Baku Mathematical Journal
2023, Vol. 2, No 2, PP. 199-207
https://doi.org/10.32010/j.bmj.2023.18

ON THE COMPLETENESS OF A SYSTEM OF DECREASING ELEMENTARY SOLUTIONS FOR ONE CLASS OF FOURTH-ORDER OPERATOR-DIFFERENTIAL EQUATIONS ON THE SEMI-AXIS

E.S. RZAYEV

Received: 05.07.2023 / Revised: 11.08.2023 / Accepted: 18.08.2023

Abstract. *In this work, we study a homogeneous fourth-order elliptic operator-differential equation with an unbounded operator under non-homogeneous boundary conditions on the semi-axis. We obtain sufficient conditions for the completeness of decreasing elementary solutions of the equation under study in the space of all its solutions from a fourth-order Sobolev type space.*

Keywords: polynomial operator pencil, self-adjoint operator, operator-differential equation, derivative chain of eigen- and adjoined vectors, system of decreasing elementary solutions

Mathematics Subject Classification (2020): 34G10, 34L10, 35J40

1. Introduction

In a separable Hilbert space H, consider a fourth-order polynomial operator pencil

$$P(\lambda) = \lambda^4 E + A^4 + \sum_{j=1}^{4} \lambda^{4-j} A_j, \tag{1}$$

where λ is the spectral parameter, E is the unit operator, A is a self-adjoint positive definite operator ($A = A^* \geq cE$, $c > 0$) with a completely continuous inverse operator A^{-1}, and the operators A_j, $j = 1,2,3,4$, are such that $A_j A^{-j}$, $j = 1,2,3,4$, are bounded in H.

As is known, the domain of definition of the operator A^α ($\alpha \geq 0$) becomes the Hilbert space H_α with respect to the scalar product $(x,y)_\alpha = (A^\alpha x, A^\alpha y)$, $x, y \in D(A^\alpha)$; for $\alpha = 0$ we assume $H_0 = H$.

Elvin S. Rzayev
Institute of Mathematics and Mechanics, Baku, Azerbaijan
E-mail: Elvin.Rzayev88@gmail.com

Below we use the notation from [1].

Let us denote by $L_2(\mathbb{R}_+;H)$ the Hilbert space of all vector-functions $f(t)$, defined almost everywhere in $\mathbb{R}_+=(0,+\infty)$, with values in H and quadratically integrable in $\mathbb{R}_+$, and

$$\|f\|_{L_2(\mathbb{R}_+;H)}=\left(\int_0^{+\infty}\|f(t)\|^2\,dt\right)^{1/2}<+\infty.$$

Following the book [6], we introduce the Hilbert space of vector-functions

$$W_2^4(\mathbb{R}_+;H)=\left\{u(t):A^4u(t)\in L_2(\mathbb{R}_+;H),\ u^{(4)}(t)\in L_2(\mathbb{R}_+;H)\right\}$$

with the norm

$$\|u\|_{W_2^4(\mathbb{R}_+;H)}=\left(\left\|A^4u\right\|^2_{L_2(\mathbb{R}_+;H)}+\left\|u^{(4)}\right\|^2_{L_2(\mathbb{R}_+;H)}\right)^{1/2}.$$

Here and further, derivatives are understood in the sense of distribution theory [6].

Further, by $L(X,Y)$ we mean the set of linear bounded operators acting from a Hilbert space X to another Hilbert space Y. Let us fix some operator $K\in L\left(H_{\frac{5}{2}},H_{\frac{3}{2}}\right)$ and define a subspace in $W_2^4(\mathbb{R}_+;H)$:

$$W_{2,K}^4(\mathbb{R}_+;H)=\left\{u(t):u(t)\in W_2^4(\mathbb{R}_+;H),\ u(0)=0,\ u''(0)=Ku'(0)\right\}.$$

From the trace theorem [6] it follows that $W_{2,K}^4(\mathbb{R}_+;H)$ is defined correctly.

We associate the boundary-value problem with pencil (1)

$$P\left(\frac{d}{dt}\right)u(t)=0,\quad t\in\mathbb{R}_+, \tag{2}$$

$$u(0)=\varphi,\ u''(0)-Ku'(0)=\psi,\ \varphi\in H_{\frac{7}{2}},\ \psi\in H_{\frac{3}{2}},\ K\in L\left(H_{\frac{5}{2}},H_{\frac{3}{2}}\right). \tag{3}$$

Definition 1. *If for any $\varphi\in H_{\frac{7}{2}}$, $\psi\in H_{\frac{3}{2}}$ one can find a vector-function $u(t)\in W_2^4(\mathbb{R}_+;H)$ satisfying equation (2) almost everywhere in $\mathbb{R}_+$, as well as the boundary conditions (in the sense of convergence)*

$$\lim_{t\to+0}\|u(t)-\varphi\|_{H_{\frac{7}{2}}}=0,\quad \lim_{t\to+0}\|u''(t)-Ku'(t)-\psi\|_{H_{\frac{3}{2}}}=0$$

and the assessment takes place

$$\|u\|_{W_2^4(\mathbb{R}_+;H)}\le const\left(\|\varphi\|_{H_{\frac{7}{2}}}+\|\psi\|_{H_{\frac{3}{2}}}\right),$$

then we will say that the boundary-value problem (2), (3) is regularly solvable, and $u(t)$ is called a regular solution of the boundary-value problem (2), (3).

Definition 2. *If the equation $P(\lambda_0)x_0=0$ has a non-trivial solution x_0, then λ_0 is called the eigenvalue of the operator pencil $P(\lambda)$, and x_0 – eigenvector of the operator pencil $P(\lambda)$ corresponding to the eigenvalue λ_0.*

Definition 3. *Let λ_0 be an eigenvalue and x_0 be one of the eigenvectors corresponding to the value λ_0. If, for the vectors $x_1,\ x_2,\ \ldots,\ x_m$, the following equalities are satisfied*

$$\sum_{k=0}^{4} \frac{1}{k!} \frac{d^k P(\lambda)}{d\lambda^k}\bigg|_{\lambda=\lambda_0} x_{p-k} = 0,$$

$$p = 0,\ 1,\ 2,\ \ldots\ ,\ m \quad (x_{-1} = x_{-2} = x_{-3} = x_{-4} = 0),$$

then the system of vectors $x_1,\ x_2,\ \ldots,\ x_m$ is called a chain of adjoined vectors to the eigenvector x_0.

Let us denote by $\sigma_\infty(H)$ the set of completely continuous operators acting in H. If $Q\in \sigma_\infty(H)$, then $(Q^*Q)^{1/2}$ is a completely continuous self-adjoint operator in H. The eigenvalues of the operator $(Q^*Q)^{1/2}$ are called s-numbers of the operator Q. Let us number the non-zero s-numbers of the operator Q in descending order, taking into account their multiplicity and denote

$$\sigma_p = \left\{ Q\ : Q\in \sigma_\infty(H),\ \sum_{j=1}^{\infty} s_j^p(Q) < \infty \right\} \quad (0 < p < \infty).$$

From the results of [5] and the assumptions that $A^{-1} \in \sigma_\infty(H)$, $A_j A^{-j} \in L(H,H)$, $j = 1,\ 2,\ 3,\ 4$, $\left(E + A_4 A^{-4}\right)^{-1} \in L(H,H)$ it follows that the spectrum of the pencil $P(\lambda)$ is discrete, which means the existence of the resolvent $P^{-1}(\lambda)$ for all $\lambda\in \mathbb{C}$, with the exception of the set of isolated eigenvalues $\{\lambda_n\}$, which can only have a limit point at infinity.

According to [5], each eigenvalue λ_n can be adjoined with a canonical system of eigen- and adjoined vectors $x_{0,n}\ ,\ x_{1,n}\ ,\ \ldots\ ,\ x_{m,n}$ of the pencil $P(\lambda)$. Then the functions

$$u_{h,n}(t) = e^{\lambda_n t}\left(x_{h,n} + \frac{t}{1!} x_{h-1,n} + \ldots + \frac{t^h}{h!} x_{0,n} \right), \quad h = 0, 1, \ldots, m, \tag{4}$$

satisfy equation (2) and are called its elementary solutions. It is clear that for $Re\lambda_n < 0$ these solutions decrease and belong to the space $W_2^4\left(\mathbb{R}_+; H\right)$.

In this paper, we obtain conditions for the completeness of decreasing elementary solutions of equation (2) in the space of all regular solutions of the boundary-value problem (2), (3).

Note that the completeness of elementary solutions in some solution spaces was previously indicated in [2]–[4], [10]–[15].

2. Main Results

In [1], the regular solvability of boundary-value problem (2) and (3) was established.

Theorem 1. *Let $A = A^* \geq cE$, $c > 0$, $B = A^{\frac{3}{2}} K A^{\frac{-5}{2}}$, $ReB \geq 0$, $A_j A^{-j} \in L(H,\ H)$, $j = 1,\ 2,\ 3,\ 4$, and the inequality holds $\sum\limits_{j=1}^{4} c_j \left\| A_j A^{-j} \right\|_{H \to H} < 1$, where $c_1 = 1, c_2 = \frac{1}{2}$, $c_3 = \frac{1}{\sqrt{2}}, c_4 = 1$. Then the boundary-value problem (2), (3) is regularly solvable.*

If we take $K = 0$ in the boundary conditions (3), then they will be rewritten in the form

$$u(0) = \varphi,\ u''(0) = \widetilde{\psi},\ \varphi \in H_{\frac{7}{2}},\ \widetilde{\psi} \in H_{\frac{3}{2}}. \tag{5}$$

Note that the same Theorem 1 for the boundary-value problem (2), (5) is given in [8].

Using solutions (4) for $Re\lambda_n < 0$, we determine the vector

$$\widetilde{x}_{h,n}^{(0,2)} = \left\{ x_{h,n}^{(0)}, x_{h,n}^{(2)} \right\} \in H_{7/2} \oplus H_{3/2},$$

where

$$x_{h,n}^{(0)} \equiv u_{h,n}(0), \quad x_{h,n}^{(2)} \equiv u_{h,n}^{''}(0), \quad h = 0, 1, \ldots, m.$$

We will call the system $\left\{ \widetilde{x}_{h,n}^{(0,2)} \right\}$ a derivative chain of eigen- and adjoined vectors of the operator pencil $P(\lambda)$ generated by the boundary-value problem (2), (5).

Note that in [9, p. 152-154] under conditions $A = A^* \geq cE$, $c > 0$, $A^{-1} \in \sigma_\infty(H)$, $A_j A^{-j} \in L(H,H)$, $j = 1,2,3,4$, $\sum\limits_{j=1}^{4} c_j \left\| A_j A^{-j} \right\|_{H \to H} < 1$, where $c_1 = 1, c_2 = \frac{1}{2}$, $c_3 = \frac{1}{\sqrt{2}}$, $c_4 = 1$, $\left(E + A_4 A^{-4}\right)^{-1} \in L(H,H)$ and fulfillment of one of the conditions $A^{-1} \in \sigma_p, 0 < p \leq 1$ or $A^{-1} \in \sigma_p$, $0 < p < \infty$, $A_j A^{-j} \in \sigma_\infty(H)$, $j = 1,2,3,4$, the completeness of the derivative chain $\left\{ \widetilde{x}_{h,n}^{(0,2)} \right\}$ has been proven in the space $H_{7/2} \oplus H_{3/2}$.

Let us define the following derivative chain:

$$\left\{ x_{h,n}^{(0,2,1)} \right\},\ x_{h,n}^{(0,2,1)} = \left\{ x_{h,n}^{(0)},\quad x_{h,n}^{(2,1)} \right\} \in H_{7/2} \oplus H_{3/2},$$

$$x_{h,n}^{(0)} = u_{h,n}(0),\ x_{h,n}^{(2,1)} \equiv x_{h,n}^{(2)} - K x_{h,n}^{(1)} = u_{h,n}''(0) - K u_{h,n}'(0), \quad K \in L\left(H_{\frac{5}{2}}, H_{\frac{3}{2}} \right).$$

It is obvious that the derivative chain $\left\{ x_{h,n}^{(0,2,1)} \right\}$ corresponds to the boundary-value problem (2), (3).

It is known [7, p. 126], that to prove the completeness of the derivative chain $\left\{ x_{h,n}^{(0,2,1)} \right\}$ in the space $H_{7/2} \oplus H_{3/2}$, it is sufficient to show its equivalence to the derivative chain $\left\{ \widetilde{x}_{h,n}^{(0,2)} \right\}$.

As noted above, the existence of a unique regular solution $u(t)$ of the boundary-value problem (2), (3) for any $\varphi \in H_{\frac{7}{2}}$, $\psi \in H_{\frac{3}{2}}$, was proven in [1] under the conditions of Theorem 1. And in [8], under the same conditions of Theorem 1, the existence of a unique regular solution $\widetilde{u}(t)$ of the boundary-value problem (2), (5) was proven for any $\varphi \in H_{\frac{7}{2}}$, $\widetilde{\psi} \in H_{\frac{3}{2}}$, in case $K = 0$. Then the inequalities hold

$$\|u\|_{W_2^4(\mathbb{R}_+;H)} \le const \left(\|\varphi\|_{H_{\frac{7}{2}}} + \|\psi\|_{H_{\frac{3}{2}}} \right),$$

$$\|\widetilde{u}\|_{W_2^4(\mathbb{R}_+;H)} \le const \left(\|\varphi\|_{H_{\frac{7}{2}}} + \left\|\widetilde{\psi}\right\|_{H_{\frac{3}{2}}} \right).$$

On the other hand, by the trace theorem [6], the following inequalities are true:

$$\|\varphi\|_{H_{\frac{7}{2}}} + \|\psi\|_{H_{\frac{3}{2}}} \le const \, \|u\|_{W_2^4(\mathbb{R}_+;H)},$$

$$\|\varphi\|_{H_{\frac{7}{2}}} + \left\|\widetilde{\psi}\right\|_{H_{\frac{3}{2}}} \le const \, \|\widetilde{u}\|_{W_2^4(\mathbb{R}_+;H)}.$$

To prove the completeness of the chain $\left\{x_{h,n}^{(0,2,1)}\right\}$, we find a bounded invertible operator acting in the space $H_{7/2} \oplus H_{3/2}$ and transferring the chain $\left\{\widetilde{x}_{h,n}^{(0,2)}\right\}$ to $\left\{x_{h,n}^{(0,2,1)}\right\}$.

Obviously, for $\left\{x_{h,n}^{(0)}, \ \ x_{h,n}^{(2)}\right\} \in H_{7/2} \oplus H_{3/2}$ the vector-function $u_{h,n}(t)$ is a solution to equation (2) with boundary conditions

$$u_{h,n}(0) = x_{h,n}^{(0)}, \ u''_{h,n}(0) = \ x_{h,n}^{(2)}.$$

That is why

$$\|u_{h,n}\|_{W_2^4(\mathbb{R}_+;H)} \le c_1 \left(\left\|x_{h,n}^{(0)}\right\|_{H_{\frac{7}{2}}} + \left\| \ x_{h,n}^{(2)}\right\|_{H_{\frac{3}{2}}} \right).$$

And by the trace theorem [6]

$$\|u_{h,n}\|_{W_2^4(\mathbb{R}_+;H)} \ge c_2 \left(\left\|x_{h,n}^{(0)}\right\|_{H_{\frac{7}{2}}} + \left\| \ x_{h,n}^{(2)}\right\|_{H_{\frac{3}{2}}} \right).$$

Similarly, for $\left\{x_{h,n}^{(0)}, \ \ x_{h,n}^{(2,1)}\right\} \in H_{7/2} \oplus H_{3/2}$ the vector-function $u_{h,n}(t)$ is a solution to equation (2) with the boundary conditions

$$u_{h,n}(0) = x_{h,n}^{(0)}, \ \ u''_{h,n}(0) - K u'_{h,n}(0) = x_{h,n}^{(2)} - K x_{h,n}^{(1)} = \ x_{h,n}^{(2,1)},$$

at that

$$\|u_{h,n}\|_{W_2^4(\mathbb{R}_+;H)} \le d_1 \left(\left\|x_{h,n}^{(0)}\right\|_{H_{\frac{7}{2}}} + \left\| \ x_{h,n}^{(2,1)}\right\|_{H_{\frac{3}{2}}} \right).$$

And based on the trace theorem [6], we have

$$\|u_{h,n}\|_{W_2^4(\mathbb{R}_+;H)} \ge d_2 \left(\left\|x_{h,n}^{(0)}\right\|_{H_{\frac{7}{2}}} + \left\| \ x_{h,n}^{(2,1)}\right\|_{H_{\frac{3}{2}}} \right).$$

Now for all $\left\{x_{h,n}^{(0)},\ \ x_{h,n}^{(2)}\right\}$ we define the operator S acting in the space $H_{7/2}\oplus H_{3/2}$ as follows:

$$S\ :\ H_{7/2}\oplus H_{3/2}\to\ H_{7/2}\oplus H_{3/2},$$

$$S\left(\left\{x_{h,n}^{(0)},\ \ x_{h,n}^{(2)}\right\}\right)=\left\{x_{h,n}^{(0)},\ \ x_{h,n}^{(2,1)}\right\}.$$

It is obvious that

$$\left\|S\left(\left\{x_{h,n}^{(0)},\ \ x_{h,n}^{(2)}\right\}\right)\right\|_{H_{7/2}\oplus H_{3/2}}=\left\|\left\{x_{h,n}^{(0)},\ \ x_{h,n}^{(2,1)}\right\}\right\|_{H_{7/2}\oplus H_{3/2}}=$$

$$=\left\|x_{h,n}^{(0)}\right\|_{H_{\frac{7}{2}}}+\left\|x_{h,n}^{(2,1)}\right\|_{H_{\frac{3}{2}}}\leq\frac{1}{d_2}\ \|u_{h,n}\|_{W_2^4(\mathbb{R}_+;H)}\leq$$

$$\leq\frac{c_1}{d_2}\left(\left\|x_{h,n}^{(0)}\right\|_{H_{\frac{7}{2}}}+\left\|\ x_{h,n}^{(2)}\right\|_{H_{\frac{3}{2}}}\right)=\frac{c_1}{d_2}\left\|\left\{x_{h,n}^{(0)},\ \ x_{h,n}^{(2)}\right\}\right\|_{H_{7/2}\oplus H_{3/2}}.$$

Consequently, the operator S is defined on the everywhere dense set $\left\{\widetilde{x}_{h,n}^{(0,2)}\right\}$ (completeness of the derivative chain $\left\{\widetilde{x}_{h,n}^{(0,2)}\right\}$ in the space $H_{7/2}\oplus H_{3/2}$ was proven in [9]) and bounded. Therefore, it can be extended by continuity to the entire space $H_{7/2}\oplus H_{3/2}$. On the other hand, it is clear that $S\left(\{0,\ 0\}\right)=\{0,\ 0\}$. From this it follows that S is one-to-one, i.e., from any $\left\{x_{h,n}^{(0)},\ \ x_{h,n}^{(2,1)}\right\}$ there exists $\left\{x_{h,n}^{(0)},\ \ x_{h,n}^{(2)}\right\}$, such that $S\left(\left\{x_{h,n}^{(0)},\ \ x_{h,n}^{(2)}\right\}\right)=\left\{x_{h,n}^{(0)},\ \ x_{h,n}^{(2,1)}\right\}$.

As a result, the operator $S\ :\ H_{7/2}\oplus H_{3/2}\ \to\ H_{7/2}\oplus H_{3/2}$ is one-to-one and continuous. Then S^{-1} is also a bounded operator.

Thus, since the derivative chain $\left\{\widetilde{x}_{h,n}^{(0,2)}\right\}$ is complete in the space $H_{7/2}\oplus H_{3/2}$ and $S\left(\left\{\widetilde{x}_{h,n}^{(0,2)}\right\}\right)=\left\{x_{h,n}^{(0,2,1)}\right\}$, then we obtain that the derivative chain $\left\{x_{h,n}^{(0,2,1)}\right\}$ is also a complete system in the space $H_{7/2}\oplus H_{3/2}$.

The above reasoning allows us to formulate the main result of the work.

Theorem 1 establishes sufficient conditions to ensure the existence of a unique solution from the space $W_2^4\left(\mathbb{R}_+;H\right)$ for the boundary-value problem (2), (3) for any $\varphi\in H_{7/2}$, $\psi\in H_{3/2}$. We denote the set of all such solutions by $W^{(0,2,1)}(P)$. From the theorem on intermediate derivatives and the trace theorem [6] it follows that the set $W^{(0,2,1)}(P)$ is a closed subspace of the space $W_2^4\left(\mathbb{R}_+;H\right)$. Now we prove the completeness of the system of decreasing elementary solutions of equation (2) in the space $W^{(0,2,1)}(P)$.

The following theorem holds.

Theorem 2. *Let all the conditions of Theorem 1 be satisfied and* $\left(E+A_4A^{-4}\right)^{-1}\in L(H,H)$. *In addition, one of the following conditions is true:*

a) $A^{-1}\in\sigma_p,\ 0<p\leq 1$;

b) $A^{-1}\in\sigma_p,\ 0<p<\infty,\ \ A_jA^{-j}\in\sigma_\infty(H),\ j=1,2,3,4.$

Then the system of decreasing elementary solutions to the boundary value problem (2), (3) is complete in the space $W^{(0,2,1)}(P)$.

Proof. As noted, the decreasing elementary solution $u_{h,n}(t)$ has the representation

$$u_{h,n}(t) = e^{\lambda_n t}\left(x_{h,n} + \frac{t}{1!}x_{h-1,n} + \ldots + \frac{t^h}{h!}x_{0,n}\right), \quad h = 0, 1, \ldots, m,$$

where $Re\lambda_n < 0$, and $x_{0,n},\ x_{1,n},\ \ldots,\ x_{m,n}$ is the canonical system of eigen- and adjoined vectors corresponding to the eigenvalue λ_n.

Taking into account that $K \in L\left(H_{\frac{5}{2}}, H_{\frac{3}{2}}\right)$, by the trace theorem [6], for any $u(t) \in W_2^4(\mathbb{R}_+; H)$ we obtain:

$$\|u(0)\|_{H_{\frac{7}{2}}} + \|u''(0) - Ku'(0)\|_{H_{\frac{3}{2}}} \leq$$

$$\leq \|u(0)\|_{H_{\frac{7}{2}}} + \|u''(0)\|_{H_{\frac{3}{2}}} + \|Ku'(0)\|_{H_{\frac{3}{2}}} \leq const\, \|u\|_{W_2^4(\mathbb{R}_+;H)}\,.$$

On the other hand, from the uniqueness of solutions to the boundary-value problem (2), (3) (see Theorem 1) we have

$$\|u\|_{W_2^4(\mathbb{R}_+;H)} \leq const\left(\|\varphi\|_{H_{\frac{7}{2}}} + \|\psi\|_{H_{\frac{3}{2}}}\right). \tag{6}$$

If we take into account the conditions of the theorem and the equivalence of systems $\left\{\widetilde{x}_{h,n}^{(0,2)}\right\}$ and $\left\{x_{h,n}^{(0,2,1)}\right\}$ in the space $H_{7/2} \oplus H_{3/2}$, then we can say that the derivative chain $\left\{x_{h,n}^{(0,2,1)}\right\}$ is complete in the space $H_{7/2} \oplus H_{3/2}$, i.e., for any $\varepsilon > 0$ there exist a number N and numbers $c_{h,n}^N$ such that

$$\left\|\varphi - \sum_{n=1}^{N}\sum_{h} c_{h,n}^N x_{h,n}^{(0)}\right\|_{H_{\frac{7}{2}}} < \varepsilon, \tag{7}$$

$$\left\|\psi - \sum_{n=1}^{N}\sum_{h} c_{h,n}^N x_{h,n}^{(2,1)}\right\|_{H_{\frac{3}{2}}} < \varepsilon. \tag{8}$$

Because $x_{h,n}^{(0)} = u_{h,n}(0)$, $x_{h,n}^{(2,1)} = x_{h,n}^{(2)} - Kx_{h,n}^{(1)} = u''_{h,n}(0) - Ku'_{h,n}(0)$, $\varphi = u(0)$, $\psi = u''(0) - Ku'(0)$, then for the solution $u(t) - \sum\limits_{n=1}^{N}\sum\limits_{h} c_{h,n}^N u_{h,n}(t)$ due to inequality (6) we have:

$$\left\|u(t) - \sum_{n=1}^{N}\sum_{h} c_{h,n}^N u_{h,n}(t)\right\|_{W_2^4(\mathbb{R}_+;H)} \leq$$

$$\leq const\left(\left\|\varphi - \sum_{n=1}^{N}\sum_{h} c_{h,n}^N x_{h,n}^{(0)}\right\|_{H_{\frac{7}{2}}} + \left\|\psi - \sum_{n=1}^{N}\sum_{h} c_{h,n}^N x_{h,n}^{(2,1)}\right\|_{H_{\frac{3}{2}}}\right). \tag{9}$$

Now, if we take into account inequalities (7) and (8) in inequality (9), we obtain:

$$\left\| u(t) - \sum_{n=1}^{N} \sum_{h} c_{h,n}^{N} u_{h,n}(t) \right\|_{W_2^4(\mathbb{R}_+;H)} < const \cdot \varepsilon = \varepsilon_1.$$

The last inequality means that the system of decreasing elementary solutions to the boundary-value problem (2), (3) is complete in the space $W^{(0,2,1)}(P)$. The theorem has been proven. ◄

Acknowledgements. The author expresses his gratitude to prof. A.R. Aliev for consultation and advice on the issue of proving the completeness of the derivative chain of eigen- and adjoined vectors.

References

1. Al-Aidarous E.S., Aliev A.R., Rzayev E.S., Zedan H.A. Fourth order elliptic operator-differential equations with unbounded operator boundary conditions in the Sobolev-type spaces. *Bound. Value Probl.*, 2015, **2015** (191), pp. 1-14.
2. Aliev A.R., Mohamed A.S. Completeness of elementary solutions for a class of fourth order operator-differential equations. *News Baku State Univer. Ser. Phys.-Math. Sci.*, 2011, (4), pp. 24-28.
3. Gasymov M.G., Mirzoev S.S. Solvability of boundary value problems for second-order operator-differential equations of elliptic type. *Differ. Equ.*, 1992, **28** (4), pp. 528-536.
4. Elbably A.L. On the completeness of a system of elementary solutions for an operator-differential equation. *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics and Mechanics*, 2012, **32** (4), pp. 35-42.
5. Keldysh M.V. On the completeness of the eigenfunctions of some classes of non-selfadjoint linear operators. *Russ. Math. Surv.*, 1971, **26** (4), pp. 15-44.
6. Lions J.L., Magenes E. *Non-Homogeneous Boundary Value Problems and Applications.* Dunod, Paris, 1968; Mir, Moscow, 1971; Springer, Berlin, 1972.
7. Mil'man V.D. Geometric theory of Banach spaces. Part I. The theory of basis and minimal systems. *Uspekhi Mat. Nauk*, 1970, **25** (3), pp. 113-174 (in Russian).
8. Mirzoev S.S. Well-posedness conditions for solution of boundary value problems for operator-differential equations. *Soviet Math. Dokl.*, 1983, **28**, pp. 629-632.
9. Mirzoev S.S. *Questions of the theory of solvability of boundary value problems for operator-differential equations in Hilbert space and spectral problems associated with them.* Thesis D.Sci., Baku, 1994, 229 pp. (in Russian).
10. Mirzoev S.S., Salimov M.Yu. Completeness of elementary solutions to a class of second order operator-differential equations. *Siberian Math. J.*, 2010, **51** (4), pp. 648–659.
11. Orazov M.B. On the completeness of systems of elementary solutions for some operator differential equations on a half-line and on an interval. *Soviet Math. Dokl.*, 1979, **20** (4), pp. 347-352.

12. Orazov M.B. On the completeness of the elementary solutions of the problem of stationary oscillations of a finite cylinder. *Russ. Math. Surv.*, 1980, **35** (5), pp. 267-268.
13. Orazov M.B. Completeness of elementary solutions in the problem of steady-state vibrations of an elastic rectangular plate. *Funct. Anal. Appl.*, 1980, **14** (1), pp. 63-64.
14. Ustinov Yu.A., Yudovich V.I. On the completeness of a system of elementary solutions of the biharmonic equation in a semi-strip. *J. Appl. Math. Mech.*, 1973, **37**, pp. 665-674.
15. Vorovich I.I., Kovalchuk V.E. On the basis properties of one system of homogeneous solutions. *Prikl. Mat. Mekh.*, 1967, **31** (5), pp. 861-869 (in Russian).