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ON THE SPECTRAL THEORY OF A FOURTH-ORDER DIFFERENTIAL PENCIL ON THE WHOLE REAL AXIS

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In memory of M. G. Gasymov on his 85th birthday

Abstract. *A pencil of fourth-order differential equations on the entire axis with multiple characteristics is considered. Using special solutions of a fourth-order differential equation, the spectrum of a differential pencil is studied. Necessary and sufficient conditions are found for a non-real number to be an eigenvalue of this pencil. It is proved that the operator pencil has no eigenvalues on the real axis.*

Keywords: pencil of fourth-order differential equations, multiple characteristics, operator pencil, spectrum, eigenvalues.

Mathematics Subject Classification (2020): 34A55

1. Introduction

Let us consider in the space $L_2(-\infty,+\infty)$ a differential operator pencil L_λ, generated by the left-hand side of the pencil of differential equations

$$\ell\left(x,\frac{d}{dx},\lambda\right)y=\left(\frac{d^2}{dx^2}+\lambda^2\right)^2y+r(x)\,y'+(\lambda p(x)+q(x))\,y=0, \qquad (1)$$

where the complex-valued functions $r(x)$, $p(x)$ and $q(x)$ satisfy the conditions

$$r(x)\in C^{(1)}(-\infty,+\infty),\quad p(x)\in C^{(1)}(-\infty,+\infty),\quad q(x)\in C(-\infty,+\infty),$$

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$$\int_{-\infty}^{+\infty} \left(1+x^4\right)\left\{\left|r^{(j)}(x)\right|+\left|p^{(j)}(x)\right|+|q(x)|\right\} dx<\infty, \quad j=0,1. \tag{2}$$

In this work, the spectral properties of pencil L_λ are studied. Note that for a pencil L_λ defined on a semi-axis, a similar problem was studied in [2], [7]. To study the pencil L_λ on the entire axis, in contrast to the case of the semi-axis, it will be necessary to use special solutions of equation (1) with asymptotics at $-\infty$. The latter circumstance required a significant modification of some reasoning characteristic of the pencil L_λ defined on the semi-axis. Some results of this work were first announced in [8].

The results of this work can be used in studying direct and inverse scattering problems for equation (1). Note that inverse problems for differential equations of high orders have been studied in the works of many authors (see [1], [4], [9], [10] and the references therein).

2. Preliminary Information

Consider equation the (1). Let's we put

$$\sigma_j^{\pm}(x)=\pm\int_x^{\pm\infty}|s|^j\left\{|p(s)|+|r(s)|+\left|s r'(s)\right|+|s q(s)|\right\} ds,$$

$$\tau^{\pm}(x)=\pm\int_x^{\pm\infty} s^2\left\{|p(s)|+|r(s)|+\left|s r'(s)\right|+|s q(s)|\right\} ds.$$

It is known [2], [7] that under conditions (2) equation (1) has special solutions $f_j^{\pm}(x,\lambda)$, $g_j^{\pm}(x,\lambda)$ representable in the form

$$\begin{aligned} f_j^{\pm}(x,\lambda) &= x^{j-1}e^{\pm i\lambda x}+\int_x^{\infty} A_j^{\pm}(x,t)\, e^{\pm i\lambda t}dt, j=1,2, \\ g_j^{\pm}(x,\lambda) &= x^{j-1}e^{\mp i\lambda x}+\int_{-\infty}^{x} B_j^{\pm}(x,t)\, e^{\mp i\lambda t}dt, \quad j=1,2, \end{aligned} \tag{3}$$

where $A_j^{\pm}(x,t)$, $B_j^{\pm}(x,t)$ are four times continuously differentiable functions and the following relations hold:

$$\begin{aligned} \left|A_j^{\pm}(x,t)\right| &\leq \frac{1}{4}\sigma_j^{+}\left(\tfrac{x+t}{2}\right)\exp\left\{\tau^{+}(x)\right\}, \\ \left|B_j^{\pm}(x,t)\right| &\leq \frac{1}{4}\sigma_j^{-}\left(\tfrac{x+t}{2}\right)\exp\left\{\tau^{-}(x)\right\}. \end{aligned} \tag{4}$$

According to relations (3), (4), for each fixed x, the functions $f_j^{\pm}(x,\lambda)$, $g_j^{\pm}(x,\lambda)$ are analytic functions in the open half-plane $\pm Im\lambda>0$ and are continuous up to the real axis. Further, from the general theory it is known that (see [3], [5], [6]) for all $\lambda\neq 0$, the equation (1) also has linearly independent solutions $\psi_j^{\pm}(x,\lambda), j = 1,2,3,4$ with asymptotics

$$\psi_j^{\pm}(x,\lambda) = e^{\mp i\omega_j\lambda x}(1+o(1)), \quad \omega_j = e^{\frac{i\pi(j-1)}{2}}, \quad x \to \pm\infty. \tag{5}$$

Now we consider the unperturbed equation

$$\left(\frac{d^2}{dx^2}+\lambda^2\right)^2 y = 0. \tag{6}$$

Obviously, for $\lambda \neq 0$, equation (6) has four linearly independent solutions of the form

$$y_1(x,\lambda) = e^{i\lambda x}, \quad y_2(x,\lambda) = xe^{i\lambda x}, \quad y_3(x,\lambda) = e^{-i\lambda x}, \quad y_4(x,\lambda) = xe^{-i\lambda x}.$$

It is easy to check that the Wronskian $W\{y_1,y_2,y_3,y_4\}$ of these solutions is equal to $16\lambda^4$.

3. Studying the Spectrum of the Pencil L_λ

Let us study the eigenvalues of the operator pencil L_λ. We denote by $W^{\pm}(\lambda)$ the Wronskian of the solutions $f_1^{\pm}(x,\lambda), f_2^{\pm}(x,\lambda), g_1^{\pm}(x,\lambda), g_2^{\pm}(x,\lambda)$ of equation (1):

$$W^{\pm}(\lambda) = W\left\{f_1^{\pm}(x,\lambda), f_2^{\pm}(x,\lambda), g_1^{\pm}(x,\lambda), g_2^{\pm}(x,\lambda)\right\}.$$

Since the Wronskian of solutions to equation (1) does not depend on x, we can write:

$$W^{\pm}(\lambda) = \begin{vmatrix} f_1^{\pm}(0,\lambda) & f_2^{\pm}(0,\lambda) & g_1^{\pm}(0,\lambda) & g_2^{\pm}(0,\lambda) \\ f_1^{\pm(1)}(0,\lambda) & f_2^{\pm(1)}(0,\lambda) & g_1^{\pm(1)}(0,\lambda) & g_2^{\pm(1)}(0,\lambda) \\ f_1^{\pm(2)}(0,\lambda) & f_2^{\pm(2)}(0,\lambda) & g_1^{\pm(2)}(0,\lambda) & g_2^{\pm(2)}(0,\lambda) \\ f_1^{\pm(3)}(0,\lambda) & f_2^{\pm(3)}(0,\lambda) & g_1^{\pm(3)}(0,\lambda) & g_2^{\pm(3)}(0,\lambda) \end{vmatrix}. \tag{7}$$

Theorem 1. *In order for a number λ with $\pm Im\lambda > 0$ to be an eigenvalue of the operator pencil L_λ, it is necessary and sufficient that it satisfies the equation:*

$$W^{\pm}(\lambda) = 0. \tag{8}$$

Proof. Let the number λ with $Im\lambda > 0$ be an eigenvalue of the operator pencil L_λ. Then equation (1) has a solution $y(x,\lambda)$ from the space $L_2(-\infty,+\infty)$. On the other hand, two (we denote them by $\psi_k^{+}(x,\lambda), \psi_m^{+}(x,\lambda)$) of solutions (5) do not belong to the space $L_2(0,+\infty)$. These solutions, together with the solutions $f_1^{+}(x,\lambda), f_2^{+}(x,\lambda)$ form a fundamental system of solutions to equation (1). Therefore, the decomposition

$$y(x,\lambda) = C_1 f_1^{+}(x,\lambda) + C_2 f_2^{+}(x,\lambda) + C_3\psi_m^{+}(x,\lambda) + C_4\psi_k^{+}(x,\lambda)$$

is valid. Since $y(x,\lambda) \in L_2(0,+\infty)$ and $Im\lambda > 0$, it follows from the last relation that $C_3 = C_4 = 0$. Therefore, the following identity is true:

$$y(x,\lambda) = C_1 f_1^{+}(x,\lambda) + C_2 f_2^{+}(x,\lambda).$$

Similarly is established the following equality

$$y(x,\lambda) = D_1 g_1^{+}(x,\lambda) + D_2 g_2^{+}(x,\lambda).$$

From the last two equalities we obtain that

$$C_1 f_1^{+}(x,\lambda) + C_2 f_2^{+}(x,\lambda) - D_1 g_1^{+}(x,\lambda) - D_2 g_2^{+}(x,\lambda) = 0.$$

On the other hand, $y(x,\lambda)$ serves as a nontrivial solution to equation (1). Therefore, one of the coefficients C_1, C_2, D_1, D_2 is necessarily different from zero. It follows that the solutions $f_1^{+}(x,\lambda), f_2^{+}(x,\lambda), g_1^{+}(x,\lambda), g_2^{+}(x,\lambda)$ are linearly dependent. As is clear from the latter, $W^{+}(\lambda) = 0$. On the contrary, it follows from equality (8) that the columns of the determinant

$$\begin{vmatrix} f_1^{\pm}(x,\lambda) & f_2^{\pm}(x,\lambda) & g_1^{\pm}(x,\lambda) & g_2^{\pm}(x,\lambda) \\ f_1^{\pm(1)}(x,\lambda) & f_2^{\pm(1)}(x,\lambda) & g_1^{\pm(1)}(x,\lambda) & g_2^{\pm(1)}(x,\lambda) \\ f_1^{\pm(2)}(x,\lambda) & f_2^{\pm(2)}(x,\lambda) & g_1^{\pm(2)}(x,\lambda) & g_2^{\pm(2)}(x,\lambda) \\ f_1^{\pm(3)}(x,\lambda) & f_2^{\pm(3)}(x,\lambda) & g_1^{\pm(3)}(x,\lambda) & g_2^{\pm(3)}(x,\lambda) \end{vmatrix}$$

are linearly dependent. In particular, we get

$$C_1 f_1^{+}(x,\lambda) + C_2 f_2^{+}(x,\lambda) + D_1 g_1^{+}(x,\lambda) + D_2 g_2^{+}(x,\lambda) = 0.$$

Using this equality, it is established that equation (1) has a nontrivial solution $y(x,\lambda)$ from the space $L_2(-\infty,+\infty)$. Therefore, the number λ with $Im\lambda > 0$ is an eigenvalue of the operator pencil L_λ. The case $Im\lambda < 0$ is treated similarly. ◀

Theorem 2. *The eigenvalues of the operator pencil L_λ in the half-plane $\pm Im\lambda > 0$ form no more than a countable set whose limit points can only be on the real axis.*

Proof. $W^{\pm}(\lambda)$ is an analytic function in the half-plane $\pm Im\lambda > 0$ and hence its zeros form a finite or countable set. Moreover, from formulas (3), (4), (7) it follows that

$$W^{\pm}(\lambda) = C^{\pm}\lambda^{N}\left(1 + O\left(\frac{1}{\lambda}\right)\right), \quad \lambda \to \infty, \quad N > 1.$$

From the last equality it follows that the zeros of the characteristic function $W^{\pm}(\lambda)$, i.e. the eigenvalues of the operator pencil L_λ form a bounded set. By virtue of the theorem on the uniqueness of an analytic function, the limit points of this set cannot be in the half-plane $\pm Im\lambda > 0$. ◀

Now we study the discrete spectrum of the operator pencil L_λ on the real axis.

Theorem 3. *On the real axis the operator pencil L_λ has no eigenvalues.*

Proof. Let the number λ, where $\lambda^2 > 0$, serve as an eigenvalue of the pencil L_λ. Then equation (1) would have a nonzero solution $y(x,\lambda) \in L_2(-\infty,+\infty)$. Note that for $\lambda^2 > 0$ the solutions $f_1^+(x,\lambda), f_2^+(x,\lambda), f_1^-(x,\lambda), f_2^-(x,\lambda)$ form a fundamental system of solutions to equation (1). Therefore, the solution $y(x,\lambda)$ can be represented in the form

$$y(x,\lambda) = C_1 f_1^+(x,\lambda) + C_2 f_2^+(x,\lambda) + C_3 f_1^-(x,\lambda) + C_4 f_2^-(x,\lambda).$$

Using (3), (4) we conclude that

$$y(x,\lambda) = (C_1 + C_2 x) e^{i\lambda x}(1 + o(1)) + (C_3 + C_4 x) e^{-i\lambda x}(1 + o(1)), \quad x \to +\infty.$$

It follows that if $y(x,\lambda) \in L_2(-\infty,+\infty)$, then necessarily $C_1 = C_2 = C_3 = C_4 = 0$. The last statement contradicts the fact that $y(x,\lambda) \neq 0$. Further, it is known that (see [3], [5]) for $\lambda = 0$, the equation (1) has four linearly independent solutions $y_j(x), j = 1, .2, 3, 4$, which satisfy the asymptotic equalities

$$y_j(x) = \frac{x^{j-1}}{(j-1)!}\left\{1 + o\left(\frac{1}{x}\right)\right\}, \quad x \to \infty.$$

It follows that the number $\lambda = 0$ is not an eigenvalue of the operator pencil L_λ. ◄

In conclusion, we note that using the expansion formula obtained in [8], we can prove that the continuous spectrum of the operator pencil $L_\lambda$fills the entire real axis.

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