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SOLVABILITY OF THE INVERSE SPECTRAL PROBLEM FOR STURM-LIOUVILLE OPERATOR WITH SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

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In memory of M. G. Gasymov on his 85th birthday

Abstract. *This paper deals with an inverse problem for the Sturm-Liouville operator with non-separated boundary conditions, one of which linearly depends on a spectral parameter. Uniqueness theorem is proved, solution algorithm is constructed and sufficient conditions for solvability of inverse problem are obtained. As spectral data, we only use the spectrum of one boundary value problem and some sequence of signs, and some number.*

Keywords: Sturm-Liouville operator, non-separated boundary conditions, spectrum, inverse problem.

Mathematics Subject Classification (2020): 34A55, 34B24, 34L05, 47E05

1. Introduction

The theory of inverse spectral problems for differential operators, due to its theoretical and applied importance, is one of the intensively developing branches of modern mathematics and attracts the attention of many researchers interested in both the theory itself and its applications. The inverse problems of spectral analysis consist in determining operators from their known spectral data, which include one, two or more spectra, a

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spectral function, a spectrum and normalization numbers, a Weyl function, etc. Depending on the choice of spectral data, various formulations of inverse problems are possible. By now, the problems of restoring the Sturm-Liouville differential equation

$$-y'' + q(x)\,y = \lambda^2 y \tag{1}$$

under separated boundary conditions. The first thorough research in this direction was undertaken by the Swedish mathematician G. Borg [8] in 1946. He proved that two spectra of Sturm-Liouville differential operators with one common boundary condition uniquely define the function $q(x)$. The complete solution of the inverse problem of recovering the Sturm-Liouville operator on a segment from two spectra is contained in the work of B.M. Levitan and M.G. Gasymov [30] (see also the books [35], [52]).

Since the 1970s, researchers have been intensively studying inverse spectral problems for differential operators with nonseparated boundary conditions. Among them, a special place is occupied by the recovery of periodic, anti-periodic, quasi-periodic and generalized periodic problems. These problems were solved by different approaches in [1], [12], [26], [27], [35], [37], [38], [44], [48], [53], [54] . In the future, V.A. Sadovnichii, V.A. Yurko, O.A. Plaksina, M.G. Gasymov, H.M. Huseynov, I.M. Nabiev, A.M. Akhtyamov, Ya.T. Sultanaev, A.S. Makin and other authors (see [3], [14], [19]–[21], [31], [43], [46], [51]). We note that in [55] a survey of the main results on inverse spectral problems for second-order differential operators and sheaves on a segment with nonseparated boundary conditions is given.

Many very important questions of the theory of oscillations in mathematical physics lead to direct and inverse problems of spectral analysis for differential operators with a spectral parameter in an equation and in boundary conditions. Such problems also arise in technical physics, mechanics, acoustics of flow channels, electrodynamics, and other areas [11], [36], [49]. In the case of separated boundary conditions, some variants of such problems are completely solved (see [5]-[7], [9], [10], [15]-[18], [22], [28], [42], [50]). Little studied are the boundary value problems in the case when the nonseparated boundary conditions depend on the spectral parameter. Only articles [2], [4], [13], [23]-[25], [32]-[34], [40], [41], [45], [47] are devoted to the study of problems of this type. Note that in these papers, for the uniqueness of recovering the unknown coefficients of the differential equation and boundary conditions, at least two spectra of related problems and other additional spectral data are required.

In present paper, we study the inverse spectral problem of recovering the Sturm-Liouville operator with nonseparated boundary conditions, one of which contains a linear function of the spectral parameter. A uniqueness theorem is proved, an algorithm for the solution is constructed, and sufficient conditions for the solvability of the inverse problem are obtained. The spectrum of one boundary value problem, some sequence of signs, and some number are used as spectral data. Note that in [39] a similar problem was completely solved in the case of other boundary conditions.

2. Some Properties of Eigenvalues

Consider a boundary value problem generated on a segment $[0,\pi]$ by the Sturm-Liouville equation (1) with a real coefficient $q(x)\in L_2[0,\pi]$ and nonseparated boundary conditions of the form

$$y(0)+i\omega y(\pi)=0,\quad -i\omega y'(0)+(\alpha\lambda+\beta)\,y(\pi)+y'(\pi)=0, \tag{2}$$

where λ is the spectral parameter, i is the imaginary unit, ω,α,β are real numbers and $\omega\neq\pm1,\alpha>0$. This problem will be denoted by L.

Let $c(x,\lambda)$, $s(x,\lambda)$ the fundamental system of solutions to equation (1), defined by the initial conditions

$$c(0,\lambda)=s'(0,\lambda)=1, c'(0,\lambda)=s(0,\lambda)=0.$$

It is easy to see that the characteristic function of the boundary value problem will be

$$\Delta\left(\lambda\right)=\omega^2c\left(\pi,\lambda\right)+\left(\alpha\lambda+\beta\right)\,s\left(\pi,\lambda\right)+s'\left(\pi,\lambda\right). \tag{3}$$

The zeros μ_k,$k=\pm1,\ \pm2,\ ...$ of the function $\Delta\left(\lambda\right)$ are the eigenvalues of the problem L. According to Theorem 1 in [39], for μ_k as $|k|\to\infty$, we have the asymptotic formula

$$\mu_k=k-\frac{1}{2}\mathrm{sign}k+a+\frac{B}{k\pi}+\frac{\tau_k}{k}, \tag{4}$$

$$a=\frac{1}{\pi}\arccos\frac{b}{\sqrt{\alpha^2+b^2}}, \tag{5}$$

$$B=A+\frac{\beta b}{\alpha^2+b^2}, \tag{6}$$

$$b=1+\omega^2,\quad A=\frac{1}{2}\int_0^\pi q\left(x\right)dx, \{\tau_k\}\in l_2.$$

It was also proved in [39] that these eigenvalues are real, nonzero, and simple under the following condition (T): for all functions $y\left(x\right)\in W_2^2\left[0,\pi\right], y\left(x\right)\not\equiv0$ satisfying conditions (2), the inequality

$$\beta\left|y(\pi)\right|^2+\int_0^\pi\left(\left|y'(x)\right|^2+q(x)\left|y(x)\right|^2\right)dx>0. \tag{7}$$

Note that this inequality is certainly satisfied if $\beta\geq0$, $q\left(x\right)>0$.

Denote by zeros λ_k $(k=\pm1,\ \pm2,...)$ the functions $s\left(\pi,\lambda\right)$ whose squares are eigenvalues of the boundary value problem L_0 generated by equation (1) and boundary conditions $y\left(0\right)=y\left(\pi\right)=0$. The numbers λ_k at $|k|\to\infty$ obey the asymptotics

$$\lambda_k=k+\frac{A}{k\pi}+\frac{\xi_k}{k},\quad \{\xi_k\}\in l_2. \tag{8}$$

Theorem 1. *If the condition (T) is satisfied, then the numbers μ_k and λ_k are interleaved in the following sense:*

$$...<\lambda_{-3}<\mu_{-3}<\lambda_{-2}<\mu_{-2}<\lambda_{-1}<\mu_{-1}<0<\mu_1<\lambda_1<\mu_2<\lambda_2<\mu_3<\lambda_3<\ldots. \tag{9}$$

Proof. Consider the solution of equation (1) of the form

$$z\left(x,\lambda\right)=\left[1+\omega c\left(\pi,\lambda\right)\right]\, s\left(x,\lambda\right)-\omega s\left(\pi,\lambda\right)\, c\left(x,\lambda\right).$$

It's obvious that

$$\begin{array}{l} z\left(0,\,\lambda\right)=-\omega s\left(\pi,\,\lambda\right)\,,\;\; z'\left(0,\,\lambda\right)=1+\omega c\left(\pi,\,\lambda\right), \\ z\left(\pi,\,\lambda\right)=s\left(\pi,\,\lambda\right)\,,\;\; z'\left(\pi,\,\lambda\right)=\omega+s'\left(\pi,\,\lambda\right). \end{array} \tag{10}$$

In what follows, a point over a function will denote the derivative of the function with respect to the parameter λ. Differentiating λ by equality

$$z''\left(x,\lambda\right)+\left[\lambda^2-q\left(x\right)\right]\, z\left(x,\lambda\right)=0\,, \tag{11}$$

and then passing to the complex conjugate in the resulting equality, for real λ we have

$$\overline{\dot{z}''\left(x,\lambda\right)}+2\lambda\,\overline{z\left(x,\lambda\right)}+\left[\lambda^2-q\left(x\right)\right]\,\overline{\dot{z}\left(x,\lambda\right)}=0.$$

Further, multiplying this equality by $z\left(x,\lambda\right)$, and relation (11) by $\overline{\dot{z}\left(x,\lambda\right)}$, subtracting the second result from the first one, and then integrating over x within $[0,\pi]$, due to (10) we obtain

$$2\lambda\int_0^{\pi}\left|z\left(x,\lambda\right)\right|^2dx-\alpha\left|z\left(\pi,\lambda\right)\right|^2=\overline{\dot{z}\left(\pi,\lambda\right)}\,z'\left(\pi,\lambda\right)-$$

$$-z\left(\pi,\lambda\right)\,\overline{\dot{z}'\left(\pi,\lambda\right)}-\overline{\dot{z}\left(0,\lambda\right)}\,z'\left(0,\lambda\right)+z\left(0,\lambda\right)\,\overline{\dot{z}'\left(0,\lambda\right)}=$$

$$=\dot{s}\left(\pi,\lambda\right)\,\left[\omega^2c\left(\pi,\lambda\right)+s'\left(\pi,\lambda\right)+\alpha\lambda s\left(\pi,\lambda\right)\right]-$$

$$-s\left(\pi,\lambda\right)\,\left[\omega^2\dot{c}\left(\pi,\lambda\right)+\dot{s}\left(\pi,\lambda\right)+\alpha\lambda\,\dot{s}\left(\pi,\lambda\right)+\alpha\, s\left(\pi,\lambda\right)\right].$$

$$=\dot{s}\left(\pi,\lambda\right)\,\left[\omega^2c\left(\pi,\lambda\right)+\left(\alpha\lambda+\beta\right)\,s\left(\pi,\lambda\right)+s'\left(\pi,\lambda\right)-\beta s\left(\pi,\lambda\right)\right]-$$

$$-s\left(\pi,\lambda\right)\,\left[\omega^2\dot{c}\left(\pi,\lambda\right)+\dot{s}\left(\pi,\lambda\right)+\left(\alpha\lambda+\beta\right)\,\dot{s}\left(\pi,\lambda\right)+\alpha\, s\left(\pi,\lambda\right)-\beta\dot{s}\left(\pi,\lambda\right)\right].$$

Hence, taking into account relation (3), we obtain

$$\dot{s}\left(\pi,\lambda\right)\varDelta\left(\lambda\right)-s\left(\pi,\lambda\right)\dot{\varDelta}\left(\lambda\right)=2\lambda\int_0^{\pi}\left|z\left(x,\lambda\right)\right|^2dx-\alpha\left|z\left(\pi,\lambda\right)\right|^2.$$

Taking this equality into account and arguing as in the proof of Theorem 2 in [39], we can verify that the function $\varDelta\left(\lambda\right)$ has two zeros between λ_{-1} and λ_1 with different signs, and between λ_{k-1} and λ_k with $k\leq-1$, as well as between λ_k and λ_{k+1} with $k\geq1$ exactly one zero. Therefore, (8) holds. ◀

3. Statement of the Inverse Problem. Uniqueness Theorem and Solution Algorithm

Let's denote $\sigma_k = sign\left[|\omega| - |s'(\pi, \lambda_k)|\right] (k = \pm 1, \pm 2, ...)$. The sequences $\{\mu_k\}$, $\{\sigma_k\}$ and the number ω will be called the spectral data of the boundary value problem L.

Consider the following inverse problem.

Inverse problem. Based on the given spectral data of the boundary value problem, construct the function $q(x)$ in equation (1) and the coefficients α, β in the boundary conditions (2).

The following uniqueness theorem is valid.

Theorem 2. *The boundary value problem L is uniquely determined by its spectral data.*

Proof. From the given spectrum $\{\mu_k\}$, one can uniquely determine the value of a, since, according to the asymptotic formula (4), we have

$$a = \lim_{k \to +\infty} \left(\mu_k - k + \frac{1}{2} \right). \tag{12}$$

Then, using (5), the parameter α is restored by the formula

$$\alpha = b \mathrm{tg} \pi a, \tag{13}$$

where $b = 1 + \omega^2$. Knowing the spectrum $\{\mu_k\}$, the parameters ω and α, the characteristic function $\Delta(\lambda)$ of the problem L can be restored as an infinite product as follows:

$$\Delta(\lambda) = \frac{b}{\cos \pi a} \prod_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{\mu_k - \lambda}{k - \frac{1}{2} \mathrm{sign} k}. \tag{14}$$

The proof of this fact is similar to the proof of Theorem 2 in [32].

Since the functions $c(\pi, \lambda)$, $s(\pi, \lambda)$ and $s'(\pi, \lambda)$ are even (see [35, p. 32]), it follows from (3) that

$$\Delta(-\lambda) = \omega^2 c(\pi, \lambda) + (-\alpha\lambda + \beta)\, s(\pi, \lambda) + s'(\pi, \lambda).$$

According to this formula and equality (3), the characteristic function $s(\pi, \lambda)$ of the boundary value problem L_0 is restored by the formula

$$s(\pi, \lambda) = \frac{\Delta(\lambda) - \Delta(-\lambda)}{2\alpha\lambda}. \tag{15}$$

From here we find the zeros λ_k $(k = \pm 1, \pm 2, ...)$ of the function $s(\pi, \lambda)$ for which the asymptotic formula (8) holds for $|k| \to \infty$. Using (8), one can uniquely determine the value A as follows:

$$A = \pi \lim_{k \to \infty} k(\lambda_k - k). \tag{16}$$

Moreover, knowing $\{\mu_k\}$ and $\{\lambda_k\}$ it is possible to recover the parameter β, since by virtue of (4), (6) and (8)

$$\beta = \frac{\pi}{b}\left(\alpha^2 + b^2\right) \lim_{k\to+\infty} k \left(\mu_k - \lambda_k + \frac{1}{2} - a\right). \tag{17}$$

Consider the functions

$$u_+(\lambda) = \omega^2 c(\pi,\lambda) + s'(\pi,\lambda), \tag{18}$$

$$u_-(\lambda) = \omega^2 c(\pi,\lambda) - s'(\pi,\lambda). \tag{19}$$

Using relation (3), we restore the function$u_+(\lambda)$ by the formula

$$u_+(\lambda) = \Delta(\lambda) - (\alpha\lambda + \beta)\, s(\pi,\lambda). \tag{20}$$

Consequently, it follows from the above reasoning that $u_+(\lambda)$, $s(\pi,\ \lambda)$, α, β are uniquely recovered from the spectrum $\{\mu_k\}$ and number ω. Now we will show that, in addition to the spectrum $\{\mu_k\}$ and the number ω, it is sufficient to specify another sequence $\{\sigma_k\}$ in order to restore the function $u_-(\lambda)$, and hence the function

$$s'(\pi,\lambda) = \frac{1}{2}\left[u_+(\lambda) - u_-(\lambda)\right], \tag{21}$$

which is the characteristic function of the boundary value problem L_1 generated by equation (1) and the boundary conditions $y(0) = y'(\pi) = 0$. Indeed, as in [19] (see also [33]), it is established that

$$u_-(\lambda_k) = (-1)^k \sigma_k \sqrt{u_+^2(\lambda_k) - 4\omega^2}. \tag{22}$$

Next, let's put

$$g(\lambda) = \frac{1-\omega^2}{4\omega^2} u_+(\lambda) + \frac{1+\omega^2}{4\omega^2} u_-(\lambda). \tag{23}$$

Hence, due to (18) and (19)

$$g(\lambda) = \frac{1}{2}\left[c(\pi,\lambda) - s'(\pi,\lambda)\right].$$

Then, according to the well-known fact [35, p. 253] the function $g(\lambda)$ is uniquely determined by the sequences $\{\lambda_k\}$, $\{\sigma_k\}$, $\{u_+(\lambda_k)\}$, according to the formula

$$g(\lambda) = 2s(\pi,\ \lambda) \sum_{k=1}^{\infty} \frac{\lambda_k g(\lambda_k)}{\left(\lambda^2 - \lambda_k^2\right) \dot{s}(\pi,\ \lambda_k)}, \tag{24}$$

where

$$g(\lambda_k) = \frac{1-\omega^2}{4\omega^2} u_+(\lambda_k) + \frac{1+\omega^2}{4\omega^2} (-1)^k \sigma_k \sqrt{u_+^2(\lambda_k) - 4\omega^2}.$$

Since, and are known, we define the function from (23):

$$u_-(\lambda) = \frac{\omega^2-1}{\omega^2+1} u_+(\lambda) + \frac{4\omega^2}{\omega^2+1} g(\lambda). \tag{25}$$

Hence, the characteristic function $s'(\pi,\lambda)$ of the boundary value problem L_1 is restored by formula (21), in which the functions $u_+(\lambda)$ and $u_-(\lambda)$ are determined by relations (18) and (25), respectively.

It is known [35, p. 248] that the zeros of the function $s'(\pi,\lambda)$ and sequence $\{\lambda_n\}$ (i.e., the spectra of problems L_0 and L_1) uniquely determine the coefficient $q(x)$ of equation (1).

Thus, given the sequences $\{\mu_k\}$, $\{\sigma_k\}$, and the number ω, the boundary value problem L is completely restored. ◀

We now present the main steps of the algorithm for solving the inverse problem.

Algorithm. Let the sequences $\{\mu_k\}$, $\{\sigma_k\}$ and the number ω, i.e. the spectral data of the boundary value problem L, be given.

1) Recover the quantities a and α from (12) and (13).

2) Using formulas (14) and (15), construct the characteristic functions $\mathit{\Delta}(\lambda)$ and $s(\pi,\lambda)$ of boundary value problems L and L_0 and find the zeros λ_k of the function $s(\pi,\lambda)$.

3) Define the quantities A and β from (16) and (17).

4) Recover function (18) from (20).

5) Find the values of the function (19) at points λ_k using (22).

6) Recover function (23) by interpolation formula (24).

7) Define (19) by (25).

8) Define the characteristic function $s'(\pi,\lambda)$ of the boundary value problem L_1 by formula (21).

9) Using the sequences of zeros of the functions $s(\pi,\lambda)$ and $s'(\pi,\lambda)$, construct the coefficient $q(x)$ of equation (1) by a well-known procedure (see [30], [35], [52]).

4. Sufficient Conditions for the Solvability of the Inverse Problem

Theorem 3. *In order for sequences of real numbers* $\{\mu_k\}$, $\{\sigma_k\}$ $\left(\sigma_k = -1, 0, 1;\ k = \pm1, \pm2, ...\right)$, *and real number* ω *were spectral data of a boundary value problem of the form* L, *it is sufficient that the following conditions are satisfied:*

1) the asymptotic formula (4) *holds, in which* B, a, b *are real numbers,* $0 < a < \frac{1}{2}$, $b = 1 + \omega^2$, $\omega \neq 0, \pm1$;

2) the terms of the given sequence $\{\mu_k\}$ *and the sequence* $\{\lambda_k\}$ *of zeros of the function* $\frac{\mathit{\Delta}(\lambda)-\mathit{\Delta}(-\lambda)}{\lambda}$ *are interleaved in the sense of* (9), *where* $\mathit{\Delta}(\lambda)$ *has the form* (14);

3) the inequality $|\mathit{\Delta}(\lambda_k)| \geq 2|\omega|$ *is true;*

4) $\sigma_k = \sigma'_k \operatorname{sign}(|\omega| - 1)$, *where* σ'_k *equal to zero if* $|\mathit{\Delta}(\lambda_k)| = 2|\omega|$, *and to* 1 *or* -1 *if* $|\mathit{\Delta}(\lambda_k)| > 2|\omega|$; *besides, there exists* $N > 0$*such that* $\sigma'_k = 1$ *for all*$|k| \geq N$.

Proof. Since the numbers μ_k obey the asymptotics (4), according to Lemma 1.1 in [21], the function (14) satisfies the representation

$$\mathit{\Delta}(\lambda) = \frac{b}{\cos\pi a}\left[\cos\pi(\lambda - a) + B\frac{\sin\pi(\lambda - a)}{\lambda - a} + \frac{g_0(\lambda - a)}{\lambda - a}\right], \tag{26}$$

where $g_0(\lambda) = M\cos\pi\lambda + \int_{-\pi}^{\pi}\tilde{g}_0(t)\, e^{it\lambda}dt,\ \tilde{g}_0(t) \in L_2[-\pi, \pi]\ ,\ g_0(0) = 0, M -$ some constant. Let $\alpha = b\,\mathrm{tg}\pi a$. Using the Paley-Wiener theorem [29, p. 47], from (26) we obtain

$$\Delta(\lambda) = b\cos\lambda\pi + \alpha\sin\lambda\pi + B_1\frac{\sin\lambda\pi}{\lambda} + B_2\frac{\cos\lambda\pi}{\lambda} + \frac{1}{\lambda}\int_{-\pi}^{\pi}\tilde{g}_1(t)\, e^{i\lambda t}dt\,, \tag{27}$$

where $B_1 = bB + \alpha M,\ B_2 = bM - \alpha B,\ g_1(t) \in L_2[-\pi, \pi]$. Therefore, the function $s(\lambda) = \frac{\Delta(\lambda)-\Delta(-\lambda)}{2\alpha\lambda}$ has the following representation:

$$s(\lambda) = \frac{\sin\lambda\pi}{\lambda} + B_3\pi\frac{\cos\lambda\pi}{\lambda^2} + \frac{1}{\lambda^2}\int_0^{\pi}\tilde{g}_2(t)\cos\lambda t dt\,, \tag{28}$$

where $B_3 = \dfrac{B_2}{\pi\alpha}$, $\tilde{g}_2(t) = \dfrac{1}{\alpha}[\tilde{g}_1(-t) + \tilde{g}_1(t)]$. By virtue of Lemma 3.4.2 in [35] (see also Lemma 12.3.3 in [36]), for the zeros λ_k $(\lambda_{-k} = -\lambda_k,\quad k = \pm 1, \pm 2, ...)$ of the function (28) for $|k| \to \infty$, the asymptotic formula

$$\lambda_k = k - \frac{B_3}{k} + \frac{\xi_k}{k}, \quad \{\xi_k\} \in l_2. \tag{29}$$

Let $\beta = B_1 + bB_3\pi$. It is easy to establish that the function

$$u_1(\lambda) = \Delta(\lambda) - (\alpha\lambda + \beta)\, s(\lambda)$$

is even. Indeed,

$$u_1(\lambda) = \Delta(\lambda) - (\alpha\lambda + \beta)\,\frac{\Delta(\lambda) - \Delta(-\lambda)}{2\alpha\lambda} = \frac{\Delta(\lambda) + \Delta(-\lambda)}{2} - \frac{\beta}{2\alpha}\cdot\frac{\Delta(\lambda) - \Delta(-\lambda)}{\lambda},$$

and therefore, $u_1(-\lambda) = u_1(\lambda)$. According to (27) and (28), for the function $u_1(\lambda)$ we get a representation of the form

$$u_1(\lambda) = b\cos\lambda\pi + (B_1 - \beta)\,\frac{\sin\lambda\pi}{\lambda} + \frac{1}{\lambda}\int_0^{\pi}p(t)\sin\lambda t dt\,, p(t) \in L_2[0, \pi]\,. \tag{30}$$

Consider the function

$$u_2(\lambda) = \frac{\omega^2 - 1}{\omega^2 + 1}u_1(\lambda) + \frac{4\omega^2}{\omega^2 + 1}r(\lambda)\,, \tag{31}$$

where $r(\lambda) = 2s(\lambda)\sum\limits_{k=1}^{\infty}\frac{\lambda_k r(\lambda_k)}{\left(\lambda^2 - \lambda_k^2\right)\dot{s}(\lambda_k)}$,

$$r(\lambda_k) = \frac{1 - \omega^2}{4\omega^2}u_1(\lambda_k) + \frac{1 + \omega^2}{4\omega^2}(-1)^k\sigma_k\sqrt{u_1^2(\lambda_k) - 4\omega^2}.$$

Taking into account relation (30) and estimates

$$\sin x = x + O\left(x^3\right), \quad \cos x = 1 + O\left(x^2\right),$$

$\sqrt{1+x} = 1 + \frac{x}{2} + O\left(x^2\right)$, $\frac{1}{1-x} = 1 + x + O\left(x^2\right)$ (as $x \to 0$), we have

$$r(\lambda_k) = \frac{1-\omega^2}{4\omega^2} b(-1)^k + \frac{\alpha_k}{k} + \frac{1+\omega^2}{4\omega^2}(-1)^k \sigma_k \sqrt{b^2 - 4\omega^2 + \frac{p_k}{k}} =$$

$$= \frac{1-\omega^2}{4\omega^2} b(-1)^k + \frac{1+\omega^2}{4\omega^2}(-1)^k \sigma'_k \operatorname{sign}(|\omega|-1) \sqrt{(\omega^2-1)^2 + \frac{p_k}{k}} + \frac{\alpha_k}{k} =$$

$$= \frac{1-\omega^4}{4\omega^2}(-1)^k + \frac{1+\omega^2}{4\omega^2}(-1)^k \left|\omega^2 - 1\right| \sigma'_k \operatorname{sign}(|\omega|-1) + \frac{m_k}{k} =$$

$$= \frac{1-\omega^4}{4\omega^2}(-1)^k + \frac{\omega^4-1}{4\omega^2}(-1)^k \sigma'_k + \frac{m_k}{k},$$

where $\{\alpha_k\}, \{p_k\}, \{m_k\} \in l_2$. Hence, by virtue of the fourth condition, we obtain that for all $|k|$ sufficiently large values of $r(\lambda_k) = \frac{m_k}{k}$. So, $\{\lambda_k r(\lambda_k)\} \in l_2$. Then it is known (see [35, p. 269]) that the function $r(\lambda)$ is an even entire function exponential type and admits the representation

$$r(\lambda) = \int_0^\pi h(t) \frac{\sin \lambda t}{\lambda} dt, \tag{32}$$

where $h(t) \in L_2[0,\pi]$. Since the function

$$s_1(\lambda) = \frac{1}{2}\left[u_1(\lambda) - u_2(\lambda)\right]$$

according to formulas (30)-(32) has the form

$$s_1(\lambda) = \cos \lambda\pi - B_3 \pi \frac{\sin \lambda\pi}{\lambda} + \frac{p_1(\lambda)}{\lambda},$$

$\left(p_1(\lambda) = \int_0^\pi \tilde{p}_1(t) \sin \lambda t dt,\ \tilde{p}_1(t) \in L_2[0,\pi]\right)$, then, by virtue of Lemma 3.4.2 in [35], its zeros ν_k $(\nu_{-k} = -\nu_k,\ k = \pm 1,\ \pm 2,\ ...)$ satisfy the asymptotic formula

$$\nu_k = k - \frac{1}{2}\operatorname{sign} k + \frac{B_3}{k} + \frac{\zeta_k}{k},\ \ \{\zeta_k\} \in l_2. \tag{33}$$

Using the second and third conditions of the theorem as in [19] (see also [24] and [39]), we can show that $\nu_m^2 < \lambda_m^2 < \nu_{m+1}^2$ $(m = 1, 2, ...)$. Thus, the zeros of the functions $\sqrt{\lambda} s\left(\sqrt{\lambda}\right)$ and $s_1\left(\sqrt{\lambda}\right)$ are interleaved and satisfy the asymptotic formulas (29) and (33), whence it follows (see Theorem 3.4.1 in [35]) that there exists a unique real function $q(x) \in L_2[0,\pi]$ such that the sequences of zeros under consideration are the spectra of boundary value problems L_0 and L_1 with the function $q(x)$ and equalities $s(\lambda) = s(\pi,\lambda)$, $s_1(\lambda) = s'(\pi,\lambda)$ are true. Taking into account these equalities, it is easy to prove that the spectrum of the constructed boundary value problem coincides with the sequence $\{\mu_k\}$. The theorem is proved.

Remark 1. It is easy to see that the conditions of Theorem 3 are also necessary under the condition (T). However, the function $q(x)$ constructed in the proof of this theorem may not satisfy inequality (7).

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