\theta}(\pi)^n x \right\| \to 0$ for all $\pi \in \widehat{G} \setminus \{\pi_0\}$ and $x \in \mathcal{H}_{\pi}$. If $\pi \in \widehat{G}$, then by Theorem 1,

$$\langle \widehat{\theta}(\pi)^n x, y \rangle = \langle \theta^n, f_{x,y}^{\pi} \rangle \to \langle m_G, f_{x,y}^{\pi} \rangle = \langle \widehat{m_G}(\pi) x, y \rangle, \quad \forall x, y \in \mathcal{H}_{\pi}.$$

From the orthogonality of π and π_0, we have $\widehat{m_G}(\pi) = 0$ for all $\pi \in \widehat{G} \setminus \{\pi_0\}$. Hence,

$$\langle \widehat{\theta}(\pi)^n x, y \rangle \to 0 \quad \text{for all } \pi \in \widehat{G} \setminus \{\pi_0\} \text{ and } x, y \in \mathcal{H}_{\pi}.$$

Since $\mathcal{H}_{\pi}$ is finite dimensional, we have

$$\left\| \widehat{\theta}(\pi)^n x \right\| \to 0 \quad \text{for all } \pi \in \widehat{G} \setminus \{\pi_0\} \text{ and } x \in \mathcal{H}_{\pi}.$$

If $\int f dm_G = c \neq 0$, then $\int h dm_G = 0$, where $h = f - c\mathbf{1}$ and $\mathbf{1}$ is the identity one function on G. Then as $\theta^n * h \to 0$ uniformly, we have $\theta^n * f \to c$ uniformly on G. ◀

It is well known that if G is a compact group, then the Haar measure m_G on G is an idempotent measure on G with $\mathrm{supp} m_G = G$.

The following result may be of some interest.

Proposition 5. *Let G be a compact group and let n be a fixed natural number. If μ is a probability measure on G with* $\mathrm{supp}\mu = G$, *then $\mu = m_G$ is the unique solution of the equation*

$$\mu^n + \mu^{n-1} + ... + \mu = n m_G. \tag{2}$$

Proof. We have

$$(m_G * \mu)(B) = \int_G m_G\left(Bg^{-1}\right) d\mu(g) = m_G(B),$$

for every Borel subsets B of G. It follows that $m_G * \mu = m_G$. Also, notice that

$$\mu * \left(\frac{1}{n}\sum_{i=0}^{n-1}\mu^i\right) - \frac{1}{n}\sum_{i=0}^{n-1}\mu^i = \frac{\mu^n - \delta_e}{n} \to 0 \text{ in the } \|\cdot\|_1\text{-norm, as } n \to \infty.$$

Since the left (or right) multiplication on $M(G)$ is separately continuous, by the Kawada-Ito theorem, we have $\mu * m_G - m_G = 0$. Consequently, we can write

$$\mu^{n+1} - \mu = (\mu - \delta_e) * \left(\mu^n + \mu^{n-1} + ... + \mu\right) =$$

$$= (\mu - \delta_e) * n m_G = n(\mu * m_G - m_G) = 0.$$

So we have $\mu^{n+1} = \mu$, which implies $\mu^{2n} = \mu^n$. If $\nu := \mu^n$, then ν is an idempotent measure with supp$\nu = G$. Since

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\nu^i = \lim_{n\to\infty}\frac{\delta_e + (n-1)\nu}{n} = \nu \text{ in the } \|\cdot\|_1 - \text{norm},$$

by the Kawada-Ito theorem, we have $\nu = m_G$ or $\mu^n = m_G$. Taking into account the identity $\mu^n = m_G$ in the equation (2), we have

$$\mu^{n-1} + \mu^{n-2} + ... + \mu = (n-1)\, m_G.$$

If we continue this process, finally we get $\mu = m_G$. ◀

References

1. Billingsley P. *Convergence of Probability Measures.* 2nd ed. John Wiley & Sons, Inc., New York, 1999.
2. Derriennic Y. Lois "zéro ou deux" pour les processus de Markov. Applications aux marches aléatoires. *Ann. Inst. H. Poincaré Sect.* B, 1976, **12** (2), pp. 111-129 (in French).
3. Derriennic Y., Lin M. Convergence of iterates of averages of certain operator representations and of convolution powers. *J. Funct. Anal.*, 1989, **85** (1), pp. 86-102.
4. Dixmier J. *Les C-algèbres et Leurs Représentations.* Gauthier-Villars, Paris, 1964 (in French).
5. Grenander U. *Probabilities on Algebraic Structures.* 2nd ed. Almqvist & Wiksell, Stockholm; John Wiley & Sons, Inc., New York-London, 1968.
6. Kawada Y., Itô K. On the probability distribution on a compact group, I. *Proc. Phys.-Math. Soc. Japan (3)*, 1940, **22**, pp. 977-998.
7. Krengel U. *Ergodic Theorems.* Walter de Gruyter & Co., Berlin, 1985.

8. Lyubich Yu.I. *Introduction to the Theory of Banach Representations of Groups.* Birkhäuser Verlag, Basel, 1988.
9. Mukherjea A. Limit theorems for probability measures on non-compact groups and semi-groups. *Z. Wahrscheinlichkeitstheorie verw Gebiete*, 1976, **33** (4), pp. 273-284.
10. Mustafayev H. Mean ergodic theorems for multipliers on Banach algebras. *J. Fourier Anal. Appl.*, 2019, **25** (2), pp. 393-426.
11. Mustafayev H. A note on the Kawada-Itô theorem. *Statist. Probab. Lett.*, 2022, **181**, Paper No. 109261, pp. 1-6.