HARDY-CESARO MAXIMAL OPERATOR IN LEBESGUE-BMO SPACES

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Abstract. In this paper defining the Lebesgue-BMO spaces. We prove the boundedness of the Hardy-Cesaro maximal operators in such spaces. Also prove necessary and sufficient condition the boundedness of the Hardy-Cesaro maximal operators in Lebesque spaces.

Keywords: maximal operator, Hardy-Cesaro maximal operator, Lebesgue space, BMO space, Lebesgue-BMO spaces

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1. Introduction

We will use the following notation. For $1 \leq p < \infty$, $L_p(\mathbb{R}^n)$ is the space of all classes of measurable functions on $\mathbb{R}^n$ for which

$$\|f\|_{L_p} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{\frac{1}{p}} < \infty,$$

and also $WL_p(\mathbb{R}^n)$, the weak $L_p$ space defined as the set of all measurable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{WL_p} = \sup_{r > 0} r \left|\left\{x \in \mathbb{R}^n : |f(x)| > r\right\}\right|^{1/p} < \infty.$$

For $p = \infty$ the space $L_\infty(\mathbb{R}^n)$ is defined by means of the usual modification

$$\|f\|_{L_\infty} = \text{ess sup}_{x \in \mathbb{R}^n} |f(x)|.$$

Functions of bounded mean oscillation were introduced by John and Nirenberg in [12], and shown to satisfy the by-now celebrated John-Nirenberg inequality. Mean oscillation measures the average deviation of a locally integrable function from its mean on a certain
set. Classically, on $\mathbb{R}^n$, the mean oscillation is measured on all cubes $Q$ with sides parallel to the axes; sometimes the cubes are restricted to lie in a fixed cube. The space of functions of bounded mean oscillation, $\text{BMO}$, has proven useful in harmonic analysis and partial differential equations as a replacement for $L_\infty$, as well as as the dual of the Hardy space $H_1$, and has important connections with the theory of quasi-conformal mappings. With the observation that an equivalent characterization of $\text{BMO}$ is obtained by replacing cubes with balls, its reach has extended beyond the Euclidean setting, to manifolds [2] and metric measure spaces [4].

This idea of replacing cubes as the sets over which mean oscillation is measured has lead various authors to consider more general $\text{BMO}$ spaces, notable among which is the strong $\text{BMO}$ space where cubes are replaced by rectangles [5], [14]. The different geometric properties of cubes versus rectangles manifest as different functional properties for the corresponding $\text{BMO}$ spaces. In particular, the constants in various inequalities are affected by this change see [13], [17] for an overview of the cases of the boundedness of the decreasing rearrangement and the absolute value operator.

Replacing familiar geometries with more general bases is not a phenomenon limited to $\text{BMO}$. The theory of maximal functions with respect to a general basis is a well developed field, with ties to the theory of differentiation of the integral. The book of de Guzman [8] is a standard reference in the area. Muckenhoupt weights have also been considered with respect to a general basis - see, for instance, the work of Jawerth [11] and Perez [16], and more recent work [7], [10], [15], which also touches on the connection between $A_p$ weights and $\text{BMO}$ in this general context. Sometimes restrictive assumptions are imposed in order to obtain strong results, such as the John-Nirenberg inequality see for example [3], [9].

It is interesting to investigate what are the strongest results one could obtain about $\text{BMO}$ under the weakest assumptions, thus isolating the features which are inherent to $\text{BMO}$. Motivated by this, in previous work [6], two of the authors developed a theory of $\text{BMO}$ spaces on a domain in $\mathbb{R}^n$ with Lebesgue measure, where the mean oscillation was taken over sets of finite and positive measure belonging to an open cover of the domain. In this work, the completeness of $\text{BMO}$ was demonstrated (without any connection to duality) and sharp constants were studied for various inequalities.

**Definition.** Let $1 < \theta < \infty$ We define the $L_{B\theta}(\mathbb{R}^n)$ space as the set of all locally integrable functions $f$ such that

$$
\|f\|_{L_{B\theta}} = \sup_{x \in \mathbb{R}^n} \left\| |B(x, r)|^{-1} \int_{B(x, r)} |f(y) - f_{B(x, r)}| \, dy \right\|_{L^\theta(0, \infty)} < \infty.
$$

For $\theta = \infty$ the space $\text{BMO}(\mathbb{R}^n)$ is define the $\text{BMO}(\mathbb{R}^n)$ space as the set of all locally integrable functions $f$ such that

$$
\|f\|_{\text{BMO}} = \sup_{x \in \mathbb{R}^n, r > 0} |B(x, r)|^{-1} \int_{B(x, r)} |f(y) - f_{B(x, r)}| \, dy < \infty,
$$

where $f_{B(x, r)}(x) = |B(x, r)|^{-1} \int_{B(x, r)} f(y) \, dy$. 
It is easy to show:
If \( \| f \|_{LB_\theta} = 0 \Leftrightarrow f = \text{constant (a.e.)} \), so the space \( LB_\theta \) is a subspace of the quotient \( L^{loc}_1(\mathbb{R}^n)/\{\text{constant functions}\} \).

If \( f \in LB_\theta(\mathbb{R}^n) \) and \( g \in LB_\theta(\mathbb{R}^n) \), then \( \| f + g \|_{LB_\theta} \leq \| f \|_{LB_\theta} + \| g \|_{LB_\theta} \).

Let \( f \) be a locally integrable function on \( \mathbb{R}^n \). The so-called of Hardy-Littlewood maximal function is defined by the formula

\[
Mf(x) = \sup_{r>0} |B(x, r)|^{-1} \int_{B(x, r)} |f(y)|dy,
\]
where \( |B(x, r)| \) is the Lebesgue measure of the ball \( B(x, r) \).

The following theorem was proved in [1].

**Theorem 1.** If \( M \) maximal operator is not identically infinite, then \( M \) maximal operator is bounded on \( BMO(\mathbb{R}^n) \).

2. Hardy-Cesaro Maximal Operator in Lebesgue and \( LB_\theta(\mathbb{R}^n) \) Spaces

Let \( f, \psi \) be a locally integrable functions on \( \mathbb{R}^n \). The so-called of Hardy-Cesaro maximal operator is defined by the formula

\[
M_\psi f(x) = \sup_{r>0} \frac{1}{|B(0, r)|} \int_{B(0, r)} |f(xy)| \psi(y)dy.
\]

**Theorem 2.** Let \( 1 \leq p < \infty \), \( \psi \) be positive measurable function. Then \( M_\psi \) Hardy-Cesaro maximal operator is bounded on \( L^p(\mathbb{R}^n) \) if and only if

\[
\sup_{t>0} \frac{1}{|B(0, t)|} \int_{B(0, t)} |y|^{-\frac{n}{p'}} \psi(y)dy < \infty. \tag{1}
\]

**Proof. Sufficiency.** Let \( f \in L^p(\mathbb{R}^n) \) and the condition (1) holds. By the Minkowski inequality we have

\[
\left( \int_{\mathbb{R}^n} (M_\psi f(z))^p dz \right)^{\frac{1}{p}} \leq \sup_{t>0} \frac{1}{|B(0, t)|} \int_{B(0, t)} \left( \int_{\mathbb{R}^n} |f(z)|^p dz \right)^{\frac{1}{p}} \psi(y)dy
\]

\[
\leq \sup_{t>0} \frac{1}{|B(0, t)|} \int_{B(0, t)} \left( \int_{\mathbb{R}^n} |f(z)|^p dz \right)^{\frac{1}{p}} |y|^{-\frac{n}{p'}} \psi(y)dy
\]

\[
\leq C \| f \|_{L^p(\mathbb{R}^n)} \sup_{t>0} \frac{1}{|B(0, t)|} \int_{B(0, t)} |y|^{-\frac{n}{p'}} \psi(y)dy.
\]
Necessity. Let $M_\psi$ Hardy-Cesaro maximal operator is bounded on $L_p(\mathbb{R}^n)$. Now, for any $\varepsilon > 0$ take
\[
f_\varepsilon(x) = \begin{cases} 
0, & |x| \leq 1 \\
|x|^{-\frac{n}{p}-\varepsilon}, & |x| > 1 
\end{cases}.
\]
Then
\[
\|f_\varepsilon\|_{L^p_p} = \frac{C}{p \varepsilon}
\]
and
\[
M_\psi f_\varepsilon(x) = \begin{cases} 
0, & |x| \leq 1 \\
|x|^{-\frac{n}{p}-\varepsilon} \sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)\setminus B(0,\frac{1}{t})} |y|^{-\frac{n}{p}-\varepsilon} \psi(y) dy, & |x| > 1
\end{cases}.
\]
By the $M_\psi$ Hardy-Cesaro maximal operator is bounded on $L_p(\mathbb{R}^n)$, we have
\[
C^p \|f_\varepsilon\|_{L^p_p} \geq \|M_\psi f_\varepsilon\|_{L^p_p}
\]
\[
= \int_{B(0,1)} |x|^{-\frac{n}{p}-\varepsilon} \left( \sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)\setminus B(0,\frac{1}{t})} |y|^{-\frac{n}{p}-\varepsilon} \psi(y) dy \right) dx
\]
\[
\geq \int_{\mathbb{R}^n} |x|^{-\frac{n}{p}-\varepsilon} \left( \sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)\setminus B(0,\varepsilon)} |y|^{-\frac{n}{p}-\varepsilon} \psi(y) dy \right) dx
\]
\[
\geq \|f_\varepsilon\|_{L^p_p} \varepsilon \sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)\setminus B(0,\varepsilon)} |y|^{-\frac{n}{p}} \psi(y) dy.
\]
In the last inequality we get the following inequality:
\[
\sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)\setminus B(0,\varepsilon)} |y|^{-\frac{n}{p}} \psi(y) dy \leq \frac{C}{\varepsilon^\varepsilon}.
\]
Letting $\varepsilon \to 0$ in the last inequality, we obtain (1).

\begin{itemize}
\item \textbf{Theorem 3.}
\item 1) If $1 < \theta < \infty$, $\psi$ be positive measurable function satisfying the condition
\[
\sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)} |y|^{-\frac{n}{\theta}} \psi(y) dy < \infty.
\]
Then $M_\psi$ Hardy-Cesaro maximal operator is bounded on $L^B_\theta(\mathbb{R}^n)$.
\item 2) If $\theta = \infty$, $\psi$ be positive measurable function satisfying the condition
\[
\sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)} \psi(y) dy < \infty.
\]
Then $M_\psi$ Hardy-Cesaro maximal operator is bounded on $BMO(\mathbb{R}^n)$.
\end{itemize}
Proof. At first estimate \((M_\phi f)_{B(a,r)}\):

\[
(M_\phi f)_{B(a,r)} = \frac{1}{|B(a,r)|} \int_{B(a,r)} M_\phi f(z) \, dz
\]

\[
= \frac{1}{|B(a,r)|} \int_{B(a,r)} \left( \sup_{t > 0} \frac{1}{|B(0,t)|} \int_{B(0,t)} |f(zy)| \, |\psi(y)| \, dy \right) \, dz
\]

\[
= \sup_{t > 0} \frac{1}{|B(0,t)|} \int_{B(0,t)} \left( \frac{1}{|B(a,r)|} \int_{B(a,r)} |f(zy)| \, dz \right) \, \psi(y) \, dy
\]

\[
= \sup_{t > 0} \frac{1}{|B(0,t)|} \int_{B(0,t)} f_B(ay,r|y|) \psi(y) \, dy. \tag{4}
\]

If \(1 < \theta < \infty\) and the condition (2) holds. Let \(f \in LB_\theta(\mathbb{R}^n)\), then by the Minkowski inequality, we have

\[
\left\| \frac{1}{|B(a,r)|} \int_{B(a,r)} |M_\phi f(z) - (M_\phi f)_{B(a,r)}| \, dz \right\|_{L_\theta(0,\infty)}
\]

\[
\leq \left\| \frac{1}{|B(a,r)|} \int_{B(a,r)} \left( \sup_{t > 0} \frac{1}{|B(0,t)|} \int_{B(0,t)} |f(zy) - f_{B(ay,r|y|)}\psi(y)\,dy| \right) \, dz \right\|_{L_\theta(0,\infty)}
\]

\[
\leq \sup_{t > 0} \frac{1}{|B(0,t)|} \int_{B(0,t)} \left| \frac{1}{|B(a,r)|} \int_{B(a,r)} |f(zy) - f_{B(ay,r|y|)}| \, dz \right| \psi(y) \, dy
\]

\[
= \sup_{t > 0} \frac{1}{|B(0,t)|} \int_{B(0,t)} \left| \frac{1}{|B(ay,r|y|)|} \int_{B(ay,r|y|)} |f(z) - f_{B(ay,r|y|)}| \, dz \right| \psi(y) \, dy
\]

\[
\leq C \|f\|_{LB_\theta} \sup_{t > 0} \frac{1}{|B(0,t)|} \int_{B(0,t)} |y|^{-\frac{n}{2}} \psi(y) \, dy.
\]

If \(\theta = \infty\) and the condition (3) holds. Let \(f \in BMO(\mathbb{R}^n)\), by the (4) and by the Minkowski inequality, we have

\[
\frac{1}{|B(a,r)|} \int_{B(a,r)} |M_\phi f(z) - (M_\phi f)_{B(a,r)}| \, dz
\]

\[
\leq \frac{1}{|B(a,r)|} \int_{B(a,r)} \left( \sup_{t > 0} \frac{1}{|B(0,t)|} \int_{B(0,t)} |f(zy) - f_{B(ay,r|y|)}\psi(y)\,dy| \right) \, dz
\]
\[
\begin{align*}
&= \sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)} \left( \frac{1}{|B(a,r)|} \int_{B(a,r)} |f(zy) - f_B(ay,r(y))|dz \right) \psi(y)dy \\
&= \sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)} \left( \frac{1}{|B(ay,r(y))|} \int_{B(ay,r(y))} |f(z) - f_B(ay,r(y))|dz \right) \psi(y)dy \\
&\leq C\|f\|_{BMO} \sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)} \psi(y)dy.
\end{align*}
\]

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References


