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HARDY-CESARO MAXIMAL OPERATOR IN LEBESGUE-BMO SPACES

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Abstract. *In this paper defining the Lebesgue-BMO spaces. We prove the boundedness of the Hardy-Cesaro maximal operators in such spaces. Also prove necessary and sufficient condition the boundedness of the Hardy-Cesaro maximal operators in Lebesgue spaces.*

Keywords: maximal operator, Hardy-Cesaro maximal operator, Lebesgue space, BMO space, Lebesgue-BMO spaces

Mathematics Subject Classification (2020): 42B20, 42B25, 42B35

1. Introduction

We will use the following notation. For $1 \le p < \infty$, $L_p(\mathbb{R}^n)$ is the space of all classes of measurable functions on $\mathbb{R}^n$ for which

$$\|f\|_{L_p} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty,$$

and also $WL_p(\mathbb{R}^n)$, the weak L_p space defined as the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{WL_p} = \sup_{r>0} r \left| \{x \in \mathbb{R}^n \ : \ |f(x)| > r\} \right|^{1/p} < \infty.$$

For $p = \infty$ the space $L_\infty(\mathbb{R}^n)$ is defined by means of the usual modification

$$\|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|.$$

Functions of bounded mean oscillation were introduced by John and Nirenberg in [12], and shown to satisfy the by-now celebrated John-Nirenberg inequality. Mean oscillation measures the average deviation of a locally integrable function from its mean on a certain

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set. Classically, on $\mathbb{R}^n$, the mean oscillation is measured on all cubes Q with sides parallel to the axes; sometimes the cubes are restricted to lie in a fixed cube. The space of functions of bounded mean oscillation, BMO, has proven useful in harmonic analysis and partial differential equations as a replacement for L_∞, as well as as the dual of the Hardy space H_1, and has important connections with the theory of quasi-conformal mappings. With the observation that an equivalent characterization of BMO is obtained by replacing cubes with balls, its reach has extended beyond the Euclidean setting, to manifolds [2] and metric measure spaces [4].

This idea of replacing cubes as the sets over which mean oscillation is measured has lead various authors to consider more general BMO spaces, notable among which is the strong BMO space where cubes are replaced by rectangles [5], [14]. The different geometric properties of cubes versus rectangles manifest as different functional properties for the corresponding BMO spaces. In particular, the constants in various inequalities are affected by this change see [13], [17] for an overview of the cases of the boundedness of the decreasing rearrangement and the absolute value operator.

Replacing familiar geometries with more general bases is not a phenomenon limited to BMO. The theory of maximal functions with respect to a general basis is a well developed field, with ties to the theory of differentiation of the integral. The book of de Guzman [8] is a standard reference in the area. Muckenhoupt weights have also been considered with respect to a general basis - see, for instance, the work of Jawerth [11] and Perez [16], and more recent work [7], [10], [15], which also touches on the connection between A_p weights and BMO in this general context. Sometimes restrictive assumptions are imposed in order to obtain strong results, such as the John-Nirenberg inequality see for example [3], [9].

It is interesting to investigate what are the strongest results one could obtain about BMO under the weakest assumptions, thus isolating the features which are inherent to BMO. Motivated by this, in previous work [6], two of the authors developed a theory of BMO spaces on a domain in $\mathbb{R}^n$ with Lebesgue measure, where the mean oscillation was taken over sets of finite and positive measure belonging to an open cover of the domain. In this work, the completeness of BMO was demonstrated (without any connection to duality) and sharp constants were studied for various inequalities.

Definition. *Let $1 < \theta < \infty$ We define the $LB_\theta(\mathbb{R}^n)$ space as the set of all locally integrable functions f such that*

$$\|f\|_{LB_\theta} = \sup_{x\in\mathbb{R}^n} \left\| |B(x,r)|^{-1} \int_{B(x,r)} |f(y) - f_{B(x,r)}|dy \right\|_{L_\theta(0,\infty)} < \infty.$$

For $\theta = \infty$ the space $BMO(\mathbb{R}^n)$ is define the $BMO(\mathbb{R}^n)$ space as the set of all locally integrable functions f such that

$$\|f\|_{BMO} = \sup_{x\in\mathbb{R}^n,\ r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y) - f_{B(x,r)}|dy < \infty,$$

where $f_{B(x,r)}(x) = |B(x,r)|^{-1} \int_{B(x,r)} f(y)dy$.

It is easy to show:

If $\|f\|_{LB_\theta} = 0 \Leftrightarrow f = constant$ (a.e.), so the space LB_θ is a subspace of the quotient $L_1^{loc}(\mathbb{R}^n)/\{constant\, functions\}$.

If $f \in LB_\theta(\mathbb{R}^n)$ and $g \in LB_\theta(\mathbb{R}^n)$, then $\|f+g\|_{LB_\theta} \le \|f\|_{LB_\theta} + \|g\|_{LB_\theta}$.

Let f be a locally integrable function on $\mathbb{R}^n$. The so-called of Hardy-Littlewood maximal function is defined by the formula

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|dy,$$

where $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$.

The following theorem was proved in [1].

Theorem 1. *If M maximal operator is not identically infinite, then M maximal operator is bounded on $BMO(\mathbb{R}^n)$.*

2. Hardy-Cesaro Maximal Operator in Lebesgue and $LB_\theta(\mathbb{R}^n)$ Spaces

Let f, ψ be a locally integrable functions on $\mathbb{R}^n$. The so-called of Hardy-Cesaro maximal operator is defined by the formula

$$M_\psi f(x) = \sup_{r>0} \frac{1}{|B(0,r)|} \int_{B(0,r)} |f(xy)|\, \psi(y) dy.$$

Theorem 2. *Let $1 \le p < \infty$, ψ be positive measurable function. Then M_ψ Hardy-Cesaro maximal operator is bounded on $L_p(\mathbb{R}^n)$ if and only if*

$$\sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)} |y|^{-\frac{n}{p}} \psi(y) dy < \infty. \tag{1}$$

Proof. Sufficiency. Let $f \in L_p(\mathbb{R}^n)$ and the condition (1) holds. By the Minkowski inequality we have

$$\left(\int_{\mathbb{R}^n} (M_\psi f(z))^p\, dz\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^n} \left(\sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)} |f(zy)|\, \psi(y) dy\right)^p dz\right)^{\frac{1}{p}}$$

$$\le \sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)} \left(\int_{\mathbb{R}^n} |f(zy)|^p\, dz\right)^{\frac{1}{p}} \psi(y) dy$$

$$\le \sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)} \left(\int_{\mathbb{R}^n} |f(z)|^p dz\right)^{\frac{1}{p}} |y|^{-\frac{n}{p}} \psi(y) dy$$

$$\le C\|f\|_{L_p(\mathbb{R}^n)} \sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)} |y|^{-\frac{n}{p}} \psi(y) dy.$$

Necessity. Let M_ψ Hardy-Cesaro maximal operator is bounded on $L_p(\mathbb{R}^n)$. Now, for any $\varepsilon > 0$ take

$$f_\varepsilon(x) = \begin{cases} 0, & |x| \le 1 \\ |x|^{-\frac{n}{p}-\epsilon}, & |x| > 1 \end{cases}.$$

Then

$$\|f_\varepsilon\|_{L_p}^p = \frac{C}{p\,\varepsilon}$$

and

$$M_\psi f_\varepsilon(x) = \begin{cases} 0, & |x| \le 1 \\ |x|^{-\frac{n}{p}-\varepsilon} \sup\limits_{t>0} \frac{1}{|B(0,t)|} \int\limits_{B(0,t)\setminus B(0,\frac{1}{|x|})} |y|^{-\frac{n}{p}-\varepsilon}\, \psi(y)dy, & |x| > 1 \end{cases}$$

By the M_ψ Hardy-Cesaro maximal operator is bounded on $L_p(\mathbb{R}^n)$, we have

$$C^p \|f_\varepsilon\|_{L_p}^p \ge \|M_\psi f_\varepsilon\|_{L_p}^p$$

$$= \int_{{}^C B(0,1)} |x|^{-\frac{n}{p}-\varepsilon} \left(\sup_{t>0} \frac{1}{|B(0,t)|} \int\limits_{B(0,t)\setminus B(0,\frac{1}{|x|})} |y|^{-\frac{n}{p}-\varepsilon}\, \psi(y)dy \right)^p dx$$

$$\ge \int_{{}^C B(0,\frac{1}{\varepsilon})} |x|^{-\frac{n}{p}-\varepsilon} \left(\sup_{t>0} \frac{1}{|B(0,t)|} \int\limits_{B(0,t)\setminus B(0,\varepsilon)} |y|^{-\frac{n}{p}-\varepsilon}\, \psi(y)dy \right)^p dx$$

$$\ge \|f_\varepsilon\|_{L_p}^p \varepsilon^\varepsilon \sup_{t>0} \frac{1}{|B(0,t)|} \int\limits_{B(0,t)\setminus B(0,\varepsilon)} |y|^{-\frac{n}{p}}\, \psi(y)dy.$$

In the last inequality we get the following inequality:

$$\sup_{t>0} \frac{1}{|B(0,t)|} \int\limits_{B(0,t)\setminus B(0,\varepsilon)} |y|^{-\frac{n}{p}}\, \psi(y)dy \le \frac{C}{\varepsilon^\varepsilon}.$$

Letting $\varepsilon \to 0$ in the last inequality, we obtain (1). ◀

Theorem 3.
1) If $1 < \theta < \infty$, ψ be positive measurable function satisfying the condition

$$\sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)} |y|^{-\frac{1}{\theta}} \psi(y)dy < \infty. \tag{2}$$

Then M_ψ Hardy-Cesaro maximal operator is bounded on $LB_\theta(\mathbb{R}^n)$.
2) If $\theta = \infty$, ψ be positive measurable function satisfying the condition

$$\sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)} \psi(y)dy < \infty. \tag{3}$$

Then M_ψ Hardy-Cesaro maximal operator is bounded on $BMO(\mathbb{R}^n)$.

Proof. At first estimate $(M_\psi f)_{B(a,r)}$:

$$(M_\psi f)_{B(a,r)} = \frac{1}{|B(a,r)|}\int_{B(a,r)} M_\psi f(z)dz$$

$$= \frac{1}{|B(a,r)|}\int_{B(a,r)}\left(\sup_{t>0}\frac{1}{|B(0,t)|}\int_{B(0,t)}|f(zy)|\,|\psi(y)dy\right)dz$$

$$= \sup_{t>0}\frac{1}{|B(0,t)|}\int_{B(0,t)}\left(\frac{1}{|B(a,r)|}\int_{B(a,r)}|f(zy)|\,dz\right)\psi(y)dy$$

$$= \sup_{t>0}\frac{1}{|B(0,t)|}\int_{B(0,t)}\left(\frac{1}{|B(ay,r|y|)|}\int_{B(ay,r|y|)}|f(z)|dz\right)\psi(y)dy$$

$$= \sup_{t>0}\frac{1}{|B(0,t)|}\int_{B(0,t)} f_{B(ay,r|y|)}\psi(y)dy. \tag{4}$$

If $1<\theta<\infty$ and the condition (2) holds. Let $f\in LB_\theta(\mathbb{R}^n)$, then by the Minkowski inequality, we have

$$\left\|\frac{1}{|B(a,r)|}\int_{B(a,r)}|M_\psi f(z)-(M_\psi f)_{B(a,r)}|dz\right\|_{L_\theta(0,\infty)}$$

$$\le \left\|\frac{1}{|B(a,r)|}\int_{B(a,r)}\left(\sup_{t>0}\frac{1}{|B(0,t)|}\int_{B(0,t)}|f(zy)-f_{B(ay,r|y|)}|\psi(y)dy\right)dz\right\|_{L_\theta(0,\infty)}$$

$$\le \sup_{t>0}\frac{1}{|B(0,t)|}\int_{B(0,t)}\left\|\frac{1}{|B(a,r)|}\int_{B(a,r)}|f(zy)-f_{B(ay,r|y|)}|dz\right\|_{L_\theta(0,\infty)}\psi(y)dy$$

$$= \sup_{t>0}\frac{1}{|B(0,t)|}\int_{B(0,t)}\left\|\frac{1}{|B(ay,r|y|)|}\int_{B(ay,r|y|)}|f(z)-f_{B(ay,r|y|)}|dz\right\|_{L_\theta(0,\infty)}\psi(y)dy$$

$$= \sup_{t>0}\frac{1}{|B(0,t)|}\int_{B(0,t)}\left\|\frac{1}{|B(ay,r)|}\int_{B(ay,r)}|f(z)-f_{B(ay,r)}|dz\right\|_{L_\theta(0,\infty)}|y|^{-\frac{1}{\theta}}\psi(y)dy$$

$$\le C\|f\|_{LB_\theta}\sup_{t>0}\frac{1}{|B(0,t)|}\int_{B(0,t)}|y|^{-\frac{1}{\theta}}\psi(y)dy.$$

If $\theta=\infty$ and the condition (3) holds. Let $f\in BMO(\mathbb{R}^n)$, by the (4) and by the Minkowski inequality, we have

$$\frac{1}{|B(a,r)|}\int_{B(a,r)}|M_\psi f(z)-(M_\psi f)_{B(a,r)}|dz$$

$$\le \frac{1}{|B(a,r)|}\int_{B(a,r)}\left(\sup_{t>0}\frac{1}{|B(0,t)|}\int_{B(0,t)}|f(zy)-f_{B(ay,r|y|)}|\psi(y)dy\right)dz$$

$$= \sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)} \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} |f\,(zy) - f_{B(ay,r|y|)}|dz \right) \psi(y)dy$$

$$= \sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)} \left(\frac{1}{|B(ay,r|y|)|} \int_{B(ay,r|y|)} |f(z) - f_{B(ay,r|y|)}|dz \right) \psi(y)dy$$

$$\leq C\|f\|_{BMO} \sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)} \psi(y)dy.$$

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