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ASYMPTOTICS OF EIGENVALUES AND EIGENFUNCTIONS OF A DISCONTINUOUS BOUNDARY VALUE PROBLEM WITH A SPECTRAL PARAMETER IN THE TRANSMISSION CONDITION

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Abstract. *We determine asymptotics of eigenvalues and eigenfunctions of a discontinuous boundary value problem with a spectral parameter in the transmission condition.*

Keywords: boundary-value problem, eigenvalue, eigenfunction, asymptotics, transmission condition

Mathematics Subject Classification (2020): 34B05, 34B08, 34E05

1. Problem Statement

Consider the following boundary value problem:

$$-y'' = \lambda y, \quad x \in \left[0, \frac{q}{p}\right) \cup \left(\frac{q}{p}, 1\right], \tag{1}$$

$$\left.\begin{array}{c} y(0) = y(1) = 0, \\ y\left(\frac{q}{p} - 0\right) = y\left(\frac{q}{p} + 0\right), \\ y'\left(\frac{q}{p} - 0\right) - y'\left(\frac{q}{p} + 0\right) = \lambda m y\left(\frac{q}{p}\right), \end{array}\right\} \tag{2}$$

here $q \in \mathbb{Z}_+, \quad p \in \mathbb{N}, \quad q < p$, $gcd(q,p) = 1$, $0 \neq m \in \mathbb{C}$.

In the sequel we will use the denotations $\lambda = \rho^2$ and $Im\rho = \tau$.

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The problem (1), (2), in special cases $q=1,\ p=2$ and $q=1,\ p=3$, were investigated in [2] and [3]. The asymptotic behavior of eigenvalues and eigenfunctions for the equation

$$-y''+q(x)\,y=\lambda y$$

with boundary conditions (2) were studied in [1] for the case of transmission condition at midpoint.

2. Main Results

The main result of the paper is the following

Theorem. *Eigenvalues of the problem* (1), (2) *are asymptotically simple and can be represented as a union of* $p-q+2$ *sequences*

$$|\lambda_{1,1}|\le|\lambda_{1,2}|\le|\lambda_{1,3}|\le\ldots,$$
$$|\lambda_{2,1}|\le|\lambda_{2,2}|\le|\lambda_{2,3}|\le\ldots,$$
$$|\lambda_{3,1}|\le|\lambda_{3,2}|\le|\lambda_{3,3}|\le\ldots,$$
$$\ldots\ldots\ldots$$
$$|\lambda_{p-q+2,0}|\le|\lambda_{p-q+2,1}|\le|\lambda_{p-q+2,2}|\le\ldots,$$

counted with their multiplicity. For these sequences the following asymptotics hold:

$$\sqrt{\lambda_{1,n}}=p\pi n,\ \sqrt{\lambda_{2,n}}=p\pi n+\frac{p}{m\,(p-q)\,q\pi n}+O\left(\frac{1}{n^3}\right),$$

$$\sqrt{\lambda_{3,n}}=\frac{p\pi}{q}n+\frac{1}{\pi mn}+O\left(\frac{1}{n^2}\right),n\ne 0\ \ (mod\ q)\,,$$

$$\sqrt{\lambda_{l,n}}=p\pi n+\frac{p}{p-q}\pi\,(l-3)+\frac{1}{m\,(p-q)\,\pi n}+O\left(\frac{1}{n^2}\right),\ 4\le l\le p-q+2,$$

as $n\to\infty$.

For the eigenfunctions $y_{1,n}(x)$, $y_{2,n}(x)$, $y_{3,n}(x)$ *and* $y_{l,n}(x)$, *that correspond to eigenvalues* $\lambda_{1,n}=\rho^2_{1,n}$, $\lambda_{2,n}=\rho^2_{2,n}$, $\lambda_{3,n}=\rho^2_{3,n}$ *and* $\lambda_{l,n}=\rho^2_{l,n}$, *respectively, the following asymptotics hold:*

$$y_{1,n}(x)=\sin p\pi nx,\quad x\in[0,1]\,;$$

$$y_{2,n}(x)=\begin{cases}\gamma_{2,n}\sin p\pi nx+O\left(\frac{1}{n}\right), & x\in\left[0,\frac{p}{q}\right),\\ \gamma'_{2,n}\sin p\pi n\,(x-1)+O\left(\frac{1}{n}\right), & x\in\left(\frac{p}{q},\ 1\right];\end{cases}$$

$$y_{3,n}(x)=\begin{cases}\gamma_{3,n}\sin\frac{p\pi n}{q}x+O\left(\frac{1}{n}\right), & x\in\left[0,\frac{p}{q}\right),\\ \gamma'_{3,n}\sin\frac{p\pi n}{q}\,(x-1)+O\left(\frac{1}{n}\right), & x\in\left(\frac{p}{q},\ 1\right];\end{cases}$$

$$y_{l,n}(x)=\begin{cases}\gamma_{l,n}\sin\left(p\pi n+\frac{p}{p-q}\pi(l-3)\right)\ \ x+O\left(\frac{1}{n}\right), & x\in\left[0,\frac{p}{q}\right),\\ \gamma'_{l,n}\sin\left(p\pi n+\frac{p}{p-q}\pi(l-3)\right)\ (x-1)+O\left(\frac{1}{n}\right), & x\in\left(\frac{p}{q},\ 1\right],\end{cases}$$

$4\le l\le p-q+2$, *as* $n\to\infty$, *where*

$$\gamma_{2,n}=(-1)^p+O\left(\frac{1}{n^2}\right),\quad \gamma'_{2,n}=\frac{q}{q-p}+O(\frac{1}{n^2}),$$

$$\gamma_{3,n}=-\cos\left(\frac{p-q}{q}\pi n\right)\ +O(\frac{1}{n}),\ \gamma'_{3,n}=O(\frac{1}{n}),$$

$$\gamma_{l,n}=(-1)^{pn-1+l}+O(\frac{1}{n^2}),$$

$$\gamma'_{l,n}=-\cos\left(\frac{q}{p-q}\pi(l-3)\right)+m\left(p\pi n+\frac{p}{p-q}\pi(l-3)\right)\sin\left(\frac{q}{p-q}\pi(l-3)\right)+O\left(\frac{1}{n^2}\right),$$

as $n\to\infty$.

Proof. It is straightforward to check that $\lambda=0$ is not an eigenvalue of the problem (1), (2). For any $\lambda\neq 0$ the solution $y(x,\lambda)$ of the problem (1), (2) is in the form

$$y(x,\lambda)=\begin{cases}C_1y_1(x,\lambda), & \text{if}\ \ 0\le x<\frac{q}{p},\\ C_2y_2(x,\lambda), & \text{if}\ \ \frac{q}{p}<x\le 1,\end{cases}$$

where $y_1(x,\lambda)=\sin\rho x$, $y_2(x,\lambda)=\sin\rho(x-1)$, and C_1 and C_2 are yet unknown complex numbers. $\lambda\neq 0$ is an eigenvalue of the problem (1), (2) if and only if C_1 and C_2 are nontrivial solutions of the following homogeneous system of linear equations:

$$\begin{cases}C_1\sin\frac{q}{p}\rho-C_2\sin\left(\frac{q}{p}-1\right)\rho=0,\\ C_1\rho\cos\frac{q}{p}\rho-C_2\rho\cos\left(\frac{q}{p}-1\right)\rho=C_1\rho^2m\sin\frac{q}{p}\rho.\end{cases}$$

Hence, eigenvalues of the problem (1), (2) are nonzero roots of the following equation:

$$\triangle(\rho)=\begin{vmatrix}A_{11}(\rho) & A_{12}(\rho)\\ A_{21}(\rho) & A_{22}(\rho)\end{vmatrix}=0,$$

where

$$A_{11}(\rho)=\sin\frac{q}{p}\rho,\quad A_{12}(\rho)=\sin\left(1-\frac{q}{p}\rho\right),\quad A_{21}(\rho)=\rho\cos\frac{q}{p}\rho-\rho^2m\sin\frac{q}{p}\rho,$$

$$A_{22}(\rho)=-\rho\cos\left(1-\frac{q}{p}\right)\rho.$$

Therefore, we obtain

$$\triangle(\rho)=-\rho\sin\rho+\rho^2\ m\ \sin\frac{q}{p}\rho\sin\left(1-\frac{q}{p}\right)\rho=0. \tag{3}$$

Now we find asymptotics of roots of the equation (3). From (3) it follows that, $\lambda_{1,n} = \rho_{1,n}^2 = (p\pi n)^2,\ n = 1,2,\dots$ are simple eigenvalues of the problem (1), (2). For sufficiently small $\alpha > 0$, set

$$Q = \bigcap_{n=0}^{\infty}\left(\left\{\rho \in \mathbb{C}\colon \left|\rho - \frac{p}{q}\pi n\right| > \alpha\pi\right\} \bigcap \left\{\rho \in \mathbb{C}\colon \left|\rho - \frac{p}{p-q}\pi n\right| > \alpha\pi\right\}\right).$$

Taking into account that for all $\rho \in Q$

$$|-\rho\sin\rho\ | = |\rho| \cdot |\sin\rho\ | \le |\rho| \cdot e^{|\tau|};$$

and

$$\left|\rho^2 m\sin\frac{q}{p}\rho\,\sin\left(1-\frac{q}{p}\right)\rho\ \right| = |\rho|^2 \cdot |m| \cdot \left|\sin\frac{q}{p}\rho\,\sin\left(1-\frac{q}{p}\right)\rho\ \right| \ge$$

$$\ge C \cdot |\rho|^2 \cdot |m| \cdot e^{\frac{q}{p}|\tau|} \cdot e^{\left(1-\frac{q}{p}\right)|\tau|} = C \cdot |\rho|^2 \cdot |m| \cdot e^{|\tau|},$$

where C is an absolute constant, by virtue of Rouche's theorem it follows that, zeroes of the function $\triangle(\rho)$ of which absolute values are sufficiently large and are different from $\rho_{1,n}$ lie in small neighborhoods of roots of the equation $\rho^2\ m\ sin\frac{q}{p}\rho\sin\left(1-\frac{q}{p}\right)\ \rho = 0$. Then, all such zeroes of $\triangle(\rho)$ lie in the set $\mathbb{C}/Q$, hence, have bounded imaginary parts. By Rouche's theorem, these zeroes of the function $\triangle(\rho)$ are asymptotically simple and are in the form $\frac{p\pi n}{q} + \alpha_n$ and $\frac{p\pi n}{p-q} + \theta_n$, where $\alpha_n \ne 0, \theta_n \ne 0$ and $\alpha_n \to 0,\ \theta_n \to 0$ as $n \to \infty$.

Now let us study asymptotic behaviour of the sequence $\{\alpha_n\}$ as $n \to \infty$. Firstly, consider the case of $n \ne 0\ (mod\ q)$. Take $\rho = \frac{p\pi n}{q} + \alpha_n$ in equation (4). We get

$$\sin\left(\frac{p\pi n}{q} + \alpha_n\right) = \left(\frac{p\pi n}{q} + \alpha_n\right) \cdot m \cdot \sin\frac{q}{p}\left(\frac{p\pi n}{q} + \alpha_n\right) \cdot \sin\frac{p-q}{p}\left(\frac{p\pi n}{q} + \alpha_n\right).$$

Hence,

$$\sin\left(\frac{q}{p}\alpha_n\right) = \frac{1}{m} \cdot \frac{1}{\frac{p\pi n}{q} + \alpha_n} \cdot \frac{\sin\left(\frac{p\pi n}{q} + \alpha_n\right)}{\sin\left(\frac{p\pi n}{q} + \frac{p-q}{p}\alpha_n\right)}. \tag{4}$$

From the last equality it follows that

$$\alpha_n = O\left(\frac{1}{n}\right), \quad \text{as} \quad n \to \infty. \tag{5}$$

(4) implies that

$$\frac{q}{p}\alpha_n + O\left((\alpha_n)^3\right) = \frac{1}{m} \cdot \left(\frac{1}{\frac{p\pi n}{q} + \alpha_n} - \frac{q}{p\pi n} + \frac{q}{p\pi n}\right) \times$$

$$\times \frac{\sin\left(\frac{p\pi n}{q} + \alpha_n\right)}{\sin\left(\frac{p\pi n}{q} + \alpha_n\right)\cos\frac{q}{p}\alpha_n - \cos\left(\frac{p\pi n}{q} + \alpha_n\right)\sin\frac{q}{p}\alpha_n} =$$

$$= \frac{1}{m} \cdot \left(\frac{-q\alpha_n}{\left(\frac{p\pi n}{q} + \alpha_n\right) \cdot p\pi n} + \frac{q}{p\pi n} \right) \times$$

$$\times \frac{1}{\cos \frac{q}{p}\alpha_n - \cot\left(\frac{p\pi n}{q} + \alpha_n\right) \cdot \sin \frac{q}{p}\alpha_n} =$$

$$= \frac{1}{m} \cdot \left(\frac{q}{p\pi n} - \frac{q \cdot \alpha_n}{\left(\frac{p\pi n}{q} + \alpha_n\right) \cdot p\pi n} \right) \times$$

$$\times \frac{1}{1 + O\left(\alpha_n^2\right) - \cot\left(\frac{p\pi n}{q} + \alpha_n\right) \cdot \sin \frac{q\alpha_n}{p}} \quad .$$

Taking into account (5) and the fact that for sufficiently large $n \neq 0$ $(mod\ q)$ $\left|\cot\left(\frac{p\pi n}{q} + \alpha_n\right)\right| \leq \alpha$ ($\alpha > 0$ is independent of n), we get

$$\frac{q}{p}\alpha_n + O\left(\frac{1}{n^3}\right) = \frac{1}{m} \cdot \left(\frac{q}{p\pi n} + O\left(\frac{1}{n^3}\right)\right) \cdot \frac{1}{1 + O\left(\frac{1}{n}\right)}.$$

Since

$$\frac{1}{1 + O\left(\frac{1}{n}\right)} = 1 + O\left(\frac{1}{n}\right), \quad \text{as} \quad n \to \infty,$$

we have

$$\alpha_n = \left(\frac{1}{m\pi n} + O\left(\frac{1}{n^3}\right)\right) \cdot \left(1 + O\left(\frac{1}{n}\right)\right) + O\left(\frac{1}{n^3}\right) = \frac{1}{m\pi n} + O\left(\frac{1}{n^2}\right).$$

Therefore,

$$\rho_{3,n} = \frac{p\pi}{q}n + \frac{1}{\pi m n} + O\left(\frac{1}{n^2}\right), n \neq 0 \ \ (mod\ q)\,, \quad n = 1, 2, ...,$$

is a subsequence of the set of roots of (3).

Now let $n = 0$ $(mod\ q)$. Then $n = qk,\ k \in \mathbb{N}$. Denoting $\beta_k = \alpha_{qk}$, from (4) we get

$$\sin\left(\frac{q}{p}\beta_k\right) = \frac{1}{m} \cdot \frac{1}{p\pi k + \beta_k} \cdot \frac{\sin\left(p\pi k + \beta_k\right)}{\sin\left(p\pi k + \frac{p-q}{p}\beta_k\right)} =$$

$$= \frac{1}{m} \cdot \frac{1}{p\pi k + \beta_k} \cdot \frac{\sin \beta_k}{\sin \frac{p-q}{p} \cdot \beta_k}.$$

The above equality implies that $\beta_k = O\left(\frac{1}{k}\right)$. Therefore, from (4) we get:

$$\frac{q}{p}\beta_k + O\left(\frac{1}{k^3}\right) = \frac{1}{m} \cdot \left(\frac{1}{p\pi k} + O\left(\frac{1}{k^3}\right)\right) \cdot \frac{\beta_k + O\left(\frac{1}{k^3}\right)}{\frac{p-q}{p}\beta_k + O\left(\frac{1}{k^3}\right)} =$$

$$= \frac{1}{m} \cdot \left(\frac{1}{p\pi k} + O\left(\frac{1}{k^3} \right) \right) \left(\frac{p}{p-q} + O\left(\frac{1}{k^2} \right) \right) = \frac{1}{m\,(p-q)\,\pi k} + O\left(\frac{1}{k^3} \right).$$

Finally, we get

$$\beta_k = \frac{p}{mq(p-q)\pi k} + O\left(\frac{1}{k^3} \right).$$

Therefore,

$$\rho_{2,n} = p\pi n + \frac{p}{mq(p-q)\pi n} + O\left(\frac{1}{n^3} \right), n \neq 0 \;\; (mod\; q)\,, \quad n = 1, 2, ...,$$

is a subsequence of the set of roots of (3).

By the same way it is proved that

$$\theta_n = \frac{1}{m(p-q)\pi n} + O\left(\frac{1}{n^2} \right).$$

Hence, for the sequence $\rho_{l,n}$ of roots of the equation (3) we obtained the following asymptotic formula

$$\rho_{l,n} = p\pi n + \frac{p}{p-q}\pi\,(l-2) + \frac{1}{m\,(p-q)\,\pi n} + O\left(\frac{1}{n^2} \right),\; 4 \leq l \leq p-q+2, \;\; n = 0, 1, 2, \ldots.$$

Now, let us study the asymptotic behavior of eigenfunctions of the problem (1), (2). From the asymptotic equalities obtained for $\rho_{1,n},\ \rho_{2,n},\ \rho_{3,n}$and $\rho_{l,n}$ $(4 \leq l \leq p-q+2)$ and the expression for $A_{22}(\rho)$, for sufficiently large n we have:

$$A_{22}\,(\rho_{1,n}) \neq 0 \text{ (for all } n),\ A_{22}(\rho_{2,n}) \neq 0,\ A_{22}(\rho_{3,n}) \neq 0 \;\text{ and }\; A_{22}(\rho_{l,n}) \neq 0.$$

Hence, for the sufficiently large n the eigenfunctions of the problem (1), (2) corresponding to eigenvalues $\lambda_{1,n} = (\rho_{1,n})^2,\ \lambda_{2,n} = (\rho_{2,n})^2,\ \lambda_{3,n} = (\rho_{3,n})^2$, and $\lambda_{l,n} = (\rho_{l,n})^2$ $(3 \leq l \leq p-q+2)$ will be

$$y_{1,n}\,(x) = \begin{cases} \frac{1}{\rho_{1,n}} A_{22}\,(\rho_{1,n})\, y_1\,(x, \lambda_{1,n})\,, & \text{for } \; x \in \left[0, \frac{q}{p}\right), \\ -\frac{1}{\rho_{1,n}} A_{21}\,(\rho_{1,n})\, y_2\,(x, \lambda_{2,n})\,, & \text{for } \; x \in \left(\frac{q}{p}, 1\right], \end{cases}$$

$$y_{2,n}\,(x) = \begin{cases} \frac{1}{\rho_{2,n}} A_{22}\,(\rho_{2,n})\, y_1\,(x, \lambda_{2,n})\,, & \text{for } \; x \in \left[0, \frac{q}{p}\right), \\ -\frac{1}{\rho_{2,n}} A_{21}\,(\rho_{2,n})\, y_2\,(x, \lambda_{2,n})\,, & \text{for } \; x \in \left(\frac{q}{p}, 1\right], \end{cases}$$

$$y_{3,n}\,(x) = \begin{cases} \frac{1}{\rho_{3,n}} A_{22}\,(\rho_{3,n})\, y_1\,(x, \lambda_{3,n})\,, & \text{for } \; x \in \left[0, \frac{q}{p}\right), \\ -\frac{1}{\rho_{3,n}} A_{21}\,(\rho_{3,n})\, y_2\,(x, \lambda_{3,n})\,, & \text{for } \; x \in \left(\frac{q}{p}, 1\right], \end{cases}$$

and

$$y_{l,n}\,(x) = \begin{cases} \frac{1}{\rho_{l,n}} A_{22}\,(\rho_{l,n})\, y_1\,(x, \lambda_{l,n})\,, & \text{for } \; x \in \left[0, \frac{q}{p}\right), \\ -\frac{1}{\rho_{l,n}} A_{21}\,(\rho_{l,n})\, y_2\,(x, \lambda_{l,n})\,, & \text{for } \; x \in \left(\frac{q}{p}, 1\right], \end{cases}$$

respectively. We prove asymptotic equality for the eigenfunction $y_{l,n}(x)$. Asymptotic equalities for eigenfunctions $y_{1,n}(x)$, $y_{2,n}(x)$ and $y_{3,n}(x)$ are proved analogously. Since,

$$\left.\begin{array}{r} cosz = 1 + O\left(z^2\right), z \to 0, \\ sinz = z + O\left(z^3\right) = O(z), z \to 0, \end{array}\right\}$$

we have:

$$\frac{1}{\rho_{l,n}} A_{22}\left(\rho_{l,n}\right) = -\frac{1}{\rho_{l,n}} \rho_{l,n} \cos\frac{p-q}{p} \rho_{l,n} = -\cos\frac{p-q}{p} (p\pi n +$$

$$+\frac{p}{p-q}\pi\left(l-3\right) + \frac{1}{m\left(p-q\right)\pi n} + O\left(\frac{1}{n^2}\right)) = -\cos((p-q)\ \pi n + \left(l-3\right)\pi + \frac{1}{pm\pi n} +$$

$$+O\left(\frac{1}{n^2}\right)\Bigg) = (-1)^{(p-q)n+(l-2)+1} \cos\left(O\left(\frac{1}{n}\right)\right) = (-1)^{(p-q)n+l-1} + O\left(\frac{1}{n^2}\right),$$

we have

$$y_1\left(x, \lambda_{l,n}\right) = \sin\rho_{l,n}\ x.$$

Finally, for $x \in \left[0, \frac{q}{p}\right)$

$$y_1\left(x, \lambda_{l,n}\right) = \left((-1)^{(p-q)n+l-1} + O\left(\frac{1}{n^2}\right)\right)\left(\sin\left(p\pi n + \frac{p}{p-q}\pi\left(l-3\right)\right) x + O\left(\frac{1}{n}\right)\right) =$$

$$= \left((-1)^{(p-q)n+l-1} + O\left(\frac{1}{n^3}\right)\right) \sin\left(p\pi n + \frac{p}{p-q}\pi(l-3)\right) x + O\left(\frac{1}{n}\right).$$

Now, let $x \in \left[\frac{q}{p}, 1\right)$.

$$-\frac{1}{\rho_{l,n}} A_{21}\left(\rho_{l,n}\right) = -\cos\frac{q}{p}\left(p\pi n + \frac{p}{p-q}\pi\left(l-3\right) + \frac{1}{m\left(p-q\right)\pi n} + O\left(\frac{1}{n^2}\right)\right) +$$

$$+\left(p\pi n + \frac{p}{p-q}\pi\left(l-3\right) + \frac{1}{m\left(p-q\right)\pi n} + O\left(\frac{1}{n^2}\right)\right) m \times$$

$$\times \sin\frac{q}{p}\left(p\pi n + \frac{p}{p-q}\pi\left(l-3\right) + \frac{1}{m\left(p-q\right)\pi n} + O\left(\frac{1}{n^2}\right)\right) =$$

$$= (-1)^{qn+1} \cos\left(\frac{q}{p-q}\pi\left(l-3\right) + O\left(\frac{1}{n}\right)\right) + (-1)^{qn}\ m\left(p\pi n + \frac{p}{p-q}\pi\left(l-3\right) +\right.$$

$$\left. + \frac{1}{m\left(p-q\right)\pi n} + O\left(\frac{1}{n^2}\right)\right) \sin\left(\frac{q}{p-q}\ \pi\left(l-3\right) + O\left(\frac{1}{n}\right)\right) =$$

$$= (-1)^{qn+1} \cos\left(\frac{q}{p-q}\pi\left(l-3\right)\right) +$$

$$+(-1)^{qn} m\left(p\pi n + \frac{p}{p-q}\pi\left(l-3\right)\right) \sin\left(\frac{q}{p-q}\ \pi\left(l-3\right)\right) + O\left(\frac{1}{n^2}\right).$$

Hence, it was shown that

$$y_2(x,\lambda_{l,n}) = (-1)^{qn}\left(-\cos\left(\frac{q}{p-q}\pi(l-3)\right)+\right.$$
$$+m\left(p\pi n+\frac{p}{p-q}\pi(l-3)\right)\sin\left(\frac{q}{p-q}\pi(l-3)\right)+$$
$$\left.+O\left(\frac{1}{n^2}\right)\right)\sin\left(p\pi n+\frac{p}{p-q}\pi(l-3)\right)(x-1)+O\left(\frac{1}{n}\right),\quad x\in\left(\frac{p}{q},\ 1\right].$$

Theorem is proved. ◀

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