Baku Mathematical Journal
2024, Vol. 3, No 1, PP. 24-34
https://doi.org/10.32010/j.bmj.2024.03

# UPPER BOUNDS FOR BEREZIN NUMBERS OF SELF-ADJOINT OPERATORS AND APPLICATIONS

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Received: 05.08.2023 / Revised: 15.11.2023 / Accepted: 28.11.2023

*In memory of M. G. Gasymov on his 85th birthday*

**Abstract.** *We prove upper bounds for Berezin symbols and Berezin numbers of operators, with a special emphasis on the quadratic weighted operator geometric mean* $A_1 \circledS_\nu A_2$ *of operators* $A_1 \in \mathscr{B}^{-1}(\mathscr{H}(\Omega))$ *and* $A_2 \in \mathscr{B}(\mathscr{H}(\Omega))$ *defined by* $A_1 \circledS_\nu A_2 = \left||A_2 A_1^{-1}|^\nu A_1\right|^2$, *for* $\nu \geq 0$. *In particular, we prove some upper bounds for Berezin symbols and Berezin numbers of some self-adjoint operators on reproducing kernel Hilbert spaces* $\mathscr{H} = \mathscr{H}(\Omega)$ *over some suitable sets. Moreover, we estimate the best possible constant in some generalized Hardy-Hilbert inequality for the series.*

**Keywords**: Berezin symbol, Berezin number, self-adjoint operator, reproducing kernel

**Mathematics Subject Classification (2020):** 47A30, 47B35, 47B20

## 1. Introduction

Let $\mathscr{B}(\mathcal{H})$ stand for the Banach algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle .,. \rangle$. An operator $A \in \mathscr{B}(\mathcal{H})$ is call self-adjoint if $A^* = A$, where $A^*$ denotes the adjoint of $A$. It is easy to see that an operator $A$ is self-adjoint if and only if $\langle Ax, x \rangle \in \mathbb{R} := (-\infty, \infty)$ for all $x \in \mathcal{H}$. An operator $A \in \mathscr{B}(\mathcal{H})$ is called positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. The spectrum $\sigma(A)$ of an operator $A$ is the set of all $\lambda \in \mathbb{C}$ such that the operator $\lambda I - A$ does not have a bounded inverse.

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Recall that a reproducing kernel Hilbert space (shortly, RKHS) is a Hilbert space $\mathscr{H} = \mathscr{H}(\Omega)$ consisting of complex-valued functions on some set $\Omega$ such that the evaluation functionals $\varphi_\lambda(f) = f(\lambda)$, $\lambda \in \Omega$, are continuous on $\mathscr{H}$ and for any $\lambda \in \Omega$ there exists $f_\lambda$ such that $f_\lambda(\lambda) \neq 0$. Then by the Riesz representation theorem for each $\lambda \in \Omega$ there exists a unique function $K_\lambda \in \mathscr{H}$ such that

$$f(\lambda) = \langle f, K_\lambda \rangle \tag{1}$$

for all $f \in \mathscr{H}$. The collection $\{K_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel of the space $\mathscr{H}$. We say that the reproducing kernel Hilbert space admits the Ber-property (in this case we will write $\mathscr{H} \in$ (Ber), if for any bounded linear operator $A$ on $\mathscr{H}(\Omega)$, $\widetilde{A}(\mu) = 0, \forall \mu \in \Omega$, implies that $A = 0$, i.e., for the Berezin symbols of operators on $\mathscr{H}(\Omega)$ the uniqueness theorem holds, (i.e. the corresponding Berezin transform is injective). In particular, the Hardy space $H^2(\mathbb{D})$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc in $\mathbb{C}$, the Bergman space $L^2_a(\mathbb{D})$, the Dirichlet space $\mathcal{D}^2(\mathbb{D})$ and the Fock space $F(\mathbb{D})$ are RKHSs with the property (Ber). A detailed presentation of the theory of RKHSs is given, for instance, in Aronzajn [1], Saitoh [27] Halmos [18].

For any $A \in \mathscr{B}(\mathscr{H})$, its Berezin symbol $\widetilde{A}$ is defined on $\Omega$ by (see Berezin [4], [5], Nordgren and Rosenthal [25], Englisx [12] and Zhu [31])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{K}_\lambda, \widehat{K}_\lambda \right\rangle, \quad \lambda \in \Omega, \tag{2}$$

where $\widehat{K}_\lambda = \frac{K_\lambda}{\|K_\lambda\|_{\mathscr{H}}}$ is the normalized reproducing kernel of $\mathscr{H}$ and the inner product $\langle ., . \rangle$ is taken with respect to the the RKHS $\mathscr{H}$. The Berezin norm, Berezin set and Berezin number of the operator $A$ are defined respectively by

$$\|A\|_{Ber} := \sup_{\lambda \in \Omega} \left\| A\widehat{K}_\lambda \right\|, \tag{3}$$

$$Ber(A) := \mathrm{Range}(\widetilde{A}) = \left\{ \widetilde{A}(\lambda) : \lambda \in \Omega \right\}, \tag{4}$$

$$ber(A) := \sup_{\lambda \in \Omega} \left| \widetilde{A}(\lambda) \right|. \tag{5}$$

It is clear that $ber(A) \leq \|A\|_{Ber} \leq \|A\|$, $Ber(A) \subset W(A)$ and $ber(A) \leq w(A)$, where

$$W(A) := \{\langle Ax, x \rangle : x \in \mathscr{H} \quad \text{and} \quad \|x\| = 1\} \tag{6}$$

is the numerical range of the operator $A$ and

$$w(A) := \sup_{\|x\|=1} |\langle Ax, x \rangle| \tag{7}$$

is its numerical radius.

Let $\mathrm{Range}(A)$ and $\ker(A)$ denote respectively the range space and the null space of the operator $A \in \mathscr{B}(\mathscr{H})$. Then $A$ is called a partial isometry if $A|_{(\ker(A))^\perp}$ is an isometry. It is also well-known that any operator $A \in \mathscr{B}(\mathscr{H})$ admits a unique polar decomposition given by $A = V_A|A|$, where $V_A$ is a partial isometry, $|A| := (A^*A)^{\frac{1}{2}}$,

$\ker(V_A) = \ker(|A|) = \ker(A)$ and $\ker(V_A^*) = \ker(A^*)$. Two operators $B$ and $A$ in $\mathscr{B}(\mathscr{H})$ are said to have similar positive parts if there exists a unitary operator $U \in \mathscr{B}(\mathscr{H})$ such that $U|A| = |B|U$. In this respect, to our best knowledge, E. Ko [24] is the first who investigated operators having similar positive parts. In particular, Theorem 4.1 in [24] provides an expression for the polar decomposition of $A$ when $A^*$ and $A$ have similar positive parts. E. Ko also discovered several connections between two operators if they have similar parts. For example, similarity of operators preserves isometries (see Corollary 2.7 of [24]). The impact of the similarity of positive parts on local spectra of operators is also studied in [24]. Following [26], let $\mathscr{B}^{-1}(\mathscr{H})$ be the class of all bounded linear invertible operators on $\mathscr{H}$. In 2016, Dragomir [9] introduced the concept of quadratic weighted operator geometric mean of operators. Namely, for $A_1 \in \mathscr{B}^{-1}(\mathscr{H})$ and $A_2 \in \mathscr{B}(\mathscr{H})$, the quadratic weighted operator geometric mean of $\{A_1, A_2\}$ is defined by

$$A_1 \circledS_\nu A_2 = \left| |A_2 A_1^{-1}|^\nu A_1 \right|^2, \quad \text{for} \quad \nu \geq 0. \tag{8}$$

Using this mean, he obtained some inequalities for certain class of operators. In 2018, Dragomir [11] pursued his study in this direction and presented some Hölder type inequalities for the quadratic weighted operator geometric mean for some operators. In the present article, we give upper bounds for Berezin symbols and Berezin numbers of some self-adjoint operators, including the operator $(A_1 \circledS_\nu A_2)$. Our paper is mainly motivated by the work of Sahoo, Das and Mishra [26]. For related results the readers can consult [2], [3], [6], [7], [9], [10], [16]-[14], [20], [21], [28]-[30].

## 2. Upper Bounds for the Berezin Number

The following result gives an upper bound for the Berezin number of the operator $\left(T_\varphi(A_1 \circledS_\nu A_2)T_\psi^*\right)$ on the Hardy Hilbert space $H^2 = H^2(\mathbb{D})$ of analytic functions $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for which $\sum_{n=0}^{\infty} |\widehat{f}(n)|^2 = \|f\|_2^2 < \infty$, where $\widehat{f}(n) = \frac{f^{(n)}(0)}{n!}$ is the $n^{th}$ Taylor coefficient of $f$, and $T_\varphi, T_\psi$ $(\varphi, \psi \in L^\infty(\partial\mathbb{D}))$ are the Toeplitz operators on $H^2$ defined by $T_\varphi f = P_+ \varphi f$ and $T_\psi f = P_+ \psi f$, $f \in H^2$, where $P_+ : L^2(\partial\mathbb{D}) \to H^2$ is the Riesz orthogonal projection. The normalized reproducing kernel of $H^2$ is the function $\widehat{k}_\lambda(z) = \frac{\sqrt{1-|\lambda|^2}}{1-\overline{\lambda}z}$, $\lambda \in \mathbb{D}$.

Let $\mathcal{H}^\infty$ be the algebra of all bounded analytic functions on $\mathbb{D}$ with supremum norm.

**Theorem 1.** *Let $\varphi, \psi$ be two functions in $\mathcal{H}^\infty$, and let $A_1 \in \mathscr{B}^{-1}(H^2)$ and $A_2 \in \mathscr{B}(H^2)$ be two positive operators such that $A_1$ and $A_2$ have similar positive parts, i.e., $|A_1| = U^*|A_2|U$ for some unitary operator $U$ on $H^2$. Further, let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ber\left(T_\varphi(A_1 \circledS_{\frac{1}{p}} A_2)T_\psi^*\right) \leq \|\varphi\|_\infty \|\psi\|_\infty \left(\frac{1}{p}\|A_2\|_{Ber}^2 + \frac{1}{q}\|A_2 U\|_{Ber}^2\right). \tag{9}$$

To prove this assertion, we shall need the following lemmas:

**Lemma 1.** *([19]) Let $a, b > 0$, $0 \le m, n \le 1$ and $p, q > 1$ be such that $n + m = 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, one has*

*(i)* $a^m b^n \le ma + nb \le (ma^r + nb^r)^{\frac{1}{r}}$, *for* $r \ge 1$;

*(ii)* $ab \le \frac{a^p}{p} + \frac{b^q}{q} \le \left(\frac{a^{pr}}{p} + \frac{b^{qr}}{q}\right)^{\frac{1}{r}}$, *for* $r \ge 1$.

**Lemma 2.** *([23]) Let $a, b > 0$ and $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab + \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(a^{\frac{p}{2}} - b^{\frac{q}{2}}\right)^2 \le \frac{a^p}{p} + \frac{b^q}{q}.$$

**Lemma 3.** *([26]) Let $A_1 \in \mathscr{B}^{-1}(\mathscr{H})$ and $A_2 \in \mathscr{B}(\mathscr{H})$. Then for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\widetilde{A_1 \circledS_{\frac{1}{p}} A_2}(\lambda) \le \left(\widetilde{|A_2|^2}(\lambda)\right)^{\frac{1}{p}} \left(\widetilde{|A_1|^2}(\lambda)\right)^{\frac{1}{q}},$$

*for all $\lambda \in \Omega$. In particular,*

$$\widetilde{A_1 \circledS A_2}(\lambda) \le \left(\widetilde{|A_2|^2}(\lambda)\right)^{\frac{1}{2}} \left(\widetilde{|A_1|^2}(\lambda)\right)^{\frac{1}{2}},$$

*for all $\lambda \in \Omega$.*

Lemma 2 is a actually a particular case of Lemma 5 in [26].

*Proof.* *(Proof of Theorem 1.)* Let $A_1 = V_1|A_1|$ and $A_2 = V_2|A_2|$ be the polar decompositions of $A_1$ and $A_2$, respectively. Since $A_1$ and $A_2$ have similar positive parts, we have $|A_1| = U^*|A_2|U$ and $|A_1|^2 = U^*|A_2|^2U$. By Lemmas 3 and 1, we have that

$$\left|T_\varphi(\widetilde{A_1 \circledS_{\frac{1}{p}} A_2)T_\psi^*}(\lambda)\right| = \left|\left\langle T_\varphi(A_1 \circledS_{\frac{1}{p}} A_2)T_\psi^* \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| = \left|\left\langle (A_1 \circledS_{\frac{1}{p}} A_2)T_\psi^* \widehat{k}_\lambda, T_\varphi^* \widehat{k}_\lambda\right\rangle\right| =$$

$$= \left|\left\langle (A_1 \circledS_{\frac{1}{p}} A_2)\overline{\psi(\lambda)}\widehat{k}_\lambda, \overline{\varphi(\lambda)}\widehat{k}_\lambda\right\rangle\right| = |\psi(\lambda)\varphi(\lambda)| \left|\left\langle (A_1 \circledS_{\frac{1}{p}} A_2)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| \le$$

$$\le |\psi(\lambda)\varphi(\lambda)| \left(\widetilde{|A_2|^2}(\lambda)\right)^{\frac{1}{p}} \left(\widetilde{|A_1|^2}(\lambda)\right)^{\frac{1}{q}} \le$$

$$\le |\psi(\lambda)\varphi(\lambda)| \left(\frac{1}{p}\widetilde{|A_2|^2}(\lambda) + \frac{1}{q}\widetilde{|A_1|^2}(\lambda)\right) =$$

$$= |\psi(\lambda)\varphi(\lambda)| \left(\frac{1}{p}\widetilde{|A_2|^2}(\lambda) + \frac{1}{q}\widetilde{U^*|A_2|^2U}(\lambda)\right) \le$$

$$\le \|\psi\|_\infty \|\varphi\|_\infty \left(\frac{1}{p}\left\langle |A_2|\widehat{k}_\lambda, |A_2|\widehat{k}_\lambda\right\rangle + \frac{1}{q}\left\langle |A_2|U\widehat{k}_\lambda, |A_2|U\widehat{k}_\lambda\right\rangle\right) =$$

$$= \|\psi\|_\infty \|\varphi\|_\infty \left(\frac{1}{p}\left\langle V_2^*V_2|A_2|\widehat{k}_\lambda, |A_2|\widehat{k}_\lambda\right\rangle + \frac{1}{q}\left\langle V_2^*V_2|A_2|U\widehat{k}_\lambda, |A_2|U\widehat{k}_\lambda\right\rangle\right) =$$

$$= \|\psi\|_\infty \|\varphi\|_\infty \left(\frac{1}{p}\left\langle V_2|A_2|\widehat{k}_\lambda, V_2|A_2|\widehat{k}_\lambda\right\rangle + \frac{1}{q}\left\langle V_2|A_2|U\widehat{k}_\lambda, V_2|A_2|U\widehat{k}_\lambda\right\rangle\right) =$$

$$= \|\psi\|_\infty \|\varphi\|_\infty \left(\frac{1}{p}\left\langle A_2\widehat{k}_\lambda, A_2\widehat{k}_\lambda\right\rangle + \frac{1}{q}\left\langle A_2U\widehat{k}_\lambda, A_2|U\widehat{k}_\lambda\right\rangle\right) =$$

$$= \|\psi\|_\infty \|\varphi\|_\infty \left(\frac{1}{p}\left\|A_2\widehat{k}_\lambda\right\|^2 + \frac{1}{q}\left\|A_2U\widehat{k}_\lambda\right\|^2\right).$$

Taking supremum over $\lambda \in \mathbb{D}$, we thus have

$$ber\left(T_\varphi(A_1 \circledS_{\frac{1}{p}} A_2)T_\psi^*\right) \le \|\varphi\|_\infty \|\psi\|_\infty \left(\frac{1}{p}\|A_2\|_{Ber}^2 + \frac{1}{q}\|A_2U\|_{Ber}^2\right).$$

◄

The following corollary follows from Theorem 1 and Lemma 3.

**Corollary 1.** *Let $A_1 \in \mathscr{B}^{-1}(H^2)$ and $A_2 \in \mathscr{B}(H^2)$ be two positive operators. Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and let $\varphi, \psi \in H^\infty$. Then*

$$ber\left(T_\varphi(A_1 \circledS_{\frac{1}{p}} A_2)T_\psi^*\right) \le$$

$$\le \|\varphi\|_\infty \|\psi\|_\infty \left[\frac{1}{p}\|A_2\|_{Ber}^2 + \frac{1}{q}\|A_1\|_{Ber}^2 - \min\left\{\frac{1}{p}, \frac{1}{q}\right\} \inf_{\lambda\in\mathbb{D}}\left(\left\|A_2\widehat{k}_\lambda\right\| - \left\|A_1\widehat{k}_\lambda\right\|\right)^2\right].$$

*Proof.* In fact, by Lemma 3, we have

$$\left|\left\langle T_\varphi(A_1 \circledS_{\frac{1}{p}} A_2)T_\psi^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| \le$$

$$\le \|\psi\|_\infty \|\varphi\|_\infty \left\langle |A_2|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{\frac{1}{p}} \left\langle |A_1|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{\frac{1}{q}} \le$$

$$\le \|\psi\|_\infty \|\varphi\|_\infty \left(\frac{1}{p}\left\langle |A_2|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle + \frac{1}{q}\left\langle |A_1|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right) -$$

$$- \|\psi\|_\infty \|\varphi\|_\infty \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(\left\langle |A_2|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{\frac{1}{2}} + \left\langle |A_1|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2,$$

where the last inequality follows from Lemma 2. Hence

$$\left|\left\langle T_\varphi(A_1 \circledS_{\frac{1}{p}} A_2)T_\psi^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| \le$$

$$\le \|\psi\|_\infty \|\varphi\|_\infty \left(\frac{1}{p}\left\langle |A_2|\widehat{k}_\lambda, |A_2|\widehat{k}_\lambda\right\rangle + \frac{1}{q}\left\langle |A_1|\widehat{k}_\lambda, |A_1|\widehat{k}_\lambda\right\rangle\right) -$$

$$- \|\psi\|_\infty \|\varphi\|_\infty \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(\left\langle |A_2|\widehat{k}_\lambda, |A_2|\widehat{k}_\lambda\right\rangle^{\frac{1}{2}} + \left\langle |A_1|\widehat{k}_\lambda, |A_1|\widehat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2 =$$

$$= \|\psi\|_\infty \|\varphi\|_\infty \left(\frac{1}{p}\left\langle V_2^*V_2|A_2|\widehat{k}_\lambda, |A_2|\widehat{k}_\lambda\right\rangle + \frac{1}{q}\left\langle V_1^*V_1|A_1|\widehat{k}_\lambda, |A_1|\widehat{k}_\lambda\right\rangle\right) - \|\psi\|_\infty \times$$

$$\times \|\varphi\|_\infty \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(\left\langle V_2^*V_2|A_2|\widehat{k}_\lambda, |A_2|\widehat{k}_\lambda\right\rangle^{\frac{1}{2}} + \left\langle V_1^*V_1|A_1|\widehat{k}_\lambda, |A_1|\widehat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2 =$$

$$= \|\psi\|_\infty \|\varphi\|_\infty \left(\frac{1}{p}\left\langle V_2|A_2|\widehat{k}_\lambda, V_2|A_2|\widehat{k}_\lambda\right\rangle + \frac{1}{q}\left\langle V_1|A_1|\widehat{k}_\lambda, V_1|A_1|\widehat{k}_\lambda\right\rangle\right) - \|\psi\|_\infty \times$$

$$\times \|\varphi\|_\infty \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(\left\langle V_2|A_2|\widehat{k}_\lambda, V_2|A_2|\widehat{k}_\lambda\right\rangle^{\frac{1}{2}} + \left\langle V_1|A_1|\widehat{k}_\lambda, V_1|A_1|\widehat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2 =$$

$$= \|\psi\|_\infty \|\varphi\|_\infty \left(\frac{1}{p}\left\langle A_2\widehat{k}_\lambda, A_2\widehat{k}_\lambda\right\rangle + \frac{1}{q}\left\langle A_1\widehat{k}_\lambda, A_1\widehat{k}_\lambda\right\rangle\right) -$$

$$- \|\psi\|_\infty \|\varphi\|_\infty \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(\left\langle A_2\widehat{k}_\lambda, A_2\widehat{k}_\lambda\right\rangle^{\frac{1}{2}} + \left\langle A_1\widehat{k}_\lambda, A_1\widehat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2 =$$

$$= \|\psi\|_\infty \|\varphi\|_\infty \left(\frac{1}{p}\left\|A_2\widehat{k}_\lambda\right\|^2 + \frac{1}{q}\left\|A_1\widehat{k}_\lambda\right\|^2\right) -$$

$$- \|\psi\|_\infty \|\varphi\|_\infty \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(\left\|A_2\widehat{k}_\lambda\right\| + \left\|A_1\widehat{k}_\lambda\right\|\right)^2.$$

Taking supremum over $\lambda \in \mathbb{D}$, we obtain

$$ber\left(T_\varphi(A_1 Ⓢ_{\frac{1}{p}} A_2)T_\psi^*\right) \le \|\varphi\|_\infty \|\psi\|_\infty \left(\frac{1}{p}\|A_2\|_{Ber}^2 + \frac{1}{q}\|A_1\|_{Ber}^2\right) -$$

$$- \|\varphi\|_\infty \|\psi\|_\infty \min\left\{\frac{1}{p}, \frac{1}{q}\right\} \inf_{\lambda\in\mathbb{D}}\left(\left\|A_2\widehat{k}_\lambda\right\| - \left\|A_1\widehat{k}_\lambda\right\|\right)^2,$$

as desired. ◀

Since $T_\omega^*\widehat{k}_\lambda = \overline{\omega(\lambda)}\widehat{k}_\lambda$ for every $\omega \in H^\infty$ and $\|T_\omega^* U\|_{Ber} \le \|\omega\|_\infty$, the following fact is an immediate consequence of Theorem 1.

**Corollary 2.** *For any $\omega \in H^\infty$, we have* $\|\omega\|_\infty \ge \sqrt{ber\left(A_1 Ⓢ_{\frac{1}{p}} T_\omega^*\right)}$.

## 3. Power Inequalities for Berezin Symbols

In this section, we use some power inequalities for numbers and certain functional calculus technique to estimate Berezin symbols and Berezin numbers of some self-adjoint operators. Our next result gives some inequalities for Berezin symbols of some self-adjoint operators on a reproducing kernel Hilbert space $\mathscr{H} = \mathscr{H}(\Omega)$.

**Theorem 2.** *Let $f$ be a continuous function defined on an interval $J \subset (0,\infty)$ and $f \ge 0$. If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\sup_{\mu,\eta\in\Omega}\left[\widetilde{f(A)}(\mu)\widetilde{f(B)}(\eta) + \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(\widetilde{f^{\frac{p}{2}}(A) - f^{\frac{q}{2}}(A)}\right)^2(\mu)\right]$$
$$\le \frac{1}{p} ber(f^p(A)) + \frac{1}{q} ber(f^q(B)) \tag{10}$$

*for all self-adjoint operators $A$ and $B$ in $\mathscr{B}(\mathscr{H})$ with spectra contained in $J$.*

*Proof.* Let $a, b > 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then by Lemma 2, we have

$$ab + \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left( a^{\frac{p}{2}} - b^{\frac{q}{2}}\right)^2 \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{11}$$

Let $x, y \in J$. By considering that $f(x) \geq 0$ for all $x \in J$ and putting $a = f(x)$ and $b = f(y)$ in (11), we get

$$f(x)f(y) + \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left( f^{\frac{p}{2}}(x) - f^{\frac{q}{2}}(y)\right)^2 \leq \frac{f^p(x)}{p} + \frac{f^q(y)}{q}, \tag{12}$$

for all $x, y \in J$. Applying certain functional calculus arguments due to Kian [22], for an operator $A$ we get from (12) that

$$f(A)f(y) + \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left( f^{\frac{p}{2}}(A) - f^{\frac{q}{2}}(y)I_{\mathscr{H}}\right)^2 \leq \frac{f^p(A)}{p} + \frac{f^q(y)}{q}I_{\mathscr{H}}, \tag{13}$$

whence

$$\begin{aligned} f(y)\left\langle f(A)\widehat{k}_{\mathscr{H},\mu}, \widehat{k}_{\mathscr{H},\mu}\right\rangle + \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left\langle\left( f^{\frac{p}{2}}(A) - f^{\frac{q}{2}}(y)I_{\mathscr{H}}\right)^2 \widehat{k}_{\mathscr{H},\mu}, \widehat{k}_{\mathscr{H},\mu}\right\rangle \\ \leq \frac{1}{p}\left\langle f^p(A)\widehat{k}_{\mathscr{H},\mu}, \widehat{k}_{\mathscr{H},\mu}\right\rangle + \frac{f^q(y)}{q}I_{\mathscr{H}}\,, \end{aligned} \tag{14}$$

for all $\mu \in \Omega$. Using the functional calculus once more to the self-adgoint operator $B$, we get from (14) that

$$\begin{aligned} &\left\langle f(A)\widehat{k}_{\mathscr{H},\mu}, \widehat{k}_{\mathscr{H},\mu}\right\rangle \left\langle f(B)\widehat{k}_{\mathscr{H},\eta}, \widehat{k}_{\mathscr{H},\eta}\right\rangle + \\ &+ \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left\langle\left( f^{\frac{p}{2}}(A) - f^{\frac{q}{2}}(B)I_{\mathscr{H}}\right)^2 \widehat{k}_{\mathscr{H},\mu}, \widehat{k}_{\mathscr{H},\mu}\right\rangle \leq \\ &\leq \frac{1}{p}\left\langle f^p(A)\widehat{k}_{\mathscr{H},\mu}, \widehat{k}_{\mathscr{H},\mu}\right\rangle + \leq \frac{1}{q}\left\langle f^q(B)\widehat{k}_{\mathscr{H},\eta}, \widehat{k}_{\mathscr{H},\eta}\right\rangle\,, \end{aligned}$$

for all $\mu, \eta \in \Omega$. This means that

$$\begin{aligned} &\widetilde{f(A)}(\mu)\widetilde{f(B)}(\eta) + \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\widetilde{\left(f^{\frac{p}{2}}(A) - f^{\frac{q}{2}}(A)\right)^2}(\mu) \leq \\ &\leq \frac{1}{p}\widetilde{f^p(A)}(\mu) + \frac{1}{q}\widetilde{f^q(B)}(\mu) \leq \frac{1}{p}ber(f^p(A)) + \frac{1}{q}ber(f^q(B))\,, \end{aligned}$$

for all $\mu, \eta \in \Omega$, and hence

$$\begin{aligned} &\sup_{\mu,\eta\in\Omega}\left[\widetilde{f(A)}(\mu)\widetilde{f(B)}(\eta) + \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\widetilde{\left(f^{\frac{p}{2}}(A) - f^{\frac{q}{2}}(A)\right)^2}(\mu)\right] \leq \\ &\leq \frac{1}{p}ber(f^p(A)) + \frac{1}{q}ber(f^q(B))\,. \end{aligned}$$

The proof is completed. ◀

Das and Sahoo [8] proved in particular the following generalization of the classical Hardy-Hilbert inequality.

**Theorem 3.** *Let* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < r, s \leq 1$, $r + s = \lambda$, $a_n, b_n \geq 0$, $A_n := \sum_{k=1}^{n} a_k$. *If* $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, *then*

$$\sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}}}{\max\{m^\lambda, n^\lambda\}} A_m \right)^p < \left( \frac{q\lambda}{rs} \right)^p \sum_{n=1}^{\infty} a_n^p, \tag{15}$$

*where the constant factor* $\left( \frac{q\lambda}{rs} \right)^p$ *is the best possible.*

As an application of Berezin symbols technique, we will prove some lower estimate for the best constant $\left( \frac{q\lambda}{rs} \right)^p$ in Theorem 3.

**Theorem 4.** *Let* $p > 1$ *be an integer. Then the best constant in inequality (15) satisfies the following inequality*

$$\left( \frac{q\lambda}{rs} \right)^p > \frac{1}{2} + 2^{(r-1-\lambda)p}. \tag{16}$$

*Proof.* Let $N \geq 2$ and $a_1, a_2, ..., a_N$ be positive scalars. Then it follows from (15) that

$$\sum_{n=1}^{2} \left( \sum_{m=1}^{2} \frac{m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}}}{\max\{m^\lambda, n^\lambda\}} A_m \right)^p < \left( \frac{q\lambda}{rs} \right)^p \left( a_1^p + a_2^p \right), \tag{17}$$

or equivalently,

$$\left( \sum_{m=1}^{2} \frac{m^{r-\frac{1}{q}-1}}{\max\{m^\lambda, 1^\lambda\}} A_m \right)^p + \left( \sum_{m=1}^{2} \frac{m^{r-\frac{1}{q}-1} 2^{s-\frac{1}{p}}}{\max\{m^\lambda, 2^\lambda\}} A_m \right)^p < \left( \frac{q\lambda}{rs} \right)^p \left( a_1^p + a_2^p \right), \tag{18}$$

that is

$$a_1 + 2^{\left(r-\frac{1}{q}-1-\lambda\right)p} \left( a_1 + a_2 \right)^p < \left( \frac{q\lambda}{rs} \right)^p \left( a_1^p + a_2^p \right). \tag{19}$$

Let $f$ be a continuous function on $J$ and let $x, y \in J$. $f(t) \geq 0$ for all $t \in J$. Let us put $a_1 = f(x)$ and $a_2 = f(y)$. Then by using the method of the proof of Theorem 2, we have from (19) that

$$f^p(x) + 2^{\left(r-\frac{1}{q}-1-\lambda\right)p} \left( f(x) + f(y) \right)^p < \left( \frac{q\lambda}{rs} \right)^p \left( f^p(x) + f^p(y) \right), \tag{20}$$

for all $x, y \in J$. Let $A$ be a bounded positive operator on $\mathscr{H} \in (Ber)$ with spectrum in $J$. Then by using the functional calculus and inequality (20), we obtain

$$f^p(A) + 2^{\left(r-\frac{1}{q}-1-\lambda\right)p} \left( f(A) + f(y)I \right)^p < \left( \frac{q\lambda}{rs} \right)^p \left( f^p(A) + f^p(y)I \right), \tag{21}$$

and hence

$$\begin{aligned}\left\langle f^p(A)\widehat{k}_{\mathscr{H},\mu}, \widehat{k}_{\mathscr{H},\mu}\right\rangle &+ 2^{\left(r-\frac{1}{q}-1-\lambda\right)p}\left\langle (f(A)+f(y)I)^p\,\widehat{k}_{\mathscr{H},\mu}, \widehat{k}_{\mathscr{H},\mu}\right\rangle \\ &< \left(\frac{q\lambda}{rs}\right)^p\left\langle (f^p(A)+f^p(y)I)\,\widehat{k}_{\mathscr{H},\mu}, \widehat{k}_{\mathscr{H},\mu}\right\rangle, \end{aligned} \tag{22}$$

for all $\mu \in \Omega$ and all $y \in J$. Using the functional calculus once more to the positive operator $B$ with spectrum contained in $J$, we get

$$\begin{aligned}\left\langle f^p(A)\widehat{k}_{\mathscr{H},\mu}, \widehat{k}_{\mathscr{H},\mu}\right\rangle &+ 2^{\left(r-\frac{1}{q}-1-\lambda\right)p}\left\langle (f(A)+f(B))^p\,\widehat{k}_{\mathscr{H},\mu}, \widehat{k}_{\mathscr{H},\mu}\right\rangle \\ &< \left(\frac{q\lambda}{rs}\right)^p\left\langle (f^p(A)+f^p(B))\,\widehat{k}_{\mathscr{H},\mu}, \widehat{k}_{\mathscr{H},\mu}\right\rangle, \end{aligned} \tag{23}$$

for all $\mu \in \Omega$. Thus, we have from (23) that

$$\widetilde{f^p(A)}(\mu) + 2^{\left(r-\frac{1}{q}-1-\lambda\right)p}\widetilde{(f(A)+f(B))^p}(\mu) < \left(\frac{q\lambda}{rs}\right)^p \widetilde{(f^p(A)+f^p(B))}(\mu), \tag{24}$$

for all self-adjoint operators $A, B \in \mathcal{B}(\mathscr{H})$ and for all $\mu \in \Omega$. Now, for $B = A$, we have from (23) that

$$\left(1 + 2^{\left(r-\frac{1}{q}-1-\lambda\right)p+p}\right)\widetilde{f^p(A)}(\mu) < 2\left(\frac{q\lambda}{rs}\right)^p \widetilde{f^p(A)}(\mu), \tag{25}$$

for all $\mu \in \Omega$. Since $f(A)$ is self-adjoint, it can not be a nilpotent operator, and thus by the uniqueness theorem of the Berezin symbol, we deduce that there exists a point $\mu_0 \in \Omega$ such that $\widetilde{f^p(A)}(\mu_0) \neq 0$. Therefore, by considering that $\widetilde{f^p(A)}(\mu_0) > 0$, we conclude from (25) that

$$2\left(\frac{q\lambda}{rs}\right)^p > 1 + 2^{\left(r-\frac{1}{q}-\lambda\right)p}, \tag{26}$$

which is equivalent to

$$\left(\frac{q\lambda}{rs}\right)^p > \frac{1}{2} + 2^{\left(r-\frac{1}{q}-\lambda\right)p-1}. \tag{27}$$

Since $\frac{p}{q} + 1 = p$, hence (27) reduces to

$$\left(\frac{q\lambda}{rs}\right)^p > \frac{1}{2} + 2^{(r-1-\lambda)p}. \tag{28}$$

This proves (16), as desired. The theorem is proven. ◂

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