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MAXIMAL OPERATOR IN WEIGHTED MORREY SPACES, ASSOCIATED WITH THE LAPLACE-BESSEL DIFFERENTIAL OPERATOR

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Abstract. *We consider the generalized shift operator, associated with the Laplace-Bessel differential operator* $\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{k} \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}$, $\gamma_i > 0$, $i = 1, \ldots, k$, $|\gamma| = \gamma_1 + \ldots + \gamma_k$. *The maximal operator* M_γ *(B-maximal operator), associated with the generalized shift operator is investigated. At first, we prove that the boundedness of maximal operator weighted Morrey spaces* $\mathcal{L}_{p,\lambda,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ *to the spaces* $\mathcal{L}_{p,\lambda,\omega_2,\gamma}(\mathbb{R}^n_{k,+})$, *where* $1 < p < \infty$, $(\omega_1, \omega_2) \in F_{p,\gamma}(\mathbb{R}^n_{k,+})$, *associated with the Laplace-Bessel differential operator.*

Keywords: B-maximal operator, generalized shift operator, weighted B-Morrey space

Mathematics Subject Classification (2020): 42B20, 42B25, 42B35

1. Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r)$ denote the open ball centered at x of radius r.

Let $f \in L_1^{\text{loc}}(\mathbb{R}^n)$. The maximal operator M is defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)|dy,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$.

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The maximal operator play an important role in real and harmonic analysis (see, for example [20], [21] and [16]).

In the theory of partial differential equations Morrey spaces $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ play an important role. They were introduced by C. Morrey in 1938 [14] (see, for example [11]) and defined as follows: *For* $0 \le \lambda \le n$, $1 \le p < \infty$, $f \in \mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ *if* $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ *and*

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty.$$

If $\lambda = 0$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, if $\lambda = n$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$, if $\lambda < 0$ or $\lambda > n$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

These spaces appeared to be quite useful in the study of the local behaviour of the solutions to elliptic partial differential equations, apriori estimates and other topics in the theory of partial differential equations.

Also by $W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ we denote the weak Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{W\mathcal{M}_{p,\lambda}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(\mathbb{R}^n)$ denotes the weak $L_p(\mathbb{R}^n)$-space.

F. Chiarenza and M. Frasca [4] studied the boundedness of the maximal operator M in Morrey spaces $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$. Their results can be summarized as follows:

Theorem A. *Let* $0 < \alpha < n$ *and* $0 \le \lambda < n$, $1 \le p < \infty$.
1) If $1 < p < \infty$, *then* M *is bounded from* $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ *to* $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$.
2) If $p = 1$, *then* M *is bounded from* $\mathcal{M}_{1,\lambda}(\mathbb{R}^n)$ *to* $W\mathcal{M}_{1,\lambda}(\mathbb{R}^n)$.

The maximal operator, singular integral operator and related topics associated with the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{k} \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad \gamma_1 > 0, \ldots, \gamma_k > 0$$

have been investigated by many researchers, see B. Muckenhoupt and E.M. Stein [15], I.A. Kipriyanov [10], K. Trimèche [23], L.N. Lyakhov [13], K. Stempak [22], I.A. Aliev and A.D. Gadzhiev [1], V.S. Guliyev [5], V.S. Guliyev and J.J. Hasanov [6], [7], R. Ayazoglu and J.J. Hasanov [2], [8], [9], A. Şerbetçi and I. Ekincioğlu [18], E.L. Shishkina [19], C. Aykol and J.J. Hasanov [3] and others.

2. Definitions, Notation and Preliminaries

Suppose that $\mathbb{R}^n$ is n-dimensional Euclidean space, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $|x|^2 = \sum\limits_{i=1}^{n} x_i^2$, $x' = (x_1, \ldots, x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-k}$, $x = (x', x'') \in \mathbb{R}^n$, $n \ge 2$, $\mathbb{R}^n_{k,+} = \{x = (x', x'') \in \mathbb{R}^n; x_1 > 0, \ldots, x_k > 0\}$, $1 \le k \le n$, $E(x, r) = \{y \in \mathbb{R}^n_{k,+}\ ;\ |x - y| < r\}$, $E_r = E(0, r)$, $\gamma = (\gamma_1, \ldots, \gamma_k)$, $\gamma_1 > 0, \ldots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \ldots + \gamma_k$, $(x')^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$.

For measurable $E \subset \mathbb{R}^n_{k,+}$ suppose $|E|_\gamma = \int_E (x')^\gamma dx$, then $|E_r|_\gamma = \omega(n,k,\gamma) r^Q$, $Q = n + |\gamma|$, where

$$\omega(n,k,\gamma) = \int_{E_1} (x')^\gamma dx = \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^{k} \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}.$$

Denote by T^x the generalized shift operator (B–shift operator) acting according to the law

$$T^x f(y) = C_{\gamma,k} \int_0^\pi \dots \int_0^\pi f\left((x',y')_\beta, x''-y''\right)\, d\nu(\beta),$$

where $(x_i,y_i)_{\beta_i} = (x_i^2 - 2x_i y_i \cos\beta_i + y_i^2)^{\frac{1}{2}}$, $1 \le i \le k$, $(x',y')_\beta = ((x_1,y_1)_{\beta_1}, \dots, (x_k,y_k)_{\beta_k})$, $d\nu(\beta) = \prod_{i=1}^{k} \sin^{\gamma_i - 1}\beta_i \, d\beta_1 \dots d\beta_k$, $1 \le k \le n$ and

$$C_{\gamma,k} = \pi^{-\frac{k}{2}} \prod_{i=1}^{k} \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)} = \frac{2^k}{\pi^k} \omega(2k,k,\gamma).$$

We remark that the generalized shift operator T^x is closely connected with the Bessel differential operator B (for example, $n=k=1$ see [12], $n>1$, $k=1$ see [10] and $n,k>1$ see [13] for details).

Let $L_{p,\varphi,\gamma}(\mathbb{R}^n_{k,+})$ be the space of measurable functions on $\mathbb{R}^n_{k,+}$ with finite norm

$$\|f\|_{L_{p,\varphi,\gamma}} = \|f\|_{L_{p,\varphi,\gamma}(\mathbb{R}^n_{k,+})} = \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \varphi^p(x) (x')^\gamma dx \right)^{1/p}, \quad 1 \le p < \infty.$$

For $p = \infty$ the space $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ is defined by means of the usual modification

$$\|f\|_{L_{\infty,\gamma}} = \|f\|_{L_{\infty,\varphi}} = \underset{x \in \mathbb{R}^n_{k,+}}{ess\ sup}\ \varphi(x)|f(x)|.$$

Definition 1. *The weight function (φ_1,φ_2) belongs to the class $F_{p,\gamma}(\mathbb{R}^n_{k,+})$ for $1 < p,t < \infty$, if*

$$\sup_{x \in \mathbb{R}^n_{k,+},\, r>0} \left(\frac{1}{|E(x,r)|_\gamma} \int_{E(x,r)} \varphi_2^p(y)(y')^\gamma dy \right)^{\frac{1}{p}} \left(\frac{1}{|E(x,r)|_\gamma} \int_{E(x,r)} \varphi_1^{-p't}(y)(y')^\gamma dy \right)^{\frac{1}{p't}} < \infty.$$

The translation operator T^y generates the corresponding B-convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y)[T^x g(y)](y')^\gamma dy,$$

for which the Young inequality

$$\|f \otimes g\|_{L_{r,\gamma}} \le \|f\|_{L_{p,\gamma}} \|g\|_{L_{q,\gamma}}, \quad 1 \le p,q \le r \le \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

holds.

Lemma 1. *For all* $x \in \mathbb{R}^n_{k,+}$ *the following equality is valid*

$$\int_{E_t} T^y g(x)(y')^\gamma dy = \int_{E((x,0),t)} g\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z''\right) d\mu(z,\overline{z'}),$$

where $E((x,0),t) = \{(z,\overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k \ : \ \left|(x-z,\overline{z'})\right| < t\}$.

Lemma 2. *Let* $0<\theta<1$ *and* ψ *positive measurable weight function. For all* $x \in \mathbb{R}^n_{k,+}$ *the following equality is valid*

$$\int_{\mathbb{R}^n_{k,+}} T^y g(x)\psi(y)\left(M_\gamma \chi_{E_r}(y)\right)^\theta (y')^\gamma dy$$
$$= \int_{\mathbb{R}^n\times(0,\infty)^k} g\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z''\right)\psi(z,\overline{z'})\left(M_\nu \chi_{E((x,0),r)}(z,\overline{z'})\right)^\theta d\nu(z,\overline{z'}),$$

where $E((x,0),t) = \{(z,\overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k \ : \ \left|(x-z,\overline{z'})\right| < t\}$.

Lemmas 1 and 2 are straightforward via the following substitutions

$$z''=x'', z_i = y_i\cos\alpha_i, \quad \overline{z_i} = y_i \sin\alpha_i, \quad 0\le\alpha_i<\pi, \ i=1,\ldots,k,$$
$$y\in\mathbb{R}^n_{k,+}, \ \overline{z'} = (\overline{z}_1,\ldots,\overline{z}_k), \ (z,\overline{z'}) \in \mathbb{R}^n\times(0,\infty)^k, \ 1\le k\le n.$$

Also, in the work [17] it was proved:

Proposition. *Let* $1\le p<\infty$, $(\varphi_1,\varphi_2)\in F_p(Y)$. *Then* M_ν *is bounded from* $L_{p,\varphi_1}(Y)$ *to* $L_{p,\varphi_2}(Y)$, *where* (Y,d,ν) *homogeneous type space.*

Definition 2. *Let* $1\le p<\infty$, $0\le\lambda\le Q$. *We denote by* $\mathcal{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ *Morrey space (≡ B-Morrey space), associated with the Laplace-Bessel differential operator the set of locally integrable functions* $f(x)$, $x\in\mathbb{R}^n_{k,+}$, *with the finite norm*

$$\|f\|_{\mathcal{L}_{p,\lambda,\gamma}} = \sup_{t>0,\, x\in\mathbb{R}^n_{k,+}}\left(t^{-\lambda}\int_{E_t} T^y[|f|\ ]^p(x)(y')^\gamma dy\right)^{1/p}.$$

Let ω positive measurable weight function. The norm in the spaces $\mathcal{L}_{p,\lambda,\omega,\gamma}(\mathbb{R}^n_{k,+})$ defined in form,

$$\|f\|_{\mathcal{L}_{p,\lambda,\omega,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0}\frac{1}{t^{\frac{\lambda}{p}}\|\omega\|_{L_p(E_t)}}\left(\int_{E_t} T^y[|f|\ ]^p(x)\omega^p(y)(y')^\gamma dy\right)^{1/p}.$$

Consider the B-maximal operator

$$M_\gamma f(x) = \sup_{r>0}|E_r|_\gamma^{-1}\int\limits_{E_r} T^y[|f|\ ](x)(y')^\gamma dy.$$

3. Two-Weighted Inequalities B-Maximal Operator in the Spaces $\mathcal{L}_{p,\lambda,\omega,\gamma}(\mathbb{R}^n_{k,+})$

Theorem. *Let $1 < p < \infty$, $0 \le \lambda \le Q$, $(\omega_1, \omega_2) \in F_{p,\gamma}(\mathbb{R}^n_{k,+})$. Then M_γ is bounded from $\mathcal{L}_{p,\lambda,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ to $\mathcal{L}_{p,\lambda,\omega_2,\gamma}$.*

Proof. We need to introduce the maximal operator defined on a space of homogeneous type (Y, d, ν). By this we mean a topological space $Y = \mathbb{R}^n \times (0,\infty)^k$ equipped with a continuous pseudometric d and a positive measure ν satisfying

$$\nu(E((x,\overline{x'}), 2r)) \le C_1 \nu(E((x,\overline{x'}), r)) \tag{1}$$

with a constant C_1 independent of $(x,\overline{x'})$ and $r > 0$. Here $E((x,\overline{x'}), r) = \{(y,\overline{y'}) \in Y : d(((x,\overline{x'}),(y,\overline{y'})) < r\}$, $d\nu(y,\overline{y'}) = (\overline{y'})^{\gamma-1} dy\, d\overline{y'}$, $(\overline{y'})^{\gamma-1} = (\overline{y_1})^{\gamma_1-1} \cdots (\overline{y_k})^{\gamma_k-1}$, $d((x,\overline{x'}),(y,\overline{y'})) = |(x,\overline{x'}) - (y,\overline{y'})| \equiv (|x-y|^2 + (\overline{x'} - \overline{y'})^2)^{\frac{1}{2}}$.

Let (Y, d, ν) be a space of homogeneous type. Define

$$M_\nu \overline{f}(x,\overline{x'}) = \sup_{r>0} \nu(E((x,\overline{x'}), r))^{-1} \int_{E((x,\overline{x'}),r)} \left|\overline{f}(y,\overline{y'})\right| d\nu(y),$$

where $\overline{f}(x,\overline{x'}) = f\left(\sqrt{x_1^2 + \overline{x}_1^2}, \dots, \sqrt{x_k^2 + \overline{x}_k^2}, x''\right)$.

It is well known that the fractional maximal operator M_ν is bounded on $L_{p,\psi_1}(Y, d\nu)$ to $L_{p,\psi_2}(Y, d\nu)$ for $1 < p < \infty$, $(\psi_1, \psi_2) \in F_p(Y)$ (see [17]). Here we are concerned with the fractional maximal operator defined by $d\nu(y,\overline{y'}) = (\overline{y'})^{\gamma-1} dy\, d\overline{y'}$. It is clear that this measure satisfies the doubling condition (1).

It can be proved that

$$M_\gamma f\left(\sqrt{z_1^2 + \overline{z}_1^2}, \dots, \sqrt{z_k^2 + \overline{z}_k^2}, z''\right) = M_\nu \overline{f}\left(\sqrt{z_1^2 + \overline{z}_1^2}, \dots, \sqrt{z_k^2 + \overline{z}_k^2}, z'', 0\right) \tag{2}$$

and

$$M_\gamma f(x) = M_\nu \overline{f}(x, 0). \tag{3}$$

Indeed, Lemma 2 and

$$\psi_1(y) = \omega_1(y)(M_\nu \chi_{E((x,0),r)}(y))^{\frac{\theta}{p}}, \quad \psi_2(y) = \omega_2(y)(M_\nu \chi_{E((x,0),r)}(y))^{\frac{\theta}{p}},$$

for any $0 < \theta < 1$, $(\psi_1, \psi_2) \in \widetilde{A}_p(Y)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n_{k,+}} T^y |f(x)|^p \omega_1^p(y) (M_\gamma \chi_{E_r}(y))^\theta (y')^\gamma \, dy \\
&= \int_Y \left|\overline{f}\left(\sqrt{y_1^2 + \overline{y}_1^2}, \dots, \sqrt{y_k^2 + \overline{y}_k^2}, y'', 0\right)\right|^p \\
&\times \omega_1^p(y,\overline{y'}) (M_\nu \chi_{E((x,0),r)}(y,\overline{y'}))^\theta \, d\nu(y,\overline{y'})
\end{aligned}$$

and

$$|E_r|_\gamma = \nu E\left(\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\right),r\right)$$

imply (2). Furthermore, taking $\overline{z}_k=0$ in (2) we get (3).

Using Lemma 2 and equality (2) we have

$$\begin{aligned}
&\int_{E_r} T^y\left(M_\gamma f(x)\right)^p \omega_2^p(y)(y')^\gamma\, dy \\
&\le \int_{\mathbb{R}^n_{k,+}} T^y\left(M_\gamma f(x)\right)^p \omega_2^p(y)(M_\gamma \chi_{E_r}(y))^\theta (y')^\gamma\, dy \\
&= \int_{\mathbb{R}^n\times(0,\infty)^k}\left(M_\gamma f\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2},z''\right)\right)^p \\
&\qquad \times\omega_2^p(z,\overline{z'})(M_\gamma \chi_{E((x,0),r)}(z,\overline{z'}))^\theta\, d\nu(z,\overline{z'}) \\
&= \int_Y\left(M_\nu \overline{f}\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\right)\right)^p \\
&\qquad \times\omega_2^p(z,\overline{z'})(M_\nu \chi_{E((x,0),r)}(z,\overline{z'}))^\theta\, d\nu(z,\overline{z'}).
\end{aligned}$$

By the Proposition we have

$$\begin{aligned}
&\left(\int_{E_r} T^y\left(M_\gamma f(x)\right)^p \omega_2^p(y)(y')^\gamma dy\right)^{\frac{1}{p}} \\
&\le \left(\int_Y\left(M_\nu \overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\dots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right)^p\right. \\
&\qquad \left.\times\omega_2^p(y,\overline{y'})(M_\nu \chi_{E((x,0),r)}(y,\overline{y'}))^\theta\, d\nu(y,\overline{y'})\right)^{\frac{1}{p}} \\
&= \left(\int_Y\left(M_\nu \overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\dots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right)^p \psi_2^p(y,\overline{y'})\, d\nu(y,\overline{y'})\right)^{\frac{1}{p}} \\
&\le C_2\left(\int_Y\left|\overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\dots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right|^p \psi_1^p(y,\overline{y'})\, d\nu(y,\overline{y'})\right)^{\frac{1}{p}} \\
&= C_2\left(\int_Y\left|\overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\dots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right|^p\right. \\
&\qquad \left.\times\omega_1^p(y,\overline{y'})(M_\nu \chi_{E((x,0),r)}(y,\overline{y'}))^\theta\, d\nu(y,\overline{y'})\right)^{\frac{1}{p}} \\
&= C_2\left(\int_Y\left|f\left(\sqrt{y_1^2+\overline{y}_1^2},\dots,\sqrt{y_k^2+\overline{y}_k^2},y''\right)\right|^p\right. \\
&\qquad \left.\times\omega_1^p(y,\overline{y'})(M_\nu \chi_{E((x,0),r)}(y,\overline{y'}))^\theta\, d\nu(y,\overline{y'})\right)^{\frac{1}{p}} \\
&= C_2\left(\int_{\mathbb{R}^n_{k,+}} T^y[|f|\,]^p(x)\omega_1^p(y)(M_\gamma \chi_{E_r}(y))^\theta (y')^\gamma dy\right)^{\frac{1}{p}}
\end{aligned}$$

$$\leq C_2 \left(\int_{E_r} T^y[|f|\,]^p(x)\omega_1^p(y)(y')^\gamma dy \right.$$

$$\left. + \sum_{j=1}^{\infty} \int_{E_{2^{j+1}r} \setminus E_{2^j r}} T^y[|f|\,]^p(x)\omega_1^p(y)(M_\gamma \chi_{E_r}(y))^\theta (y')^\gamma dy \right)^{\frac{1}{p}}$$

$$\leq C_2 \left(\int_{E_r} T^y[|f|\,]^p(x)\omega_1^p(y)(y')^\gamma dy \right.$$

$$\left. + \sum_{j=1}^{\infty} \int_{E_{2^{j+1}r} \setminus E_{2^j r}} T^y[|f|\,]^p(x)\omega_1^p(y)\frac{r^{Q\theta}}{(|y|+r)^{Q\theta}}(y')^\gamma dy \right)^{\frac{1}{p}}$$

$$\leq C_3 \; r^{\frac{\lambda}{p}} \|\omega_1\|_{L_p(E_r)} \|f\|_{\mathcal{M}_{p,\lambda,\omega_1,\gamma}}.$$

Then

$$\|M_\gamma f\|_{\mathcal{L}_{p,\lambda,\omega_2,\gamma}} = \sup_{x \in \mathbb{R}^n_{k,+},\, r>0} \frac{1}{r^{\frac{\lambda}{p}} \|\omega_1\|_{L_p(E_r)}} \|T^{\cdot}(M_\gamma f(x))\|_{L_{p,\omega_2,\gamma}(E_r)}$$

$$\leq C_4 \|f\|_{\mathcal{L}_{p,\lambda,\omega_1,\gamma}}.$$

The Theorem is proved. ◀

References

1. Aliev I.A., Gadzhiev A.D. Weighted estimates of multidimensional singular integrals generated by the generalized shift operator. *Russian Acad. Sci. Sb. Math.*, 1994, **77** (1), pp. 37-55.
2. Ayazoglu R., Hasanov J.J. On the boundedness of a B-Riesz potential in the generalized weighted B-Morrey spaces. *Georgian Math. J.*, 2016, **23** (2), pp. 143-155.
3. Aykol C. Hasanov J.J. On the boundedness of B-maximal commutators, commutators of B-Riesz potentials and B-singular integral operators in modified B-Morrey spaces. *Acta Sci. Math. (Szeged)*, 2020, **86** (3-4), pp. 521-547.
4. Chiarenza F., Frasca M. Morrey spaces and Hardy-Littlewood maximal function. *Rend. Mat. Appl.* (7), 1987, **7** (3-4), pp. 273-279.
5. Guliev V.S. On maximal function and fractional integral, associated with the Bessel differential operator. *Math. Inequal. Appl.*, 2003, **6** (2), pp. 317-330.
6. Guliyev V.S., Hasanov J.J. Sobolev-Morrey type inequality for Riesz potentials, associated with the Laplace-Bessel differential operator. *Fract. Calc. Appl. Anal.*, 2006, **9** (1), pp. 17-32.
7. Guliyev V.S., Hasanov J.J. Necessary and sufficient conditions for the boundedness of B-Riesz potential in the B-Morrey spaces. *J. Math. Anal. Appl.*, 2008, **347** (1), pp. 113-122.
8. Hasanov J.J. A note on anisotropic potentials associated with the Laplace-Bessel differential operator. *Oper. Matrices*, 2008, **2** (4), pp. 465-481.

9. Hasanov J.J., Ayazoglu R., Bayrakci S. B-maximal commutators, commutators of B-singular integral operators and B-Riesz potentials on B-Morrey spaces. *Open Math.*, 2020, **18** (1), pp. 715-730.
10. Kiprijanov I.A. Fourier-Bessel transforms and imbedding theorems for weight classes. *Proc. Steklov Inst. Math.*, 1967, **89**, pp. 149-246.
11. Kufner A., John O., Fučík S. *Function Spaces.* Noordhoff Int. Publ., Leiden; Academia, Prague, 1977.
12. Levitan B.M. Expansion in Fourier series and integrals with Bessel functions. *Uspehi Matem. Nauk*, 1951, **6** (2(42)), pp. 102-143 (in Russian).
13. Lyakhov L.N. Multipliers of the mixed Fourier-Bessel transform. *Proc. Steklov Inst. Math.*, 1996, **214**, pp. 227-242.
14. Morrey Ch.B., Jr. On the solutions of quasi-linear elliptic partial differential equations. *Trans. Amer. Math. Soc.*, 1938, **43** (1), pp. 126-166.
15. Muckenhoupt B., Stein E.M. Classical expansions and their relation to conjugate harmonic functions. *Trans. Amer. Math. Soc.*, 1965, **118**, pp. 17-92.
16. Samko S.G., Kilbas A.A., Marichev O.I. *Fractional Integrals and Derivatives. Theory and Applications.* Gordon & Breach Sci. Publ., Yverdon, 1993.
17. Sawyer E., Wheeden R.L. Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces. *Amer. J. Math.*, 1992, **114** (4), pp. 813-874.
18. Şerbetçi A., Ekincioğlu I. Boundedness of Riesz potential generated by generalized shift operator on Ba spaces. *Czechoslovak Math. J.*, 2004, **54** (3), pp. 579-589.
19. Shishkina E.L. Hyperbolic Riesz B-potential and solution of an iterated non-homogeneous B-hyperbolic equation. *Lobachevskii J. Math.*, 2020, **41** (5), pp. 895-916.
20. Stein E.M. *Singular Integrals and Differentiability Properties of Functions.* Princeton Univ. Press, Princeton, NJ, 1970.
21. Stein E.M., Weiss G. *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton Univ. Press, Princeton, NJ, 1971.
22. Stempak K. Almost everywhere summability of Laguerre series. *Studia Math.*, 1991, **100** (2), pp. 129-147.
23. Trimèche K. Inversion of the Lions transmutation operators using generalized wavelets. *Appl. Comput. Harmon. Anal.*, 1997, **4** (1), pp. 97-112.