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INVERSE NODAL PROBLEMS FOR PENCILS OF SINGULAR STURM-LIOUVILLE OPERATORS

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In memory of M. G. Gasymov on his 85th birthday

Abstract. *In this study, some properties of the pencils of singular Sturm-Liouville operators are investigated. Firstly, the behaviors of eigenvalues and eigenfunctions is learned, then for each discontinuity point*

$$a \in \Re = \left\{ r\pi : r = \frac{p}{q},\ p < q,\ p, q \in \mathbb{N} \right\},$$

a solution of the inverse problem is given to determine the potential function and parameters β, h and H with the help of a dense set of nodes. And finally, a constructive method is given for solving the given inverse problem.

Keywords: singular diffusion equation, inverse spectral problems, inverse nodal problems

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1. Introduction

Solvable models of quantum mechanics are investigated in detail in the study [1]. As can be seen, these models are generally expressed with Hamilton operators or Schrödinger operators with singular coefficients. Many of the problems expressed by these models are related to the solution of spectral inverse problems for differential operators with singular coefficients. However, many problems in mathematical physics are reduced to the study of differential operators whose coefficients are generalized functions.

For example, the stationary vibrations of a spring-tied homogeneous wire fixed at both ends, density $R'(x) = a\delta(x - x_0)$ ($\delta(x)$-Dirac function) and stiffness $R(x)$ at point

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x_0, whose domain set is

$$D(L_o) = \Big\{ y(x) \in W_2^2[0,1] : y'(x_0+) - y'(x_0-) = ay(x_0), \\ x_0 \in (0,1); y(0) = 0 = y(1) \Big\}$$

and is expressed by the differential operator given as $L_o = -\dfrac{d^2}{dx^2}$ in Hilbert space $L_2[0,1]$. There is detailed information about the correct (regular) definition of such operators and the examination of their spectral properties in [2], [14], [19], [20] studies.

We consider the following quadratic pencils of Sturm-Liouville equation of the form

$$\ell y := -y'' + [\lambda p(x) + q(x)] y = \lambda^2 y, \quad x \in [0,\pi] \setminus \{a\}, \tag{1}$$

with the boundary conditions

$$U(y) := y'(0) - hy(0) = 0, \tag{2}$$

$$V(y) := y'(\pi) + Hy(\pi) = 0, \tag{3}$$

where $q(x)$ is a real function belonging to the space $L_2[0,\pi]$, λ is a spectral parameter, $p(x) = \beta\delta(x-a)$, h, a, H and β are real numbers.

Definition. *Any function $y(x) \in W_2^2(\Omega)$ ($\Omega = [0,\pi] \setminus \{a\}$), satisfying the Sturm-Liouville equation*

$$-y'' + q(x)y(x) = \lambda^2 y, \tag{4}$$

and the discontinuity condition at the point a:

$$y'(a+0) - y'(a-0) = \lambda\beta y(a), \tag{5}$$

is called the solution of the equation (1).

Next, suppose that for all functions $y(x) \in W_2^2(\Omega), y(x) \neq 0$, satisfying conditions (2),(3) and (5), we have

$$h|y(0)|^2 + H|y(\pi)|^2 + \int_0^\pi \left\{ |y'(x)|^2 + q(x)|y(x)|^2 \right\} dx > 0.$$

Here we denote by $W_2^n(\Omega)$ the space of functions $f(x)$,$x \in \Omega$, such that the derivatives $f^{(m)}(x)$, $\left(m = \overline{1, n-1}\right)$ are absolute continuous and $f^{(n)}(x) \in L_2(\Omega)$.

We denote the boundary value problem (1)-(3), (5) by $L = L(q,\beta)$.

Quadratic pencils of Sturm-Liouville equations with singular coefficient appear frequently in various models of classical and quantum mechanics.

In studies [11], [13], [18], the spectral properties of the operator produced by the regular differential equation given with non-separated boundary conditions containing the spectral parameter were examined and the uniqueness theorems related to the solution of the spectral inverse problem were proved. In studies [5]-[8], the spectral properties of the operator produced by the Schrödinger equation with the singular coefficient given with the boundary conditions depending on the spectral parameter were examined and the solution of the inverse spectral problems according to different spectral data was given.

2. Preliminaries

Let $y(x,\lambda)$ and $z\left(x,\lambda\right)$ be continuously differentiable solutions on $(0,a)\cup(a,\pi)$ of equation (4), satisfying the discontinuity condition (5), then

$$\left\langle y,z\right\rangle_{x=a-0}=\left\langle y,z\right\rangle_{x=a+0},$$

i.e. the function $\left\langle y,z\right\rangle$ is continuous on $(0,\pi)$.

Let $\varphi\left(x,\lambda\right)$ be solution of equation (4), satisfying the initial conditions

$$\varphi\left(0,\lambda\right)=1,\quad \varphi'\left(0,\lambda\right)=h \tag{6}$$

and the discontinuity condition (5).

The characteristic function of the problem (1)-(3) is in the form

$$\varDelta\left(\lambda\right)=\varphi'(\pi,\lambda)+H\varphi\left(\pi,\lambda\right)$$

with the function $\varphi(x,\lambda)$ being the solution of equation (1) satisfying the initial conditions (6).

It is also clear that this is an entire function [9], so this problem has a countable number of eigenvalues. We can also prove the following propositions from the methods used in [12].

In addition, using the methods used in study [10], the following propositions are proved:

Lemma 1. *The eigenvalues of the problem (1)-(3) are real and not equal to zero.*

Lemma 2. *The eigenvalues of problem (1) are simple.*

Let $\varDelta_0\left(\lambda\right)$ be the characteristic function of the problem corresponding to the case is $q(x)\equiv 0$ problem (1)-(3). In this case, it becomes

$$\varDelta_0\left(\lambda\right)=\varphi_0'(\pi,\lambda)+H\varphi_0\left(\pi,\lambda\right),$$

where $\varphi_0\left(x,\lambda\right)$ is the solution of the equation (4), satisfying initial conditions (2) and discontinuity condition (5).

Lemma 3. *Let $G_\delta=\left\{\lambda:\left|\lambda-\lambda_n^0\right|\geq\delta, n=1,2,...\right\}$ be a small enough number $\delta<\dfrac{r}{2}$. The zeros of the $\varDelta_0\left(\lambda\right)$ function λ_n^0 are discrete, so*

$$\inf_{n\neq k}\left|\lambda_n^0-\lambda_k^0\right|=r>0.$$

Lemma 4. *There is a constant $C_\delta>0$ so that the inequality*

$$\left|\varDelta_0\left(\lambda\right)\right|\geq C_\delta\left|\lambda\right|e^{\left|\operatorname{Im}\lambda\right|\pi},\quad \lambda\in G_\delta,$$

is satisfied.

Theorem 1. *When $\lambda_n, n = 1, 2, ...$ eigenvalues of problem (1)-(3) are $n \to \infty$,*

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + o\left(\frac{1}{\lambda_n^0}\right), \qquad (7)$$

has behavior, where

$$d_n = \frac{1}{\dot{\Delta}_0\left(\lambda_n^0\right)} \left\{ \left(\omega_0\left(\pi\right) + H\right) \sin \lambda_n^0 \pi + \left(\frac{H}{2}\beta - \omega_1\left(\pi\right)\right) \cos \lambda_n^0 \pi + \right.$$
$$\left. + \left(\omega_2\left(\pi\right) - \frac{H}{2}\beta\right) \cos \lambda_n^0 (2a - \pi) + \omega_3\left(\pi\right) \sin \lambda_n^0 (2a - \pi) \right\}$$

is the bounded sequence. Where $\dot{\Delta}\left(\lambda_n^0\right) = \left[\frac{d}{d\lambda}\Delta_0\left(\lambda\right)\right]_{\lambda = \lambda_n^0}$.

Proof. It is clear from the definition given above that the problem (1)-(3) is equivalent to the problem (4)-(5), (2)-(3), that is, each solution of the problem (1)-(3) is equivalent to the solution of the problem (4) satisfying the (2),(3) boundary and (5) discontinuity conditions. Let us denote the problem of seeking the solution of (4) equation satisfying (2)-(3) boundary conditions and (5) discontinuity condition with L. By applying the method in the study [17], we obtain the solution of the problem L that satisfies the initial conditions for (6), while $|\lambda| \to \infty$, according to the x variable,

$$\varphi\left(x, \lambda\right) = \cos \lambda x + \left(h + \frac{1}{2}\int_0^x q(t)dt\right) \frac{\sin \lambda x}{\lambda} + o\left(\frac{\exp\left(|\tau|\, x\right)}{|\lambda|}\right), \qquad (8)$$

$$\varphi'\left(x, \lambda\right) = -\lambda \sin \lambda x + \left(h + \frac{1}{2}\int_0^x q(t)dt\right) \cos \lambda x + o\left(\exp\left(|\tau|\, x\right)\right),$$

in the case of $x < a$ and

$$\varphi\left(x, \lambda\right) = \cos \lambda x + \frac{1}{2}\beta\left(\sin \lambda x - \sin \lambda\left(2a - x\right)\right) + \omega_0\left(x\right) \frac{\sin \lambda x}{\lambda} + \omega_1\left(x\right) \frac{\cos \lambda x}{\lambda}$$
$$+ \left(1 - 2\beta^2\right) \int_0^a q(t)dt \frac{\sin \lambda(2a - x)}{4\lambda} + \omega_2\left(x\right) \frac{\cos \lambda(2a - x)}{2\lambda} + o\left(\frac{\exp\left(|\tau|\, x\right)}{|\lambda|}\right), \qquad (9)$$

$$\varphi'\left(x, \lambda\right) = \lambda \left[-\sin \lambda x + \frac{1}{2}\beta\left(\cos \lambda x + \cos \lambda\left(2a - x\right)\right)\right] + \omega_0\left(x\right) \cos \lambda x - \omega_1\left(x\right) \sin \lambda x$$
$$+\omega_2\left(x\right) \sin \lambda(2a - x) - \frac{1}{4}\left(1 - 2\beta^2\right) \int_0^a q(t)dt \cos \lambda(2a - x) + o\left(\exp\left(|\tau|\, x\right)\right)$$

in the case of $x > a$ are valid. Here

$$\omega_0\left(x\right) = h - \frac{1}{4}\beta \int_0^a q(t)dt + \frac{1}{2}\int_0^x q(t)dt, \quad \omega_1\left(x\right) = -\frac{1}{2}\beta\left(h - \frac{1}{2}\int_0^a q(t)dt + \int_0^x q(t)dt\right),$$

$$\omega_2(x)=\frac{1}{2}\beta\left(h-\frac{3}{2}\int_0^a q(t)dt+\int_0^x q(t)dt\right),\quad \omega_3(x)=\frac{1}{4}(1-\beta^2)\int_0^a q(t)dt.$$

In this case,

$$\Delta(\lambda)=\lambda\left[-\sin\lambda\pi+\frac{1}{2}\beta\left(\cos\lambda\pi+\cos\lambda\left(2a-\pi\right)\right)\right]+$$
$$+\left(H+\omega_0(\pi)\right)\cos\lambda\pi+\left(\frac{H}{2}\beta-\omega_1(\pi)\right)\sin\lambda\pi+ \tag{10}$$
$$+\left(-\frac{H}{2}\beta+\omega_2(\pi)\right)\sin\lambda\left(2a-\pi\right)-\omega_3(\pi)\cos\lambda\left(2a-\pi\right)+o\left(\exp\left(|\tau|\pi\right)\right)$$

is for $|\lambda|\to\infty$.

Let

$$\Delta_0(\lambda)=\lambda\left[-\sin\lambda\pi+\frac{1}{2}\beta\left(\cos\lambda\pi+\cos\lambda\left(2a-\pi\right)\right)\right]$$

be a function. Using the [12] study, for the roots of the equation $\Delta_0(\lambda)=0$,

$$\lambda_n^0=n+h_n,\quad \sup_n|h_n|=h<+\infty$$

we obtain the following equality.

If we use the method given in the study [12] for the characteristic equation $\Delta(\lambda)=0$, it is obtained from (10) that

$$\lambda_n=\lambda_n^0+o(1)$$

according to Rouche's theorem.

Denote $G_n:=\left\{\lambda:|\lambda|=\left|\lambda_n^0\right|+\delta/2\right\}$. On the other hand [16], since

$$\Delta(\lambda)-\Delta_0(\lambda)=O\left(\exp\left(|\mathrm{Im}\,\lambda|\pi\right)\right),\quad |\lambda|\to\infty,$$

for sufficiently large values of hand $\lambda\in G_n$, we get

$$|\Delta(\lambda)-\Delta_0(\lambda)|<\frac{1}{2}C_\delta\exp\left(|\mathrm{Im}\,\lambda|\pi\right).$$

Thus, for $\lambda\in G_n$,

$$|\Delta_0(\lambda)|\geq C_\delta|\lambda|\exp\left(|\mathrm{Im}\,\lambda|\pi\right)>\frac{1}{2}C_\delta|\lambda|\exp\left(|\mathrm{Im}\,\lambda|\pi\right)>|\Delta(\lambda)-\Delta_0(\lambda)|$$

such that n is sufficiently large natural number. It follows from that for sufficiently large values n, functions $\Delta_0(\lambda)$ and $\Delta_0(\lambda)+(\Delta(\lambda)-\Delta_0(\lambda))=\Delta(\lambda)$ have the same number of zeros counting multiplicities inside contour G_n according to Rouches theorem. So, they have the $(n+1)$ number of zeros $\lambda_0,\lambda_1,...,\lambda_n$. Analogously, it is shown by Rouche's teorem that for sufficiently large values of n, function $\Delta(\lambda)$ has a unique of zero λ_n inside each circle $C(\delta)=\left\{\lambda:\left|\lambda-\lambda_n^0\right|\leq\delta\right\}$. Since δ is orbitrary sufficiently small number, we must have

$$\lambda_n=\lambda_n^0+\varepsilon_n,\quad \varepsilon_n=o(1),\quad n\to\infty. \tag{11}$$

Since function $\Delta_0(\lambda)$ is type of "sine" [12, p. 119], the number $\gamma_\delta > 0$ exsists such that for all n, $\left|\dot{\Delta}_0\left(\lambda_n^0\right)\right| \geq \gamma_\delta > 0$. Since λ_n are zeros of $\Delta(\lambda)$, from (10) we get

$$\varepsilon_n = -\frac{1}{\lambda_n^0 \dot{\Delta}_0(\lambda_n^0)} \left\{ (\omega_0(\pi) + H) \sin\lambda_n^0\pi + \left(\frac{H}{2}\beta - \omega_1(\pi)\right)\cos\lambda_n^0\pi + \right. \tag{12}$$
$$\left. + \left(\omega_2(\pi) - \frac{H}{2}\beta\right)\cos\lambda_n^0(2a-\pi) + \omega_3(\pi)\sin\lambda_n^0(2a-\pi)\right\} + o\left(\frac{1}{\lambda_n^0}\right).$$

Substituting (12) into (11), we get (7). ◂

3. Inverse Nodal Problems

In this section, the solution of the nodal inverse problem for the diffusion operator with $p(x) = \beta\delta(x-a)$-Dirac delta potential and any of the set of nodal points dense in the interval $(0,\pi)$ of the constants β, h, H and $q(x)$ function, an algorithm for determining with the help of subsequence will be given. Such problems have been studied in studies of [3], [15], [21], [22] for the regular diffusion operator.

In the [4] investigate inverse nodal problems for energy-dependant p-Laplacian equations and of the study applies the Tikhonov regularization method to reconstruct potential functions by only using zeros of one eigenfunction and show that the space of the p-Laplacian operator is homeomorphic to the partition set of the space of nodal sequences.

The eigenfunctions of the boundary value problem (1)-(3) or (4), (2), (3), (5) have the form $y_n(x) = \varphi(x,\lambda_n)$. We note that $y_n(x)$ are real-valued functions. Substituting (7) into (8) and (9) we obtain the following asimptotic formulae for $n\to\infty$ uniformly in x:

$$\varphi_n(x) = \cos\lambda_n^0 x + \left\{-d_n x + h + \frac{1}{2}\int_0^x q(t)dt\right\}\frac{\sin\lambda_n^0 x}{\lambda_n^0} + o\left(\frac{\exp(|\tau|x)}{\lambda_n^0}\right), \quad x<a, \tag{13}$$

$$\varphi_n(x) = \left(1 - \frac{1}{2}\beta\sin 2a\lambda_n^0\right)\cos\lambda_n^0 x + \frac{1}{2}\beta\left(1+\cos 2a\lambda_n^0\right)\sin\lambda_n^0 x$$
$$+ \left[\omega_0(x) - d_n x - \frac{1}{4}\left(1-2\beta^2\right)\cos 2a\lambda_n^0\int_0^x q(t)dt + \left(\omega_2(x) - \frac{1}{2}\beta d_n x\right)\sin 2a\lambda_n^0\right]\frac{\sin\lambda_n^0 x}{\lambda_n^0}$$
$$+ \left[\omega_1(x) + \frac{1}{2}\beta d_n x + \frac{1}{4}(1-2\beta^2)\sin 2a\lambda_n^0\int_0^x q(t)dt + \left(\omega_2(x) - \frac{1}{2}\beta d_n x\right)\cos 2a\lambda_n^0\right]\frac{\cos\lambda_n^0 x}{\lambda_n^0}$$
$$+ o\left(\frac{\exp(|\tau|x)}{\lambda_n^0}\right), \quad x>a. \tag{14}$$

Denote

$$A_n = 1 - \frac{1}{2}\beta\sin 2a\lambda_n^0, B_n = \frac{1}{2}\beta\left(1+\cos 2a\lambda_n^0\right),$$

$$C_n(x) = \omega_0(x) - d_n x - \frac{1}{4}\left(1-2\beta^2\right)\cos 2a\lambda_n^0 \int_0^x q(t)dt + \left(\omega_2(x) - \frac{1}{2}\beta d_n x\right)\sin 2a\lambda_n^0,$$

$$D_n(x) = \omega_1(x) + \frac{1}{2}\beta d_n x + \frac{1}{4}(1-2\beta^2)\sin 2a\lambda_n^0 \int_0^x q(t)dt + \left(\omega_2(x) - \frac{1}{2}\beta d_n x\right)\cos 2a\lambda_n^0,$$

in this case for $x > a$

$$\varphi_n(x) = A_n \cos\lambda_n^0 x + B_n \sin\lambda_n^0 x +$$
$$+ C_n(x)\frac{\sin\lambda_n^0 x}{\lambda_n^0} + D_n(x)\frac{\cos\lambda_n^0 x}{\lambda_n^0} + o\left(\frac{\exp(|\tau| x)}{\lambda_n^0}\right).$$

Theorem 2. *The following asymptotics expression is provided*

$$x_n^{j(n)} = \frac{\left(j-\frac{1}{2}\right)\pi}{\lambda_n^o} + \left\{h + \frac{1}{2}\int_0^{x_n^{j(n)}} q(t)dt - d_n x_n^{j(n)}\right\}\frac{1}{\left(\lambda_n^o\right)^2} + \tag{15}$$
$$+ o\left(\frac{1}{\left(\lambda_n^o\right)^2}\right), \quad x_n^{j(n)} \in (0, a),$$

$$x_n^{j(n)} = \frac{\left(j-\frac{1}{2}\right)\pi}{\lambda_n^o} + \frac{1}{\lambda_n^o}\arctan\left(\frac{B_n}{A_n}\right) - \tag{16}$$
$$- \left\{\frac{B_n}{A_n^2} D_n\left(x_n^{j(n)}\right) - \frac{C_n\left(x_n^{j(n)}\right)}{A_n}\right\}\frac{1}{\left(\lambda_n^o\right)^2} + o\left(\frac{1}{\left(\lambda_n^o\right)^2}\right), \quad x_n^{j(n)} \in (a, \pi),$$

for sufficiently large n, uniformly with respect to $j(n)$.

Proof. Use the asymptotic formulas for the case $x < a$ and $x > a$ respectively (13) and (14) to get

$$0 = \varphi\left(x_n^{j(n)}, \lambda_n\right) = \cos\lambda_n^o x_n^{j(n)} + \left\{h + \frac{1}{2}\int_0^{x_n^{j(n)}} q(t)dt - d_n x_n^{j(n)}\right\}\frac{\sin\lambda_n^o x_n^{j(n)}}{\lambda_n^o} +$$
$$+ o\left(\frac{\exp\left(|\tau| x_n^{j(n)}\right)}{\lambda_n^0}\right), \quad x_n^{j(n)} \in (0, a),$$

$$0 = \varphi\left(x_n^{j(n)}, \lambda_n\right) = A_n \cos\lambda_n^0 x_n^{j(n)} + B_n \sin\lambda_n^0 x_n^{j(n)} + C_n(x_n^{j(n)})\frac{\sin\lambda_n^0 x_n^{j(n)}}{\lambda_n^0} +$$
$$+ D_n\left(x_n^{j(n)}\right)\frac{\cos\lambda_n^0 x_n^{j(n)}}{\lambda_n^0} + o\left(\frac{\exp\left(|\tau| x_n^{j(n)}\right)}{\lambda_n^0}\right), \quad x_n^{j(n)} \in (a, \pi),$$

and so

$$\tan\left(\lambda_n^o x_n^{j(n)}+\frac{\pi}{2}\right)=\left\{h+\frac{1}{2}\int\limits_0^{x_n^{j(n)}} q(t)dt-d_n x_n^{j(n)}\right\}\frac{1}{\lambda_n^o}+o\left(\frac{1}{\lambda_n^0}\right), x_n^{j(n)}\in(0,a)\,, \quad (17)$$

$$\tan\left(\lambda_n^o x_n^{j(n)}+\frac{\pi}{2}\right)=\frac{B_n}{A_n}-\left\{\frac{B_n}{(A_n)^2}D_n\left(x_n^{j(n)}\right)-\frac{1}{A_n}C_n(x_n^{j(n)})\right\}\frac{1}{\lambda_n^o}+ +o\left(\frac{1}{\lambda_n^0}\right),\quad x_n^{j(n)}\in(a,\pi)\,, \quad (18)$$

if we apply the identity

$$\arctan\alpha-\arctan\beta=\arcsin\left(\frac{|\alpha-\beta|}{\sqrt{1+\alpha^2}\sqrt{1+\beta^2}}\right),$$

we get the asymptotic formulae (15) and (16) for the nodal points from the equations (17) and (18). ◀

It is clear from the expression of $\{\lambda_n^o\}_{n\geq 1}$ that $\{h_n\}_{n\geq 1}$ is a real and bounded sequence. Since $\sup\limits_n |h_n| \leq M < +\infty$, let's choose subsequence $\{n_k\}_{k\geq 0}\subset\mathbb{N}$ as $\lim\limits_{k\to\infty} h_{n_k}=h_o<+\infty$. Let's define the set $\Re=\left\{r\pi : r=\dfrac{p}{q}, p<q, p,q\in\mathbb{N}\right\}$. It is clear that the set $\Re$ is dense in the range $(0,\pi)$ and consists of irrational numbers in the form $r\pi, r\in(0,1)\cap\mathbb{Q}$, in this range.

Let's take any point $a\in\Re\subset(0,\pi)$ and choose the sequence $\{n_k\}_{k\geq 0}$ with $n_k = qm_k, \left(m_k\in\mathbb{N}, \lim\limits_{k\to\infty} m_k=+\infty\right)$. In this case, since $\sin 2a\lambda_{n_k}^o=\sin 2ah_{n_k}$, $\cos 2a\lambda_{n_k}^o = \cos 2ah_{n_k}$ and

$$\frac{1}{\lambda_{n_k}^o}=\frac{1}{n_k}-\frac{h_{n_k}}{(n_k)^2}+o\left(\frac{1}{(n_k)^2}\right),$$

we get following asymptotic formulae for the nodal points of the problem L, for $k\to\infty$ uniformly in $j(n_k)$:

$$x_{n_k}^{j(n_k)}=\frac{\left(j(n_k)-\frac{1}{2}\right)\pi}{n_k}-\frac{\left(j(n_k)-\frac{1}{2}\right)\pi}{(n_k)^2}h_{n_k}+ +\left[h+Q\left(x_{n_k}^{j(n_k)}\right)-d_{n_k}x_{n_k}^{j(n_k)}\right]\frac{1}{(n_k)^2}+o\left(\frac{1}{(n_k)^2}\right),\quad x_{n_k}^{j(n_k)}\in(0,a)\,,$$

$$x_{n_k}^{j(n_k)}=\frac{\left(j(n_k)-\frac{1}{2}\right)\pi}{n_k}-\frac{\left(j(n_k)-\frac{1}{2}\right)\pi}{(n_k)^2}h_{n_k}+\frac{1}{n_k}\arctan\left(\frac{B_{n_k}}{A_{n_k}}\right)- -\frac{h_{n_k}}{(n_k)^2}\arctan\left(\frac{B_{n_k}}{A_{n_k}}\right)-\left\{\frac{B_{n_k}}{(A_{n_k})^2}D_{n_k}\left(x_{n_k}^{j(n_k)}\right)-\frac{1}{A_{n_k}}C_{n_k}\left(x_{n_k}^{j(n_k)}\right)\right\}\frac{1}{(n_k)^2}+$$

$$+o\left(\frac{1}{(n_k)^2}\right), \quad x_{n_k}^{j(n_k)} \in (a,\pi).$$

Let $X_o(L) = \left\{x_{n_k}^{j(n_k)} : n_k = 1,2,...,j(n_k) = \overline{1,n_k}\right\}$ be a subsequence of the numbers $x_n^{j(n)}$ that is dense on $(0,\pi)$. According to above result, the existence of such set is obvious.

Theorem 3. *For $x \in (0,\pi)$, let $X_o(L) \subset X(L)$ and $\lim\limits_{k\to\infty} x_{n_k}^{j(n_k)} = x$. Then, for any point $a \in \Re$ the following limits exist and are finite:*

$$f_1(x) := \lim_{k\to\infty}\left[n_k x_{n_k}^{j(n_k)} - \left(j(n_k) - \frac{1}{2}\right)\pi\right], \quad x_{n_k}^{j(n_k)} \in (0,a), \tag{19}$$

$$g_1(x) := \lim_{k\to\infty} n_k \left[n_k x_{n_k}^{j(n_k)} - \left(j(n_k) - \frac{1}{2}\right)\pi + \frac{\left(j(n_k) - \frac{1}{2}\right)\pi}{n_k}\right], \tag{20}$$

$x_{n_k}^{j(n_k)} \in (0,a)$

$$f_2(x) := \lim_{k\to\infty}\left[n_k x_{n_k}^{j(n_k)} - \left(j(n_k) - \frac{1}{2}\right)\pi\right], x_{n_k}^{j(n_k)} \in (a,\pi), \tag{21}$$

$$\begin{aligned} g_2(x) := \lim_{k\to\infty} n_k \Big[& n_k x_{n_k}^{j(n_k)} - \left(j(n_k) - \frac{1}{2}\right)\pi - \arctan\left(\frac{B_{n_k}}{A_{n_k}}\right) + \\ & + \frac{\left(j(n_k) - \frac{1}{2}\right)\pi}{n_k} h_{n_k}\Big], \quad x_{n_k}^{j(n_k)} \in (a,\pi), \end{aligned} \tag{22}$$

and

$$f_1(x) = -xh_o, \quad x \in [0,a),$$

$$g_1(x) = h + \frac{1}{2}\int_0^x q(t)dt - d_o x, \quad x \in [0,a),$$

$$f_2(x) = -xh_o + \arctan\left(\frac{B_o}{A_o}\right), \quad x \in (a,\pi],$$

$$g_2(x) = -h_o \arctan\left(\frac{B_o}{A_o}\right) - \frac{B_o}{A_o^2} D_o(x) + \frac{1}{A_o} C_o(x,), \quad x \in (a,\pi],$$

where

$$A_o = \lim_{k\to\infty} A_{n_k} = 1 - \frac{1}{2}\beta\sin 2ah_o, B_o = \lim_{k\to\infty} B_{n_k} = \frac{1}{2}\beta\left(1 + \cos 2ah_o\right),$$

$$C_o(x) = \lim_{k\to\infty} C_{n_k}\left(x_{n_k}^{j(n_k)}\right) = \omega_o(x) - d_o x - \frac{1}{4}(1 - 2\beta^2)\cos 2ah_o \int_0^x q(t)dt +$$

$$+\left(\omega_2(x)-\frac{1}{2}\beta d_o x\right)\sin 2ah_o,$$

$$D_o(x)=\lim_{k\to\infty}D_{n_k}\left(x_{n_k}^{j(n_k)}\right)=\omega_1(x)+\frac{1}{2}\beta d_o x+$$

$$+\frac{1}{4}(1-2\beta^2)\sin 2ah_o\int_0^x q(t)dt+\left(\omega_2(x)-\frac{1}{2}\beta d_o x\right)\cos 2ah_o,$$

$$d_o=\frac{1}{\pi\cos\pi h_o+\frac{1}{2}\beta\left[\pi\sin\pi h_o+(2a-\pi)\sin(2a-\pi)h_o\right]}\left\{(\omega_o(\pi)+H)\sin\pi h_o+\right.$$

$$\left.+\left(\frac{1}{2}\beta H-\omega_1(\pi)\right)\cos\pi h_o+\left(\omega_2(\pi)-\frac{1}{2}\beta H\right)\cos(2a-\pi)h_o+\omega_3(\pi)\sin(2a-\pi)h_o\right\}.$$

Proof. Let $a\in\Re\subset(0,\pi)$ any point. For each fixed $x\in(0,\pi)\setminus\{a\}$, there exists a sequence $\left(x_n^{j(n)}\right)_{n\geq 1}$ converges to x. For $n_k=q.m_k$, $m_k\in\mathbb{N}$, the subsequence $\left(x_{n_k}^{j(n_k)}\right)_{n\geq 1}$ converges also to x. Therefore we get from the asymptotics in Theorem 2 (15) and (16), the limits (19)-(21) exists and are finite. ◀

Let us now state a uniqueness theorem and present a constructive procedure for solving inverse nodal problem.

Theorem 4. *Let $X_o(L)\subset X(L)$ be a subset of nodal points which is dense in $(0,\pi)$. Then, for any $a\in\Re$ the specification of $X_o(L)$ uniquely determines the potential $q(x)-\langle q\rangle$ a.e. on $(0,\pi)$ and the coefficients h and H of the boundary conditions and coefficient β. The potential $q(x)-\langle q\rangle$ and the numbers h,H and β can be constructed via the following algorithm.*

1. For each $x\in(0,\pi)$, we choose a sequence $\left\{x_{n_k}^{j(n_k)}\right\}\subset X_o(L)$ such that $\lim\limits_{n\to\infty}x_n^{j(n)}=x$.

2.From (20), we find the function $g_1(x)$ and calculate value for $g_1(x)$ at $x=0$, i.e.

$$h=g_1(0). \tag{23}$$

3. From (19) and (21) we find

$$\beta=\frac{\tan(f(a+0)-f(a-0))}{2\left[\cos ah_o+\tan(f(a+0)-f(a-0))\sin ah_o\right]\cos ah_o}, \tag{24}$$

where

$$f(x)=\begin{cases}f_1(x),\ x\in[0,a),\\ f_2(x),\ x\in(a,\pi].\end{cases}$$

4. The function $q(x)-\langle q\rangle$ can be determined as

$$q(x)-\langle q\rangle=2g_1'(x)+$$

$$+\frac{2(H+h)\left[\sin \pi h_o + +\frac{1}{2}\beta \cos \pi h_o\right] - \frac{1}{2}\beta\left(H-h\right)\cos(2a-\pi)h_o}{\pi\left[\cos \pi h_o + \frac{1}{2}\beta \sin \pi h_o\right] + \frac{1}{2}\beta(2a-\pi)\sin(2a-\pi)h_o}, x \in [0,a)\,, \qquad (25)$$

where

$$\langle q\rangle = \frac{1}{\pi\left[\cos \pi h_o + \frac{1}{2}\beta \sin \pi h_o\right] + \frac{1}{2}\beta(2a-\pi)\sin(2a-\pi)h_o}\times$$

$$\times\left\{\left[-\frac{1}{2}\beta\left(\sin \pi h_o + \cos \pi h_o\right.\right.\right. +$$

$$\left.\left.+\frac{\beta}{2}\cos(2a-\pi)h_o\right) + \frac{1}{2}\left(1-\beta^2\right)\sin(2a-\pi)h_o\right]\int\limits_0^a q(t)dt +$$

$$\left. + \left[\sin \pi h_o + \beta\cos \pi h_o + \frac{1}{2}\beta\cos(2a-\pi)h_o\right]\int\limits_0^\pi q(t)dt\right\},$$

$$\begin{aligned} q(x) - \langle q\rangle = {} & \frac{1}{1+\frac{1}{2}\beta\sin 2ah_o - \left(\frac{1}{2}-\beta^2\right)\left(\cos 2ah_o + \frac{1}{2}\beta \sin 2ah_o\right)}\times \\ & \times\left\{2\left(1-\frac{1}{2}\beta\sin 2ah_o\right)^2 g_2'(x) - \right. \\ & \left.-\frac{2(H+h)\left[\sin \pi h_o + \frac{1}{2}\beta\cos \pi h_o\right] - \frac{1}{2}\beta\left(H-h\right)\cos(2a-\pi)h_o}{\pi\left[\cos \pi h_o + \frac{1}{2}\beta \sin \pi h_o\right] + \frac{1}{2}\beta(2a-\pi)\sin(2a-\pi)h_o}\right\}, x\in(a,\pi]\,, \end{aligned} \qquad (26)$$

where

$$\langle q\rangle = \frac{\left[1+\frac{1}{2}\beta\sin 2ah_o - \left(\frac{1}{2}-\beta^2\right)\left(\cos 2ah_o + \frac{1}{2}\beta\sin 2ah_o\right)\right]^{-1}}{\pi\left[\cos \pi h_o + \frac{1}{2}\beta\sin \pi h_o\right] + \frac{1}{2}\beta(2a-\pi)\sin(2a-\pi)h_o}\times$$

$$\times\left\{\left[-\frac{1}{2}\beta(\sin \pi h_o + \cos \pi h_o\right.\right. +$$

$$\left. + \frac{\beta}{2}\cos(2a-\pi)h_o) + \frac{1}{2}\left(1-\beta^2\right)\sin(2a-\pi)h_o\right]\int\limits_0^a q(t)dt +$$

$$\left. + \left[\sin \pi h_o + \beta\cos \pi h_o + \frac{1}{2}\beta\cos(2a-\pi)h_o\right]\int\limits_0^\pi q(t)dt\right\}.$$

Proof. Formulas (23), (24), (25) and (26) can be derived from (19), (20), (21) and (22) step by step. We obtain the following reconstruction procedure:

i) Taking value for $g_1(x)$ at $x=0$, then it yields $h=g_1(0)$.

ii) Using the expression of the $f(x)$ function, the coefficient of β is found with the formula (7).

iii) After hand β are reconstructed on take derivatives of the functions $g_i(x)$, $(i=1,2)$, we have (25) and (26). ◀

Let the function $\psi(x,\lambda)$ be the solution of (4) under the initial conditions $\psi(\pi,\lambda)=1, \psi'(\pi,\lambda)=-H$, and discontinouty conditions (5). İt is clear that $\psi(x,\lambda_n)=\beta_n\varphi(x,\lambda_n)$, where $\beta_n=\psi'(0,\lambda_n)$.

To complete the proof, consider a sequence $\left\{x_n^{j(n)}\right\}\subset X_o(L)$ that converges to π and write equation (4) for $\psi(x,\lambda_n)$ and $\widetilde{\psi}\left(x,\widetilde{\lambda}_n\right)$ as follows

$$-\widetilde{\psi}''\left(x,\widetilde{\lambda}_n\right)+q(x)\widetilde{\psi}\left(x,\widetilde{\lambda}_n\right)=\widetilde{\lambda}_n\widetilde{\psi}\left(x,\widetilde{\lambda}_n\right),$$

$$-\psi''(x,\lambda_n)+q(x)\psi(x,\lambda_n)=\lambda_n\psi(x,\lambda_n).$$

If these equations are (i): Multiplied by $\psi(x,\lambda_n)$ and $\widetilde{\psi}\left(x,\widetilde{\lambda}_n\right)$, respectively; (ii): Subtracted from each other and (iii): Integrated over the interval $\left(x_n^{j(n)},\pi\right)$, the equality

$$\psi'(\pi,\lambda_n)\widetilde{\psi}\left(\pi,\widetilde{\lambda}_n\right)-\widetilde{\psi}^{!}\left(x,\widetilde{\lambda}_n\right)\psi(\pi,\lambda_n)=\left(\lambda_n-\widetilde{\lambda}_n\right)\int_{x_n^{j(n)}}^{\pi}\widetilde{\psi}\left(x,\widetilde{\lambda}_n\right)\psi(x,\lambda_n)\,dx$$

is obtained. Using (8), we get the following estimate for sufficiently large n

$$H-\widetilde{H}=\left[2\left(d_n-\widetilde{d}_n\right)+o(1)\right]\int_{x_n^{j(n)}}^{\pi}\widetilde{\psi}\left(x,\widetilde{\lambda}_n\right)\psi(x,\lambda_n)\,dx.$$

Since the sequences (d_n) and $\left(\widetilde{d}_n\right)$ are bounded, then $H=\widetilde{H}$. This completes the proof.

Corollary 1. *Let* $a=\frac{\pi}{2}$ *and* $H=\infty$. *Then* $h_o=-\frac{\alpha}{\pi}=-\frac{1}{\pi}\arctan\left(\frac{2}{\beta}\right)$. *In this case, we get the following equalities:*

$$1.\ \ x_n^j=\frac{\left(j-\frac{1}{2}\right)\pi}{n}+\frac{\alpha}{\pi n}\frac{\left(j-\frac{1}{2}\right)\pi}{n}+\left\{-c_0x_n^j+h+\frac{1}{2}\int_0^{x_n^j}q(t)dt\right\}\frac{1}{n^2}+o\left(\frac{1}{n^2}\right),$$
$$x_n^j\in\left(0,\frac{\pi}{2}\right),\quad n=2m,$$

$$x_n^j = \frac{\left(j-\frac{1}{2}\right)\pi}{n} + \frac{\alpha}{\pi n}\frac{\left(j-\frac{1}{2}\right)\pi}{n} + \left\{-c_1 x_n^j + h + \frac{1}{2}\int_0^{x_n^j} q(t)dt\right\}\frac{1}{n^2} + o\left(\frac{1}{n^2}\right),$$
$$x_n^j \in \left(0, \frac{\pi}{2}\right), \quad n = 2m-1,$$

$$x_n^j = \frac{\left(j-\frac{1}{2}\right)\pi}{n} + \frac{\alpha\left(j-\frac{1}{2}\right)\pi}{\pi n^2} + \frac{1}{n}\arctan\left(\frac{1}{2}\beta\right) + \frac{\alpha\arctan\left(\frac{1}{2}\beta\right)}{\pi n^2} -$$
$$-\frac{\frac{1}{2}\beta B^{(0)}\left(x_n^j\right) - A^{(0)}\left(x_n^j\right)}{n^2} + o\left(\frac{1}{n^2}\right), \quad x_n^j \in \left(\frac{\pi}{2}, \pi\right), \quad n = 2m,$$

$$x_n^j = \frac{\left(j-\frac{1}{2}\right)\pi}{n} + \frac{\alpha\left(j-\frac{1}{2}\right)\pi}{\pi n^2} + \frac{1}{n}\arctan\left(\frac{1}{2}\beta\right) + \frac{\alpha\arctan\left(\frac{1}{2}\beta\right)}{\pi n^2} -$$
$$-\frac{\frac{1}{2}\beta B^{(1)}\left(x_n^j\right) - A^{(1)}\left(x_n^j\right)}{n^2} + o\left(\frac{1}{n^2}\right), \quad x_n^j \in \left(\frac{\pi}{2}, \pi\right), \quad n = 2m-1,$$

where

$$A^{(t)}(x) = \frac{1}{A^{(t)}}\left\{\omega_0(x) - xc_t - (-1)^t \sin\alpha\left(\omega_2(x) - \frac{\pi - x}{2}\beta c_t\right) - (-1)^t \omega_3 \cos\alpha\right\},$$

$$B^{(t)}(x) = \frac{1}{A^{(t)}}\left\{\omega_1(x) - \frac{x}{2}\beta c_t + (-1)^t \cos\alpha\left(\omega_2(x) - \frac{\pi - x}{2}\beta c_t\right) - (-1)^t \omega_3 \sin\alpha\right\},$$

$$c_t = \frac{1}{\pi\sqrt{1+\left(\frac{1}{2}\beta\right)^2}}\left\{\frac{\beta\omega_1(\pi) - 2\omega_0(\pi)}{2\sqrt{1+\left(\frac{1}{2}\beta\right)^2}} + (-1)^t \omega_2(\pi)\right\},$$

$$A^{(t)} = 1 + (-1)^t\cos\alpha = 1 + \frac{(-1)^t\beta}{2\sqrt{1+\left(\frac{1}{2}\beta\right)^2}}, \quad t = 0, 1.$$

2. For $x < \frac{\pi}{2}$

$$\lim_{n\to\infty}\left(nx_n^j - \left(j-\frac{1}{2}\right)\pi\right) \overset{def}{=} f_1(x),$$

$$\lim_{n\to\infty}\left[nx_n^j - \left(j-\frac{1}{2}\right)\pi - \frac{\alpha}{\pi}\frac{\left(j-\frac{1}{2}\right)\pi}{n}\right] \overset{def}{=} g_1^t(x),$$

for $x > \frac{\pi}{2}$

$$\lim_{n\to\infty}\left(nx_n^j - \left(j-\frac{1}{2}\right)\pi\right) \overset{def}{=} f_2(x),$$

$$\lim_{n\to\infty}\left[nx_n^j-\left(j-\frac{1}{2}\right)\pi-\frac{\alpha\left(j-\frac{1}{2}\right)\pi}{\pi n}-\arctan\left(\frac{1}{2}\beta\right)\right]n\overset{def}{=}g_2^t\left(x\right),$$

and

$$f_1\left(x\right)=\frac{\alpha}{\pi}x,\quad x\in\left[0,\frac{\pi}{2}\right),\tag{27}$$

$$g_1^t\left(x\right)=-c_tx+h+\frac{1}{2}\int\limits_0^x q(t)dt,\ x\in\left[0,\frac{\pi}{2}\right),\tag{28}$$

$$f_2\left(x\right)=\frac{\alpha}{\pi}x+\arctan\left(\frac{1}{2}\beta\right),\quad x\in\left(\frac{\pi}{2},\pi\right],$$

$$g_2^t\left(x\right)=\frac{\alpha}{\pi}\arctan\left(\frac{1}{2}\beta\right)+A^{(t)}\left(x\right)-\frac{1}{2}\beta B^t\left(x\right),\quad x\in\left(\frac{\pi}{2},\pi\right].\tag{29}$$

3.From (28), we find the function $g_1^t\left(x\right)$ and calculate value for $g_1^t\left(x\right)$ at $x=0$, i.e.

$$h=g_1^t\left(0\right).$$

From (27), we find the function $f_1\left(x\right)$ and calculate value for $f_1\left(x\right)$ at $x=1$, i.e.

$$\beta=\frac{2}{\tan\left(\pi f_1\left(1\right)\right)}.$$

4. From (28) and (29). The function $q(x)-\langle q^t\rangle$ can be determined as

$$q(x)-\left\langle q^t\right\rangle=2\left(g_1^t\left(x\right)\right)'+\frac{g_1^t\left(0\right)}{\pi\left[1+2\left(\frac{1}{2}\beta\right)^2\right]}\left[\left(-1\right)^t\beta-\sqrt{1+\left(\frac{1}{2}\beta\right)^2}\right],\quad x\in\left[0,\frac{\pi}{2}\right),$$

where

$$\langle q^t\rangle=-\frac{2}{\pi\left[1+2\left(\frac{1}{2}\beta\right)^2\right]}\left\{\frac{1}{2}\beta\left[\left(\beta-1\right)-3\left(-1\right)^t\sqrt{1+\left(\frac{1}{2}\beta\right)^2}\right]\int\limits_0^a q(t)dt+\right.$$
$$\left.+\left[\frac{1}{2}+\left(\frac{1}{2}\beta\right)^2-\left(-1\right)^t\beta\sqrt{1+\left(\frac{1}{2}\beta\right)^2}\right]\int\limits_0^\pi q(t)dt\right\},$$

$$q(x)-\langle q^t\rangle=\frac{\left(2\sqrt{1+\left(\frac{1}{2}\beta\right)^2}+\left(-1\right)^t\beta\right)\left(g_2^t\left(x\right)\right)'}{\sqrt{1+\left(\frac{1}{2}\beta\right)^2}\left(1+2\left(\frac{1}{2}\beta\right)^2\right)-\left(-1\right)^t\beta\left(1+\left(\frac{1}{2}\beta\right)^2\right)}+$$
$$+\frac{g_1^t\left(0\right)}{\pi\left[1+2\left(\frac{1}{2}\beta\right)^2\right]}\left[\left(-1\right)^t\beta-\sqrt{1+\left(\frac{1}{2}\beta\right)^2}\right],\quad x\in\left(\frac{\pi}{2},\pi\right],$$

where

$$\langle q^t\rangle = -\frac{2}{\pi\left[1+2\left(\frac{1}{2}\beta\right)^2\right]}\frac{2\left[1+\left(\frac{1}{2}\beta\right)^2\right]+(-1)^t\beta\sqrt{1+\left(\frac{1}{2}\beta\right)^2}}{\left[1+2\left(\frac{1}{2}\beta\right)^2-(-1)^t\beta\sqrt{1+\left(\frac{1}{2}\beta\right)^2}\right]}\times$$
$$\times\left\{\frac{1}{2}\beta\left[(\beta-1)-3(-1)^t\sqrt{1+\left(\frac{1}{2}\beta\right)^2}\right]\int\limits_0^a q(t)dt+\right.$$
$$\left.+\left[\frac{1}{2}+\left(\frac{1}{2}\beta\right)^2-(-1)^t\beta\sqrt{1+\left(\frac{1}{2}\beta\right)^2}\right]\int\limits_0^\pi q(t)dt\right\}.$$

Corollary 2. *In the (1)-(3) problem, if the interval* $[0,1]$ *is taken instead of the interval* $[0,\pi]$*, it must be* $a\in(0,1)\cap\mathbb{Q}$ *for the inverse nodal problem to be solvable.*

Corollary 3. *In the (1)-(3) problem, if the interval* $[0,T]$ *is taken instead of the interval* $[0,\pi]$*, it must be* $\frac{a}{T}\in(0,1)\cap\mathbb{Q}$ *for the inverse nodal problem to be solvable.*

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