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ON THE ASYMPTOTICS OF EIGENVALUES OF THE NEUMANN PROBLEM FOR THE SCHRÖDINGER OPERATOR

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In memory of M. G. Gasymov on his 85th birthday

Abstract. *In this paper, we study the change in the eigenvalues of the Neumann problem for the Schrödinger equation with respect to the radius of the ball. We prove the self-adjointness of the Schrödinger operator with a spherically symmetric homogeneous potential and obtain asymptotic formulas for the eigenvalues of the Neumann problem as the radius of the ball tends to zero.*

Keywords: Schrödinger operator, Neumann problem, spherical-symmetric potential, asymptotic formulas for eigenvalues

Mathematics Subject Classification (2020): 33C55, 34B24, 35J10

1. Introduction

Let us denote by H_a^W the Schrödinger operator acting in $L^2(B(0,a))$ according to the formula

$$H_a^W u = -\triangle u + W(r)\,u\,,$$

with domain

$$D\left(H_a^W\right) = \left\{ u(\xi) \in L^2(B(0,a)) : \ \left.\frac{\partial u(\xi)}{\partial r}\right|_{|\xi|=a} = 0, \forall \varphi \in S^{n-1}, \ \ H_a^W u \in L^2(B(0,a)) \right\},$$

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where $\xi = (\xi_1, \xi_2, ..., \xi_n)$, $B(0,a)$ is a ball in n-dimensional Euclidean space $\mathbb{R}^n$ of radius a with center at the origin, S^{n-1} is $(n-1)$–dimensional unit sphere, i.e.

$$S^{n-1} = \left\{ (\eta_1, \eta_2, ..., \eta_n) \in R^n : \quad \eta_1^2 + \eta_2^2 + ... \eta_n^2 = 1 \right\},$$

$r = |\xi|$, $\triangle = \sum\limits_{k=1}^{n} \frac{\partial^2}{\partial \xi_k^2}$ is the classical Laplace operator, $W(r)$ is a sufficiently smooth real homogeneous function, i.e. $W(tr) = t^\gamma W(r)$, $\gamma > -2$.

Our goal in this work is to study the asymptotic behavior of the eigenvalues of the operator H_a^W as the radius of the ball $B(0,a)$ tends to zero. Note that this work is motivated by the work of M. Dauge and B. Helffer [2], [3].

In [2], the authors studied the behavior of the eigenvalues of the Sturm-Liouville operator for the Dirichlet and Neumann problems with respect to the length of the interval. In [3], in the two-dimensional case, they studied the behavior of the eigenvalues of the Neumann problem for the Schrödinger operator when a crack propagates in a plate, and in the three-dimensional case, they studied the asymptotic forms of the eigenvalues of the Neumann problem associated with the Schrödinger operator on spherical sectors as the diameter of the region tends to zero or infinity.

Note that similar studies for general Laplace operators without potential have been known for a long time: the first work in this direction was apparently Hadamard's work [7] published in 1908 for the Dirichlet problem.

In [4] and [8], they studied the asymptotic behavior of eigenvalues for arbitrary (separated or coupled) self-adjoint regular boundary-value problems for the Sturm-Liouville operator with respect to the length of the interval.

It is known that the eigenvalues of the Dirichlet problem decrease with increasing the domain of definition [1]. This result, generally speaking, is not valid for the Neumann problem. Uchiyama in [13] showed that in this case all situations can be encountered.

As mentioned above, the cases $n = 1, 2, 3$ were studied in [2]-[4], [8]. Therefore, we will assume that the dimension of the Euclidean space $\mathbb{R}^n$ is greater than three.

2. Self-adjointness of the Operator H_a^W

Theorem 1. *Let the dimension of the Euclidean space $\mathbb{R}^n$ be greater than three. Then the operator H_a^W is self-adjoint in $L^2(B(0,a))$.*

Proof. Using tensor representation (see [11, p. 160])

$$L^2(B(0,a)) = L_{2,r^{n-1}}(0,a) \otimes L^2\left(S^{n-1}\right)$$

of the space $L^2(B(0,a))$ and direct sum expansion of the Hilbert space $L^2\left(S^{n-1}\right)$ (see [6, p. 27])

$$L^2\left(S^{n-1}\right) = \bigoplus_{l=0}^{\infty} E_l,$$

the operator H_a^W is represented in the following form:

$$H_a^W = \bigoplus_{l=0}^{\infty} \left(L_{a,l}^W \otimes \triangle_{S^{n-1}}^l \right),$$

where E_l are eigensubspaces of spherical harmonics of the Laplace-Beltrami operator $\triangle_{S^{n-1}}$ on S^{n-1}, corresponding to eigenvalues $l\,(l+n-2)\,,\ l = 0, 1, 2, \ldots,\ \triangle_{S^{n-1}}^l$ is restriction of the Laplace-Beltrami operator to eigensubspaces E_l, $L_{a,l}^W$ is the operator into $L_{2,r^{n-1}}(0,a)$, acting according to the rule

$$L_{a,l}^W u = -\frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial u\,(r)}{\partial r} \right) + \frac{l\,(l+n-2)}{r^2} u\,(r) + W\,(r)\,u\,(r)\ ,$$

with the domain

$$D\left(L_{a,l}^W\right) = \left\{ u'\,(a) = 0, u\,(r) \in L_{2,r^{n-1}}(0,a)\,, u'\,(r) \in L_{2,r^{n-1}}(0,a) \right\}.$$

Because the operators $\triangle_{S^{n-1}}^l$ ($l \in \mathbb{N}^* = 0, 1, 2, ...,$) are self-adjoint, then to prove the theorem it is enough to prove that for any elements of the set $\mathbb{N}^*$ the operators $L_{a,l}^W$ are self-adjoint in the space $L_{2,r^{n-1}}(0,a)$.

Let $0 \le x \le 1$. Assume $r = ax$. Using a unitary transformation $Uf = x^{-\frac{n-1}{2}} f$ we move from the operator $L_{a,l}^W$ to the operator $L_{p,h}^W$, acting in $L_2(0,1)$, according to the rule

$$L_{p,h}^W y = -y''\,(x) + \frac{p^2 - \frac{1}{4}}{x^2} y\,(x) + hW\,(x)\,y\,(x)\ ,$$

with the domain

$$D\left(L_{p,h}^W\right) = \left\{ y\,(x) \in L_2(0,1):\quad \frac{xy'\,(x) - \frac{n-1}{2} y\,(x)}{x} \in L_2(0,1)\,, \right.$$
$$\left. y'\,(1) = \frac{n-1}{2} y\,(1)\,,\quad L_{p,h}^W y \in L_2(0,1) \quad \right\},$$

where $h = a^{\gamma+2}$, $p = \frac{n}{2} + l - 1$. It is obvious that the operators $L_{a,l}^W$ and $L_{p,h}^W$ are unitary equivalent.

We introduce in $L_2(0,1)$ a minimal operator $\tilde{L}_p^0$, generated by the differential expression

$$l_p^0 \equiv -\frac{d^2}{dx^2} + \frac{p^2 - \frac{1}{4}}{x^2}.$$

It is known that (see [10, p. 285]) in the case of $p \ge 1$, the defect index of the operator $\tilde{L}_p^0$ is $(1,1)$. From the equality $p = \frac{n}{2} + l - 1$ it follows that if $n \ge 4$, then for any $l \in \mathbb{N}^*$ the defect index of the operator L_p^0 is $(1,1)$. The conditions $\frac{xy'(x) - \frac{n-1}{2} y(x)}{x} \in L_2(0,1)$ and $y'\,(1) = \frac{n-1}{2} y\,(1)$ mean that the operator L_p^0 , acting in $L_2(0,1)$ according to the rule $L_p^0 y = l_p^0 y$, with the domain

$$D\left(L_p^0\right) = \left\{ y\,(x) \in L_2(0,1):\quad \frac{xy'\,(x) - \frac{n-1}{2} y\,(x)}{x} \in L_2(0,1)\,, \right.$$

$$\left. y'(1) = \frac{n-1}{2} y(1), \quad L_p^0 y \in L_2(0,1) \right\}$$

is one of the self-adjoint extensions of the operator $\tilde{L}_p^0$. If the function $W(x)$ is sufficiently smooth on the interval [0,1], then the self-adjointness of the operator $L_{p,h}^W$ is evident. If the function $W(x)$ is sufficiently smooth on the interval (0,1], but has a singularity at zero of type $|x|^\gamma$, $-2 < \gamma < 0$, then the self-adjointness of the operator $L_{p,h}^W$ is proved as in work [3]. The self-adjointness of the operator H_a^W follows from the representation $H_a^W = \bigoplus_{l=0}^{\infty} \left(L_{a,l}^W \otimes \triangle_{S^{n-1}}^l \right)$ and from the unitary equivalence of the operators $L_{a,l}^W$ and $L_{p,h}^W$. ◄

3. Spectral Problem for the Operator $L_{p,h}^W$

Consider the following problem:

$$-\triangle\psi(\xi) + W(r)\psi(\xi) = \mu\psi(\xi), \xi \in B(0,a), \tag{1}$$

$$\left. \frac{\partial\psi(\xi)}{\partial r} \right|_{r=a} = 0, \forall\varphi \in S^{n-1}, \tag{2}$$

where μ is the spectral parameter.

To solve problem (1), (2) it is convenient to switch to polar coordinates. Knowing that in polar coordinates the Laplace operator has the form (see [6, p. 19])

$$\triangle = -\frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial}{\partial r} \right) + \frac{-\triangle_{S^{n-1}}}{r^2},$$

and moving to polar coordinates, we rewrite problem (1) and (2) in the following form:

$$\begin{cases} -\frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial\psi(r,\varphi)}{\partial r} \right) + \frac{-\triangle_{S^{n-1}}\psi(r,\varphi)}{r^2} + W(r)\psi(r,\varphi) = \mu\psi(r,\varphi), r \in (0,a), \\ \left. \frac{\partial\psi(r,\varphi)}{\partial r} \right|_{r=a} = 0, \forall\varphi \in S^{n-1}. \end{cases}$$

Considering that the non-negative operator $-\triangle_{S^{n-1}}$ has eigenvalues of the form $l(l+n-2)$, $l = 0,1,2,...,$ (see [6, p. 27]), then it is sufficient to study the asymptotic forms of the eigenvalues and eigenfunctions as $a \to 0$ of the following problem:

$$\begin{cases} lu \equiv -\frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial u(r)}{\partial r} \right) + \frac{l(l+n-2)}{r^2} u(r) + W(r) u(r) = \mu u(r), \quad r \in (0,a), \\ u'(a) = 0, u(r) \in L_{2,r^{n-1}}(0,a), u'(r) \in L_{2,r^{n-1}}(0,a), lu \in L_{2,r^{n-1}}(0,a) \ . \end{cases} \tag{3}$$

Let $0 \leq x \leq 1$. Assume $r = ax$ and $V(x) = u(ax)$. Taking into account the homogeneity of the function $W(r)$, from (3) for $V(x)$ we obtain the following equation:

$$\frac{1}{a^2} \left\{ -\frac{1}{x^{n-1}} \frac{d}{dx} \left(x^{n-1} \frac{dV(x)}{dx} \right) + \frac{l(l+n-2)}{x^2} V(x) + a^{\gamma+2} W(x) V(x) \right\} = \mu V(x).$$

Setting $h = a^{\gamma+2}$, $\lambda^2 = \mu a^2$, we finally obtain the following problem for $V(x)$ in the space $L_{2,x^{n-1}}(0,1)$:

$$\begin{cases} l_h V \equiv -\frac{1}{x^{n-1}}\frac{d}{dx}\left(x^{n-1}\frac{dV(x)}{dx}\right) + \frac{l(l+n-2)}{x^2}V(x) + hW(x)V(x) = \lambda^2 V(x), \ x \in (0,1), \\ V'(1) = 0, V(x) \in L_{2,x^{n-1}}(0,1), \quad V'(x) \in L_{2,x^{n-1}}(0,1), l_h V \in L_{2,x^{n-1}}(0,1) \ . \end{cases} \tag{4}$$

Setting

$$V(x) = e^{-\frac{1}{2}\int \frac{n-1}{x}dx} y(x) = x^{-\frac{n-1}{2}} y(x), \tag{5}$$

from (4) we obtain

$$l_p^W y \equiv -y''(x) + \frac{p^2 - \frac{1}{4}}{x^2} y(x) + hW(x)y(x) = \lambda^2 y(x), \ x \in (0,1), \tag{6}$$

$$y'(1) = \frac{n-1}{2} y(1), y(x) \in L_2(0,1), \frac{xy'(x) - \frac{n-1}{2}y(x)}{x} \in L_2(0,1), l_p y \in L_2(0,1), \tag{7}$$

where

$$p = \sqrt{\left(\frac{n-2}{2}\right)^2 + l(l+n-2)} = \frac{n}{2} + l - 1.$$

Indeed, referring to the relation

$$V'(x) = x^{-\frac{n-1}{2}} \frac{xy'(x) - \frac{n-1}{2}y(x)}{x} \tag{8}$$

and taking into account that $V'(x) \in L_{2,x^{n-1}}(0,1)$, we obtain

$$\int_0^1 x^{-(n-1)} \left(\frac{xy'(x) - \frac{n-1}{2}y(x)}{x}\right)^2 x^{n-1} dx = \int_0^1 \left(\frac{xy'(x) - \frac{n-1}{2}y(x)}{x}\right)^2 dx < +\infty.$$

Turning again to formulas (5) and (8), we obtain

$$V(1) = y(1),$$

$$V'(1) = y'(1) - \frac{n-1}{2} y(1),$$

From the condition $V'(1) = 0$ it follows that $y'(1) - \frac{n-1}{2}y(1) = 0$.

The remaining conditions included in (7) are easily verified.

To solve problem (6), (7), we first study the corresponding unperturbed boundary-value problem.

4. Eigenvalues and Eigenfunctions of the Unperturbed Problem

Theorem 2. *The eigenvalues $\lambda_k, k = 1, 2, ...,$ of the operator L_p^0 are the solutions to the equation*

$$tg\lambda = \frac{1}{n-1}\lambda, \tag{9}$$

and the corresponding eigenfunctions $y_k(x)$, $k = 1, 2, 3, ...,$ have the following form:

$$y_k(x) = x^{\frac{n-1}{2}+l}\lambda_k{}^{\frac{n}{2}+l-1}\left[1-\frac{(\lambda_k x)^2}{2\,(n+2l)}+...\right], \quad k = 1, 2, 3, ... \,. \tag{10}$$

Proof. It is known (see [12]) that for odd n (in this case $p \notin \mathbb{N}^*$) the fundamental solutions of the equation

$$-\frac{1}{x^{n-1}}\frac{d}{dx}\left(x^{n-1}\frac{dV\,(x)}{dx}\right) + \frac{l\,(l+n-2)}{x^2}V\,(x) = \lambda^2 V\,(x)$$

are the functions $V_1(x) = x^{-\frac{n-2}{2}}J_{\frac{n}{2}+l-1}(\lambda x)$ and $V_2(x) = x^{-\frac{n-2}{2}}J_{-(\frac{n}{2}+l-1)}(\lambda x)$, where $J_\nu(t)$ is Bessel function of the 1st kind. And for even n (in this case $p \in \mathbb{N}^*$) the fundamental solutions of equation (9) are the functions $V_1(x) = x^{-\frac{n-2}{2}}J_{\frac{n}{2}+l-1}(\lambda x)$ and $V_2(x) = x^{-\frac{n-2}{2}}Y_{\frac{n}{2}+l-1}(\lambda x)$, where $Y_\nu(\tau)$ is Bessel function of the 2nd kind. From this and relation (5) it follows that for odd n the fundamental solutions of the equation

$$-y''\,(x) + \frac{p^2-\frac{1}{4}}{x^2}y\,(x) = \lambda^2 y\,(x) \tag{11}$$

are the functions $y_{(1)}(x) = x^{\frac{n-1}{2}}V_1(x) = x^{\frac{1}{2}}J_{\frac{n}{2}+l-1}(\lambda x)$ and $y_{(2)}(x) = x^{\frac{n-1}{2}}V_2(x) = x^{\frac{1}{2}}J_{-(\frac{n}{2}+l-1)}(\lambda x)$, and for even n the fundamental solutions of equation (11) are the functions $y_{(1)}(x) = x^{\frac{1}{2}}J_{\frac{n}{2}+l-1}(\lambda x)$ and $y_{(2)}(x) = x^{\frac{1}{2}}Y_{\frac{n}{2}+l-1}(\lambda x)$.

Using the behavior of functions $J_{\frac{n}{2}+l-1}(\lambda x)$ and $Y_{\frac{n}{2}+l-1}(\lambda x)$ for small positive values of x (see [9, p. 172, formulas (5.16.1) and (5.16.2)]), i.e.

$$J_{\frac{n}{2}+l-1}(\lambda x) \sim \frac{(\lambda x)^{\frac{n}{2}+l-1}}{2^{\frac{n}{2}+l-1}\Gamma\left(\frac{n}{2}+l\right)}, \tag{12}$$

$$Y_{\frac{n}{2}+l-1}(\lambda x) \sim \frac{2^{\frac{n}{2}+l-1}\Gamma\left(\frac{n}{2}+l-1\right)}{\pi(\lambda x)^{\frac{n}{2}+l-1}},$$

where $\Gamma\left(\frac{n}{2}+l\right)$ is Euler's gamma function, we find that only the function $y_{(1)}(x)$ satisfies conditions (10). Indeed, from (12) it follows that the following asymptotic formulas hold:

$$y_{(1)}(x) \sim \frac{\lambda^{\frac{n}{2}+l-1}}{2^{\frac{n}{2}+l-1}\Gamma\left(\frac{n}{2}+l\right)}x^{\frac{n-1}{2}+l},$$

$$y'_{(1)}(x) \sim \frac{\lambda^{\frac{n}{2}+l-1}}{2^{\frac{n}{2}+l-1}\Gamma\left(\frac{n}{2}+l\right)}\left(\frac{n-1}{2}+l\right)x^{\frac{n-3}{2}+l}.$$

From these asymptotic formulas we have

$$\frac{xy'_{(1)}(x)-\frac{n-1}{2}y_{(1)}(x)}{x} \sim l\cdot\frac{\lambda^{\frac{n}{2}+l-1}}{2^{\frac{n}{2}+l-1}\Gamma\left(\frac{n}{2}+l\right)}x^{\frac{n-3}{2}+l}.$$

From formula (12) it immediately follows that if $n \geq 4$, then for any $l \in \mathbb{N}^*$

$$\frac{xy'_{(1)}(x)-\frac{n-1}{2}y_{(1)}(x)}{x} \in L_2\,(0,1)\,.$$

Now let's find the eigenvalues and eigenfunctions of the operator L_p^0. It is easy to check that the function $y_{(1)}(x,\lambda)$, which is a solution to equation (11), satisfies the integral equation

$$y_{(1)}(x,\lambda) = \sin\lambda x\ +\frac{p^2-\frac{1}{4}}{\lambda}\int_x^1\frac{\sin\lambda\,(\xi-x)}{\xi^2}y_{(1)}(\xi,\lambda)d\xi. \qquad (13)$$

From equation (13) we have

$$y'_{(1)}(x) = \lambda\cos\lambda x\ +\left(p^2-\frac{1}{4}\right)\int_x^1\frac{\cos\lambda\,(\xi-x)}{\xi^2}y_{(1)}(\xi,\lambda)d\xi. \qquad (14)$$

From relations (13) and (14), and from the boundary condition

$$y'\,(1,\lambda)-\frac{n-1}{2}y\,(1,\lambda)=0$$

it follows that the eigenvalues of the operator L_p^0 are a solution to equation (9). From the expansion of the Bessel function $J_\nu\,(\tau)$ in powers of τ [12] it follows that the eigenfunctions of the operator L_p^0 will be in the form (10). ◀

5. Asymptotic Formulas for the Eigenvalues

Theorem 3. *The following asymptotic formulas are valid for the eigenvalues of the operator* H_a^W*:*

$$\mu_k\,(a)=\frac{\beta_{k,0}}{a^2}+\beta_{k,1}a^\gamma+\beta_{k,2}a^{2\gamma+2}+\beta_{k,3}a^{3\gamma+6}+O\left(a^{4\gamma+8}\right),$$

$$k=1,2,...,(0<a\to 0)\,, \qquad (15)$$

where the coefficients $\beta_{k,j}$*,* $j=0,1,2,3$*, are determined by formulas* (16)-(18)*.*

Proof. Knowing the eigenvalues and eigenfunctions of the operator L_p^0, using the Rayleigh-Schrödinger method (see [5]) we find the asymptotic forms of the eigenvalues and eigenfunctions of the operator L_p^W as $h \to 0$ in the following form:

$$\lambda_k^2(h) = \beta_{k,0} + \beta_{k,1}h + \beta_{k,2}h^2 + \beta_{k,3}h^3 + O\left(h^4\right),$$

$$y_{k,h}(x) = y_k(x) + y_{k+1}(x)\,h + y_{k+2}(x)\,h^2 + y_{k+3}(x)\,h^3 + O\left(h^4\right),$$

where $\beta_{k,0}$ is the kth eigenvalue of the operator L_p^0,

$$\beta_{k,1} = \int_0^1 W(x)\, y_k^2(x)\, dx, \tag{16}$$

$$\left(L_p^0 - \beta_{k,0}\right) y_{k+1}(x) = \beta_{k,1} y_k(x) - W(x)\, y_k(x),$$

$$\beta_{k,2} = \int_0^1 W(x)\, y_k(x)\; y_{k+1}(x)\, dx, \tag{17}$$

$$\left(L_p^0 - \beta_{k,0}\right) y_{k+2}(x) = \beta_{k,1} y_{k+1}(x) - W(x)\, y_{k+1}(x) + \beta_{k,2} y_k(x),$$

$$\beta_{k,3} = \int_0^1 W(x)\, y_k(x)\; y_{k+2}(x)\, dx + \frac{1}{2}\beta_{k,1}\int_0^1 W(x)\, y_{k+1}^2(x)\, dx, \tag{18}$$

$$\left(L_p^0 - \beta_{k,0}\right) y_{k+3}(x) = \beta_{k,2} y_{k+2}(x) - W(x)\, y_{k+2}(x) + \beta_{k,3} y_{k+1}(x).$$

Considering that the eigenvalues and eigenfunctions of problem (1) and (2) are related to the eigenvalues and eigenfunctions of problem (6) and (7) by the relations

$$\mu_k(a) = \frac{\lambda_k^2}{a^2}, \quad k = 1, 2, ..., u_k(r) = \left(\frac{r}{a}\right)^{-\frac{n-1}{2}} y_k\left(\frac{r}{a}\right),$$

from (16) taking into account $h = a^{\gamma+2}$ we finally obtain formula (15). ◀

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