Baku Mathematical Journal
2024, Vol. 3, No 1, PP. 35-41
https://doi.org/10.32010/j.bmj.2024.04

ON THE UNIQUENESS OF INVERSE PROBLEMS FOR "WEIGHTED" STURM-LIOUVILLE OPERATOR WITH DELTA INTERACTION

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Received: 30.07.2023 / Revised: 17.11.2023 / Accepted: 29.11.2023

In memory of M. G. Gasymov on his 85th birthday

Abstract. *We establish various uniqueness result of inverse problems for "weighted" Sturm-Liouville operator with δ - interaction point.*

Keywords: Uniqueness results, "weigthed" Sturm-Liouville operator, point delta interaction.

Mathematics Subject Classification (2020): 34A55, 34B24, 34L05

1. Introduction

Consider the following Sturm-Liouville problem $L := L(q(x))$

$$\ell\,[y] := -y'' + q(x)y = \lambda y,\ x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right), \tag{1}$$

with the boundary conditions

$$U(y) := y'(0) = 0,\ V(y) := y(\pi) = 0, \tag{2}$$

and conditions at the point $x = \frac{\pi}{2}$,

$$I(y) := \begin{cases} y(\frac{\pi}{2}+0) = y(\frac{\pi}{2}-0) \equiv y(\frac{\pi}{2}), \\ y'(\frac{\pi}{2}+0) - y'(\frac{\pi}{2}-0) = -\alpha\lambda y(\frac{\pi}{2}), \end{cases} \tag{3}$$

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where $q(x)$ is real-value function in $W_2^1(0,\pi)$ and $\alpha > 0$; λ is spectral parameter. It is known [9] that the problem has a discrete spectrum consisting of simple real and bounded below eigenvalues.

Notice that, we can understand problem (1), (3) as studying the equation

$$-y'' + q(x)y = \lambda\rho(x)y,\ x \in (0,\pi)\,, \tag{4}$$

when $\rho(x) = 1 + \alpha\delta(x - \frac{\pi}{2})$, where $\delta(x)$ is the Dirac function (see [1]). In this aspect, various spectral problems for the equation (4) have been investigated in [7].

Here we will confine ourselves to the problem of recovering L from the given sets of spectral characteristics. This paper can be viewed as a continuation of [9], in which we gave a self-adjoint operator-theoretic formulation in an appropriate produce space $L_2[0,\pi] \oplus \mathbb{C}$ for this problem, obtained some properties of the spectrum and proved some uniqueness theorems for the inverse problems of L.

Boundary value problems with discontinuities within the interval often appear in mathematics, mechanics, physics, geophysics and other branches of naturel sciences. Usually,such problems involve discontinuous material properties. For example, the discontinuous Sturm-Liouville problem (1)-(3) appears in the one-dimensional wave equation corresponding to a string with finitely many embedded point masses (see [2]), Moreover, boundary valve problems with discontinuities in an interior point also appear in geophysical models for oscillations of the Earth (see [6]). Here the main discontinuity is coursed by reflection of the shear waves at the base of the cruck. The inverse problem of reconstructing the material properties of a medium from data collected outside of the medium is of central importance.

Inverse Sturm-Liuville problems with interior point conditions depending on the spectral parameter are less to investigate, and nowadays there are only a rather limited number of papers in this direction (see [5], [8], [10]-[12], and the references therein).

In the present paper, we prove some uniqueness theorems three inverse problems of recovering L from the Weyl function, or from spectral data, or from two spectra. The obtained results here are natural generalizations of the well-known results on the classical inverse problems since the interior point conditions (3) involve the spectral parameter.

2. Formulation of the Inverse Problem. Uniqueness Theorem

In this section, we give the spectral properties of the boundary value problem L and state the relationship between these spectral characteristics. We also formulate the inverse problem of reconstructing the problem L: from the Weyl function, from the spectral data, and from two spectra.

Let $\varphi(x,\lambda)$, $\psi(x,\lambda)$ be the solutions of equation (1) under the initial conditions

$$\varphi(0,\lambda) = 1,\ \varphi^{'}(0,\lambda) = 0;\ \psi(\pi,\lambda) = 0,\ \psi^{'}(\pi,\lambda) = 1, \tag{5}$$

and the interior point conditions (3) respectively. For each fixed $x \in \left(0,\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\pi\right)$ the functions $\varphi(x,\lambda)$, $\psi(x,\lambda)$ together with their derivatives with respect to x are entire in λ. For convenience, we denote $\rho = \sqrt{\lambda} = \sigma + i\tau$. It is known [9] that the following asymptotic estimates hold uniformly with respect $x \in (0,\pi)$, as $|\rho| \to \infty$

$$\begin{cases} \varphi(x,\lambda) = \cos\rho x + \frac{\sin\rho x}{2\rho}\int\limits_0^x q(t)dt + O(\frac{1}{\rho^2}\exp(|\tau|\, x)),\ x < \frac{\pi}{2}, \\ \varphi(x,\lambda) = -\frac{\alpha}{2}\rho\sin\rho x + \frac{\alpha}{2}\rho\sin\rho(\pi - x) + \cos\rho x[1 + \frac{\alpha}{4}\int\limits_0^x q(t)dt] \\ \qquad + \frac{\alpha}{4}\cos\rho(\pi - x)[\int\limits_{\pi/2}^{\pi} q(t)dt - \int\limits_0^{\pi/2} q(t)dt] + O(\frac{1}{\rho}\exp(|\tau|\, x)),\ x > \frac{\pi}{2}, \end{cases} \tag{6}$$

$$\begin{cases} \psi(x,\lambda) = -\frac{\sin\rho(\pi-x)}{\rho} + \frac{\cos\rho(\pi-x)}{2\rho^2}\int\limits_x^{\pi} q(t)dt + O(\frac{1}{\rho^3}\exp(|\tau|\,(\pi - x))),\ x > \frac{\pi}{2}, \\ \psi(x,\lambda) = -\frac{\alpha}{2}\cos\rho x - \frac{\alpha}{2}\cos\rho(\pi - x) - \frac{\alpha}{4\rho}[\sin\rho x + \sin\rho(\pi - x)]\int\limits_x^{\pi/2} q(t)dt] \\ \qquad + O(\frac{1}{\rho^2}\exp(|\tau|\,(\pi - x))),\ x < \frac{\pi}{2}. \end{cases} \tag{7}$$

Define the Wronskians determinant $< y, z > (x) := (yz' - y'z)(x)$ for the functions $y(x)$ and $z(x)$ which are all continuously differentiable on $(0,\pi)\backslash(\frac{\pi}{2})$. It is easy to verify from (3) and (5) that $< \varphi,\psi > (\frac{\pi}{2} - 0,\lambda) =< \varphi,\psi > (\frac{\pi}{2} + 0,\lambda)$, which implies that the Wronskians $< \varphi,\psi > (x,\lambda)$ is continuous on $(0,\pi)$. From this fact and by virtue of Liouville's formula (see [3, p. 83]), we infer that $< \varphi(x,\lambda),\psi(x,\lambda) >$ does not depend on x. Denote

$$\Delta(\lambda) =< \varphi(x,\lambda),\psi(x,\lambda) > . \tag{8}$$

Substituting $x = 0$ and $x = \pi$ into (8), one has $\Delta(\lambda) = -V(\varphi) = U(\psi)$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be the zeros of the function $\Delta(\lambda)$. Then the numbers $\{\lambda_n\}_{n=1}^{\infty}$ coincide with the eigenvalues of the problem L defined by (1)-(3) and therefore the function $\Delta(\lambda)$ is called the *characteristic function* of L. The functions $\varphi(x,\lambda_n)$, $\psi(x,\lambda_n)$ are eigenfunctions, and there exists a sequence $\{\beta_n\}$ such that

$$\psi(x,\lambda_n) = \beta_n\varphi(x,\lambda_n), \quad \beta \neq 0,$$

where $\beta_n = \psi(0,\lambda_n)$. Throughout this paper we use the notation

$$\alpha_n = \int\limits_0^{\pi} \varphi^2(x,\lambda_n)dx + \alpha\varphi^2\left(\frac{\pi}{2},\lambda_n\right),$$

as the *norming constant* corresponding to eigenvalue λ_n. The data $\Omega = \{\lambda_n,\alpha_n\}_{n=1}^{\infty}$ are called the spectral data associated with problem (1)-(3).

Lemma 1. *The following statements hold.*

i. *For each eigenvalue* λ_n, *we have*

$$\dot{\varDelta}(\lambda_n) = \alpha_n \beta_n,$$

where $\dot{\varDelta}(\lambda_n) = \frac{d\varDelta(\lambda)}{d\lambda}$.

ii. *Let* λ_n *be n th eigenvalue of the problem (1)-(3). Let* $\{\rho_n^0\}_{n=1}^{\infty}$ *be zeros of the entire function*

$$\varDelta_0(\lambda) = \tfrac{\alpha}{2}\rho \sin \rho\pi.$$

Then as $n \to \infty$

$$\rho_n = n - 1 + \frac{1}{\pi n}(\omega_1 + (-1)^{n-1}\omega_2) + \frac{\xi_n}{n},\ \xi_n \in \ell_2,$$

where $\omega_1 = \frac{2}{\alpha} + \frac{1}{2}\int\limits_0^{\pi} q(t)dt$, $\omega_2 = \frac{1}{2}[\int\limits_0^{\pi/2} q(t)dt - \int\limits_{\pi/2}^{\pi} q(t)dt]$, *(see [2]).*

iii. *Fix* $\delta > 0$. *Then there exists a constant* C_δ *such that*

$$|\varDelta(\lambda)| > C_\delta \left|\rho\right|^2 \exp(|\tau|\,\pi),\ \rho \in G_\delta,\ |\rho| \geq \rho^*, \tag{9}$$

for sufficiently large ρ^*, *where* $G_\delta = \{\rho : \left|\rho - \rho_n^0\right| \geq \delta, n \geq 1\}$.

iv. *The equality*

$$\alpha_n = \alpha_n^0 + o(1)\ ,\ n \to \infty$$

holds. Here

$$\alpha_n^0 = \int\limits_0^{\pi} \varphi^2(x, \lambda_n^0)dx + \alpha\varphi^2(\tfrac{\pi}{2}, \lambda_n^0).$$

We define the Weyl function by

$$M(\lambda) = \frac{\psi(0,\lambda)}{\varDelta(\lambda)}, \tag{10}$$

Here the function $\psi(0,\lambda)$ is the characteristic function of the boundary value problem for equation (1) subject to the boundary conditions $y(0) = V(y) = 0$ and the point conditions (3). Let $\{\mu_n\}_{n=1}^{\infty}$ be the zeros of the entire function $\psi(0,\lambda)$. Obviously, $\psi(0,\lambda)$ and $\varDelta(\lambda)$ have no common zeros. Thus, the Weyl function $M(y)$ is meromophic with in $\{\lambda_n\}_{n=1}^{\infty}$ and zeros in $\{\mu_n\}_{n=1}^{\infty}$.

The following lemma provides the relationship among the spectral characteristics of L: the Weyl function $M(\lambda)$, the spectral data $\varOmega$ and two spectra $\{\lambda_n, \mu_n\}_{n=1}^{\infty}$.

Lemma 2. *Let* $M(\lambda)$, $\varOmega$ *and* $\varDelta(\lambda)$ *be defined as above. Then the following representation holds*

$$M(\lambda) = \sum_{n=1}^{\infty} \frac{1}{\alpha_n(\lambda - \lambda_n)}. \tag{11}$$

Moreover, if condition (9) holds, then

$$\Delta(\lambda) = -\pi\alpha(\lambda_1 - \lambda)\prod_{n=2}^{\infty}\frac{\lambda_n - \lambda}{\rho_n}. \tag{12}$$

We omit the proofs of Lemmas 1-2 since they are similar to those for the classical Sturm-Liouville operators (see [4]).

We shall consider the following inverse problems of recovering L:
(i) - Inverse problem 1: Give the Weyl function $M(\lambda)$, construct $q(x)$.
(ii) - Inverse problem 2: Given the spectral data $\{\lambda_n, \mu_n\}_{n=1}^{\infty}$, construct $q(x)$.
(iii) - Inverse problem 3: Suppose (9) holds, given two spectra $\{\lambda_n, \mu_n\}_{n=1}^{\infty}$, construct $q(x)$.

This gives us the desired uniqueness result for the solutions of the inverse problems above. Moreover, according to (10), (11) and (12), the inverse problems of recovering the L from the spectral data and from two spectra are particular inverse problem of recovering the L from Weyl function. Consequently, inverse problems 1-3 are equivalent.

First, let us prove the uniqueness theorems for the solutions of problems (i)-(iii). For this purpose, we agree to consider together with L a boundary value problem $\widetilde{L}$ of the same of, but with different coefficient $\widetilde{q}(x)$.). If in the following a certain symbol e denotes an object related to L, then the corresponding symbol $\widetilde{e}$ with tilde denotes the analogous object related to $\widetilde{L}$.

Theorem 1. *If $M(\lambda) = \widetilde{M}(\lambda)$, then $L = \widetilde{L}$. Thus, the specification of the Weyl function $M(\lambda)$ uniquely determines L.*

Proof. Let us define the matrix $P(x,\lambda) = [P_{jk}(x,\lambda)]_{j,k=1,2}$ by the formula

$$P(x,\lambda)\begin{pmatrix}\widetilde{\varphi}(x,\lambda) & \widetilde{\Phi}(x,\lambda)\\ \widetilde{\varphi}'(x,\lambda) & \widetilde{\Phi}(x,\lambda)'\end{pmatrix} = \begin{pmatrix}\widetilde{\varphi}(x,\lambda) & \widetilde{\Phi}(x,\lambda)\\ \widetilde{\varphi}'(x,\lambda) & \widetilde{\Phi}'(x,\lambda)\end{pmatrix},$$

where

$$\Phi(x,\lambda) = \frac{\psi(x,\lambda)}{\Delta(x,\lambda)} = S(x,\lambda) + M(\lambda)\varphi(x,\lambda), \tag{13}$$

and $S(x,\lambda)$ is the solution of equation (1) with the initial conditions $S(x,\lambda) = 0$, $S'(0,\lambda) = 1$ and the point conditions (3). The function $\Phi(x,\lambda)$ is called the Weyl solution for L. By virtue of (13), we calculate

$$\begin{cases} P_{j1}(x,\lambda) = \varphi^{(j-1)}(x,\lambda)\widetilde{\Phi}'(x,\lambda) - \Phi^{(j-1)}(x,\lambda)\widetilde{\varphi}'(x,\lambda),\\ P_{j2}(x,\lambda) = \Phi^{(j-1)}(x,\lambda)\widetilde{\varphi}(x,\lambda) - \varphi^{(j-1)}(x,\lambda)\widetilde{\varphi}(x,\lambda),\end{cases} \tag{14}$$

and

$$\begin{cases} \varphi(x,\lambda) = P_{11}(x,\lambda)\widetilde{\varphi}(x,\lambda) + P_{12}(x,\lambda)\widetilde{\varphi}'(x,\lambda),\\ \Phi(x,\lambda) = P_{11}(x,\lambda)\widetilde{\Phi}(x,\lambda) + P_{12}(x,\lambda)\widetilde{\Phi}'(x,\lambda).\end{cases} \tag{15}$$

Taking (6), (7) and (9) into account, we infer that

$$|P_{11}(x,\lambda)-1| \le C_\delta |\rho|^{-1}, \ |P_{12}(x,\lambda)-1| \le C_\delta |\rho|^{-1}, \ \rho \in G_\delta, \ |\rho| \ge \rho^*, \qquad (16)$$

where G_δ is defined in (9) and C_δ is a constant.

On the other hand according to (10) and (14)

$$\begin{cases} P_{11}(x,\lambda) = \varphi(x,\lambda)\widetilde{S}'(x,\lambda) - S(x,\lambda)\widetilde{\varphi}'(x,\lambda) + (\widetilde{M}(\lambda) - M(\lambda))\varphi(x,\lambda)\widetilde{\varphi}'(x,\lambda), \\ P_{12}(x,\lambda) = S(x,\lambda)\widetilde{\varphi}(x,\lambda) - \varphi(x,\lambda)\widetilde{S}(x,\lambda) + (M(\lambda) - \widetilde{M}(\lambda))\varphi'(x,\lambda)\widetilde{\varphi}(x,\lambda). \end{cases}$$

Since $M(\lambda) = \widetilde{M}(\lambda)$, it follows that for each x the functions $P_{1k}(x,\lambda)$, $k = 1,2$ are entire in λ. With the help of (16) and well-known Liouville's theorem, this yields $P_{11}(x,\lambda) \equiv 1$, $P_{12}(x,\lambda) \equiv 0$. Substituting into (15), we obtain $\varphi(\lambda) = \widetilde{\varphi}(\lambda)$, $\Phi(\lambda) = \widetilde{\Phi}(\lambda)$ for all $x \in (0,\pi)\backslash\left\{\frac{\pi}{2}\right\}$ and λ. Taking this into account, from (1) we obtain $q(x) = \widetilde{q}(x)$ a.e. on $(0,\pi)$. Consequently, $L = \widetilde{L}$. ◀

Theorem 2. *If $\lambda_n = \widetilde{\lambda_n}$, $\alpha_n = \widetilde{\alpha_n}$, $n = 1,2...$, $L = \widetilde{L}$. Thus the specification of the spectral data $\Omega = \{\lambda_n, \alpha_n\}_{n=1}^{\infty}$ uniquely determines the operator L.*

Proof. Under the hypothesis of the theorem we obtain, in view of (11), that $M(\lambda) = \widetilde{M}(\lambda)$ on consequently by Theorem 1, $L = \widetilde{L}$. ◀

Theorem 3. *If $\lambda_n = \widetilde{\lambda_n}$ and $\mu_n = \widetilde{\mu_n}$, $n = 1,2...$, then $L = \widetilde{L}$. Thus the specification of two spectra $\{\lambda_n, \mu_n\}_{n=1}^{\infty}$ uniquely determines L.*

Proof. It is obvious that characteristic functions $\Delta(\lambda)$ and $\psi(0,\lambda)$ are uniquely determined by the sequences $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$, respectively. If $\lambda_n = \widetilde{\lambda_n}$, $\mu_n = \widetilde{\mu_n}$, $n = 1,2,...$,then $\Delta(\lambda) = \widetilde{\Delta}(\lambda)$, $\psi(\lambda) = \widetilde{\psi}(\lambda)$. Together with (10) this yields $M(\lambda) = \widetilde{M}(\lambda)$. By Theorem 1 we obtain $L = \widetilde{L}$. ◀

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