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NONLINEAR INVERSE BOUNDARY VALUE PROBLEM FOR A SECOND ORDER PARABOLIC EQUATION WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. *The paper studies the classical solvability of an inverse boundary value problem for a second order parabolic equation with nonlocal boundary conditions. For this purpose, first, the considered problem is reduced to an auxiliary equivalent problem in a certain sense. Then, using the Fourier method the auxiliary problem is presented as a system of integral equations. Further, by means of the contraction mappings principle the unique existence of the solution of the obtained system of integral equations is shown. At the end of investigation the existence and uniqueness theorem for the classical solution of the original inverse boundary value problem is proved based on the equivalence of these problems.*

Keywords: inverse problem, parabolic equation, nonlocal boundary conditions, integral overdetermination condition, classical solution

Mathematics Subject Classification (2020): 35R30, 35K10, 35A01, 35A02, 35A09

1. Introduction and Formulation of the Problem

It is known that mathematical modeling of many real processes occurring during experiments in the field of some natural sciences leads to the study of inverse boundary value problems. Simultaneous determination of unknown coefficients and/or right-hand side of partial differential equations with respect to some additional measurements is called inverse boundary value problem in the theory of equations of mathematical physics.

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The applied importance of inverse problems is so great (they arise in various fields of human activity such as seismology, mineral exploration, biology, medicine, desalination of seawater, movement of liquid in a porous medium, etc.) which puts them a series of the most actual problems of modern mathematics. The presence in the inverse problems of additional unknown functions requires that in the complement to the boundary conditions that are natural for a particular class of differential equations, impose some additional conditions - overdetermination conditions. The basics of the theory and practice of investigating inverse problems of mathematical physics were established and developed in the fundamental works of the outstanding mathematicians A.N.Tikhonov [25], M.M.Lavrent'ev et al. [21], V.K.Ivanov et al. [16], A.M.Denisov [10], M.I.Ivanchov [14], A.I.Kozhanov [20], and their followers.

Nowadays, in the modern mathematical literature, the theory of inverse boundary value problems for parabolic equations is stated rather satisfactory. The inverse and ill-posed problems associated with the parabolic/heat equation has drawn the attention of many authors. A more detailed bibliography and a classification of problems are found in monographs or books (see for example, [2], [3], [6], [8], [9], [11], [12], [18], [22], [23], [24], and the references therein).

Let us now browse the content of some related works devoted to inverse boundary value problems for parabolic/heat equations. A.Y. Akhundov studied the well-posedness of the inverse problem for determining the unknown coefficient of higher derivatives of a quasilinear parabolic equation of divergent type in a multidimensional domain in his work [1]. In [4], [5], the identification of the unknown lowest coefficient and the right-hand side in a second-order parabolic equation with integral overdetermination conditions is studied, and sufficient conditions for the existence and uniqueness of the classical solution to the considered inverse problem are established. J.R. Cannon and Y.P. Lin [7] studied the inverse problem of simulantenousely finding the evolution parameter $p(t)$ and the solution $u(x,t)$ in quasi-linear parabolic equation, and demonstrated the existence, uniqueness, and continuous dependence upon the data of the solution (u,p). The authors M.I. Ismailov and F. Kanca [13] investigate the inverse boundary value problem of finding the time-dependent coefficient of heat capacity together with the solution of heat equation with nonlocal boundary and integral overdetermination conditions. In addition, in this work are studied the existence, uniqueness, and continuous dependence of solution upon the data and the numerical procedure for the solution of considered inverse problem are presented with the examples. In the article published by M.I. Ivanchov and N.V. Pabyrivs'ka [15], the existence and uniqueness conditions for a solution of the inverse problem for a parabolic equation with nonlocal boundary and integral overdetermination conditions are established. In the paper of V.L.Kamynin [17], existence and uniqueness theorems for the solution of the inverse problem of simultaneously determination of the right-hand side and the coefficient of a lower-order derivative in a parabolic equation under the integral observation condition were proved. Moreover, explicit estimates for the maximum absolute value of the unknown right-hand side and an unknown coefficient of the equation with constants are expressed by the input data of the problem.

It should be noted that the problem statement and the proof techniques used in this paper are different from previous published works. More precisely, the technique used in this paper is based on the passing from the original inverse problem to the new equivalent one, the study of the solvability of the equivalent problem, and then in the reverse transition to the original problem. A distinctive feature of this article is the consideration of an inverse boundary-value problem with both spatial and time nonlocal conditions.

Let $T > 0$ be a fixed time moment and let D_T denotes the rectangular region defined by the inequalities $0 \le x \le 1$ and $0 \le t \le T$. We consider the problem of determining the unknown functions $u(x,t) \in C^{2,1}(D_T)$ and $a(t), b(t) \in C[0,T]$ such that the triple $\{u(x,t), a(t), b(t)\}$ satisfies the following parabolic equation

$$c(t)u_t(x,t) = u_{xx}(x,t) + a(t)u(x,t) + b(t)g(x,t) + f(x,t) \; (x,t) \in D_T, \tag{1}$$

with the nonlocal condition

$$u(x,0) + \delta u(x,T) + \int_0^T p(t)u(x,t)dt = \varphi(x), \; 0 \le x \le 1, \tag{2}$$

the boundary condition

$$u_x(0,t) = u(1,t) = 0, \; 0 \le t \le T, \tag{3}$$

and the overdetermination conditions

$$u(0,t) = h_1(t), \; 0 \le t \le T, \tag{4}$$

$$\int_0^1 H(x)u(x,t)dx = h_2(t), \; 0 \le t \le T, \tag{5}$$

where $\delta \ge 0$ is any fixed number, $0 < c(t), f(x,t), g(x,t), 0 \le p(t), \varphi(x), H(x)$, and $h_i(t)$ $(i = 1,2)$ are given functions.

Definition. *The triple $\{u(x,t), a(t), b(t)\}$ is said to be a classical solution of the problem* (1)–(5)*, if the functions $u(x,t) \in C^{2,1}(D_T)$ and $a(t), b(t) \in C[0,T]$ satisfy Eq.* (1) *in D_T, the condition* (2) *on* $[0,1]$*, and the statements* (3)–(5) *on the interval* $[0,T]$ *in the classical (usual) sense.*

The following lemma is valid.

Lemma 1. *[4] Suppose that $\delta \ge 0, c(t), a(t) \in C[0,T]$ and $p(t) \in C[0,T]$ holds. Then the problem*

$$c(t)y'(t) = a(t)y(t), \; 0 \le t \le T,$$

$$y(0) + \delta y(T) + \int_0^T p(\tau)y(\tau)d\tau = 0,$$

has a unique trivial solution.

We have the following theorem.

Theorem 1. *Assume that* $c(t) \in C[0,T], f(x,t), g(x,t) \in C(D_T), \varphi(x) \in C[0,1], H(x) \in C[0,1], h_i(t) \in C^1[0,T] (i=1,2), h(t) \equiv h_1(t) \int_0^1 H(x)g(x,t)dx - g(0,t)h_2(t) \neq 0, t \in [0,T]$*, and the compatibility conditions*

$$h_1(0) + \delta h_1(T) + \int_0^T p(t)h_1(t)dt = \varphi(0), \tag{6}$$

$$h_2(0) + \delta h_2(T) + \int_0^T p(t)h_2(t)dt = \int_0^1 H(x)\varphi(x)dx, \tag{7}$$

holds. Then the problem of finding a classical solution of (1)–(5) *is equivalent to the problem of determining the functions* $u(x,t) \in C^{2,1}(D_T)$ *and* $a(t), b(t) \in C[0,T]$*, satisfying the Eq.*(1)*, the conditions* (2), (3)*, and the relations*

$$c(t)h_1(t) = u_{xx}(0,t) + a(t)h_1(t) + b(t)g(0,t) + f(0,t),\ 0 \le t \le T, \tag{8}$$

$$c(t)h'(t) = \int_0^1 H(x)u_{xx}(x,t)dx + a(t)h(t) + \int_0^1 H(x)f(x,t)dx,\ 0 \le t \le T. \tag{9}$$

Proof. Let $\{u(x,t), a(t), b(t)\}$ be a classical solution of (1)–(5). Further, assuming $h_i(t) \in C^1[0,T]$ $(i=1,2)$ and differentiating (4) and (5), we have

$$u_t(0,t) = h_1'(t),\ 0 \le t \le T, \tag{10}$$

and

$$\int_0^1 H(x)u_t(x,t)dx = h_2'(t),\ 0 \le t \le T, \tag{11}$$

respectively.

Setting $x=0$ in Eq. (1), the procedure yields

$$c(t)u_t(0,t) = u_{xx}(0,t) + a(t)u(0,t) + a(t)g(0,t) + f(0,t),\ 0 \le t \le T. \tag{12}$$

From (12), taking into account (4) and (10), we conclude that condition (8) is satisfied. Further, multiplying both parts of Eq.(1) by the function $H(x)$ and integrating with respect to x over the interval $[0,1]$, we obtain

$$c(t)\frac{d}{dt}\int_0^1 H(x)u(x,t)dx = \int_0^1 H(x)u_{xx}(x,t)dx$$

$$+a(t)\int_0^1 H(x)u(x,t)dx + b(t)\int_0^1 H(x)g(x,t)dx + \int_0^1 H(x)f(x,t)dx,\ 0 \le t \le T. \tag{13}$$

Hence, from (13), taking into account (5) and (11), we arrive at (9).

Now suppose that the triple $\{u(x,t), a(t), b(t)\}$ is a solution to the problem (1)–(3), (8), (9). Then from (8) and (12), we get

$$c(t)\frac{d}{dt}(u(0,t)-h_1(t)) = a(t)(u(0,t)-h_1(t)),\ 0 \le t \le T. \tag{14}$$

Using (2) and the compatibility condition (8), we obtain the following relation

$$u(0,0)-h_1(0)+\delta(u(0,T)-h_1(T))+\int_0^T p(t)(u(0,t)-h_1(t))dt$$

$$= u(0,0)+\delta u(0,T)+\int_0^T p(t)u(0,t)dt-(h_1(0)+\delta+h_1(T))+\int_0^T p(t)h_1(t)dt$$

$$= \varphi(0)-\left(h_1(0)+\delta h_1(T)+\int_0^T p(t)h_1(t)dt\right) = 0. \tag{15}$$

From (14), (15), by virtue of Lemma 1, we conclude that condition (4) is satisfied.

In turn, from (9) and (13) we have

$$c(t)\frac{d}{dt}\left(\int_0^1 H(x)u(x,t)dx-h(t)\right) = a(t)\left(\int_0^1 H(x)u(x,t)dx-h(t)\right),\ 0 \le t \le T. \tag{16}$$

By virtue of (2) and the compatibility condition (7), we may write

$$\int_0^1 H(x)u(x,0)dx-h(0)+\delta\left(\int_0^1 H(x)u(x,T)dx-h(T)\right)$$

$$+\int_0^T p(t)\left(\int_0^1 H(x)u(x,t)dx-h(t)\right)dt$$

$$= \int_0^1 H(x)\left[u(x,0)+\delta u(x,T)+\int_0^T p(t)u(x,t)dt\right]dx$$

$$-\left(h(0)+\delta h(T)+\int_0^T p(t)h(t)dt\right)$$

$$= \int_0^1 H(x)\varphi(x)dx-\left(h(0)+\delta h(T)+\int_0^T p(t)h(t)dt\right) = 0. \tag{17}$$

From (16), (17), by virtue of Lemma 1, we conclude that condition (5) is satisfied. The proof is complete.

2. Classical Solvability of Inverse Boundary Value Problem

Let us now consider problem (1)–(3), (8), (9), and assume that $a(t)$ and $b(t)$ are known functions. Since the system $\{\cos\lambda_k x\}_{k=1}^{\infty}$, for $\lambda_k = \frac{\pi}{2}(2k-1)$ is a complete orthogonal system in $L_2(0,1)$, then

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\cos\lambda_k x,\ \ \lambda_k = \frac{\pi}{2}(2k-1), \tag{18}$$

where

$$u_k(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx,\ \ k = 1,2,....$$

Applying the formal scheme of the Fourier method, from (1) and (2), we have

$$c(t)u_k'(t) + \lambda_k^2 u_k(t) = f_k(t) + a(t)u_k(t) + b(t)g_k(t),\ \ k = 1,2,...;\ \ 0 \le t \le T, \tag{19}$$

$$u_k(0) + \delta u_k(T) + \int_0^T p(t)u_k(t)dt = \varphi_k,\ \ k = 1,2,\ldots, \tag{20}$$

where

$$f_k(t) = 2\int_0^1 f(x,t)\cos\lambda_k x dx,\ \ g_k(t) = 2\int_0^1 g(x,t)\cos\lambda_k x dx,$$

$$\varphi_k = 2\int_0^1 \varphi(x)\cos\lambda_k x dx, k = 1,2,\ldots.$$

Solving problem (19), (20), we find

$$u_k(t) = \frac{e^{-\int_0^t \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\left(\varphi_k - \int_0^T p(t)u_k(t)dt\right)$$

$$-\frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\int_0^T \frac{1}{c(\tau)}F_k(\tau;u,a,b)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}d\tau$$

$$+\int_0^t \frac{1}{c(\tau)}F_k(\tau;u,a,b)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}d\tau,\ k = 1,2,\ldots, \tag{21}$$

where $F_k(\tau;u,a,b) = f_k(t) + a(t)u_k(t) + b(t)g_k(t),\ \ k = 1,2,\ldots.$

Substituting the expressions of $u_k(t),\ k=1,2,...$ into (18), we get

$$u(x,t)=\sum_{k=1}^{\infty}\left\{\frac{e^{-\int_0^t \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\left(\varphi_k-\int_0^T p(t)u_k(t)dt\right)-\frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\right.$$

$$\left.\times\int_0^T \frac{1}{c(\tau)}F_k(\tau;u,a,b)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}d\tau+\int_0^t \frac{1}{c(\tau)}F_k(\tau;u,a,b)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}d\tau\right\}\cos\lambda_k x. \quad (22)$$

Now from (8) and (9), taking into account (18), respectively, we have

$$a(t)=[h(t)]^{-1}\left\{(c(t)h_1'(t)-f(0,t))\int_0^1 H(x)g(x,t)dx\right.$$

$$-(c(t)h_2'(t)-\int_0^1 H(x)f(x,t)dx)g(0,t))$$

$$\left.+\sum_{k=1}^{\infty}\lambda_k^2 u_k(t)\left(\int_0^1 H(x)g(x,t)dx-g(0,t)\int_0^1 H(x)\cos\lambda_k x dx\right)\right\},\ 0\le t\le T, \quad (23)$$

$$b(t)=[h(t)]^{-1}\left\{(c(t)h_2'(t)-\int_0^1 H(x)f(x,t)dx)h_1(t))-(c(t)h_1'(t)-f(0,t)h_2(t))\right.$$

$$\left.+\sum_{k=1}^{\infty}\lambda_k^2 u_k(t)\left(h_1(t)\int_0^1 H(x)\cos\lambda_k x dx-h_2(t)\right)\right\},\ 0\le t\le T, \quad (24)$$

where

$$h(t)\equiv h_1(t)\int_0^1 H(x)g(x,t)dx-g(0,t)h_2(t)\neq 0,\ 0\le t\le T.$$

Furthermore, in order to obtain the second and third components of the triple $\{u(x,t),a(t),b(t)\}$, we substitute the expression (21) into (23) and (24)

$$a(t)=[h(t)]^{-1}\left\{(c(t)h_1'(t)-f(0,t))\int_0^1 H(x)g(x,t)dx\right.$$

$$-(c(t)h_2'(t)-\int_0^1 H(x)f(x,t)dx)g(0,t))$$

$$+\sum_{k=1}^{\infty}\lambda_k^2\left[\frac{e^{-\int_0^t \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\left(\varphi_k-\int_0^T p(t)u_k(t)dt\right)\right.$$

$$-\frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\int_0^T \frac{1}{c(\tau)}F_k(\tau;u,a,b)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}d\tau$$

$$\left.+\int_0^t \frac{1}{c(\tau)}F_k(\tau;u,a,b)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}d\tau\right]$$

$$\left.\times\left(\int_0^1 H(x)(g(x,t)-g(0,t)\cos\lambda_k x)dx\right)\right\},\ 0\le t\le T, \tag{25}$$

$$b(t)=[h(t)]^{-1}\left\{(c(t)h_2'(t)-\int_0^1 H(x)f(x,t)dx)h_1(t))-(c(t)h_1'(t)-f(0,t)h_2(t))\right.$$

$$+\sum_{k=1}^{\infty}\lambda_k^2\left[\frac{e^{-\int_0^t \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\left(\varphi_k-\int_0^T p(t)u_k(t)dt\right)\right.$$

$$-\frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\int_0^T \frac{1}{c(\tau)}F_k(\tau;u,a,b)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}d\tau$$

$$\left.+\int_0^t \frac{1}{c(\tau)}F_k(\tau;u,a,b)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}d\tau\right]$$

$$\left.\times\left(h_1(t)\int_0^1 H(x)\cos\lambda_k x dx-h_2(t)\right)\right\},\ 0\le t\le T. \tag{26}$$

Thus, finding a solution to the problem (1)–(3), (8), (9) was reduced to finding a solution of system (22), (25), (26), with respect to the unknown functions $u(x,t)$ and $a(t)$.

The following lemma plays an important role to study the uniqueness of the solution of problem (1)–(3), (8), (9).

Lemma 2. *If $\{u(x,t),a(t),b(t)\}$ is any solution of (1)–(3), (8), (9) then the functions*

$$u_k(t)=2\int_0^1 u(x,t)\cos\lambda_k x dx,\ k=1,2,...,$$

satisfy the system (21) *on the interval* $[0,T]$.

Obviously, if $u_k(t) = 2\int\limits_0^1 u(x,t)\cos\lambda_k x dx,\ k = 1,2,...$ is a solution to system (21), then the functions $u(x,t) = \sum\limits_{k=1}^{\infty} u_k(t)\cos\lambda_k x\ (\lambda_k = \frac{\pi}{2}(2k-1)),\ a(t)$ and $a(t)$ is also a solution of system (22), (25), (26).

It follows from Lemma 2 that

Corollary. *Suppose that system* (22), (25), (26) *has a unique solution. Then the problem* (1)–(3), (8), (9) *couldn't have more than one solution, in other words, if problem* (1)–(3), (8), (9) *has a solution, then it is a unique.*

In order to study the problem (1)–(3), (8), (9), we introduce the following functional spaces: Let $B_{2,T}^3$ denote the set of all functions of the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\cos\lambda_k x,\ \ \lambda_k = \frac{\pi}{2}(2k-1),$$

considered in domain D_T, where the functions $u_k(t),\ k = 1,2,...$ are continuous on $[0,T]$, and satisfy the condition

$$J(u) = \left(\sum_{k=1}^{\infty} (\lambda_k^3 \left\| u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}} < +\infty.$$

The norm in the space $B_{2,T}^3$ is defined as follows:

$$\left\| u(x,t)\right\|_{B_{2,T}^3} = J(u);$$

We denote by E_T^3 the topological product of $B_{2,T}^3 \times C[0,T] \times C[0,T]$. The norm of the vector function $z(x,t) = \{u(x,t), a(t), b(t)\}$ is determined by the formula

$$\left\| z(x,t)\right\|_{E_T^3} = \left\| u(x,t)\right\|_{B_{2,T}^3} + \left\| a(t)\right\|_{C[0,T]} + \left\| b(t)\right\|_{C[0,T]}.$$

It is known that the spaces $B_{2,T}^3$ and E_T^3 are Banach spaces [19].

Now, consider the operator

$$\Phi(u,a) = \{\Phi_1(u,a,b),\ \Phi_2(u,a,b),\ \Phi_3(u,a,b)\}$$

in the space E_T^3, where

$$\Phi_1(u,a,b) = \tilde{u}(x,t) \equiv \sum_{k=1}^{\infty} \tilde{u}_k(t)\cos\lambda_k x,\quad \Phi_2(u,a,b) = \tilde{a}(t),\quad \Phi_3(u,a,b) = \tilde{b}(t),$$

and the functions $\tilde{u}_k(t),\ k = 1,2,...,\ a(t)$, and $b(t)$ are equal to the right-hand sides of (21), (25), (26), respectively.

We impose the following conditions on the data of problem (1)–(3), (8), (9):

$H_1)$ $\varphi(x) \in C^2[0,1],\ \varphi^{(3)}(x) \in L_2(0,1),\ \varphi'(0) = \varphi(1) = \varphi''(1) = 0;$
$H_2)$ $f(x,t),\ f_x(x,t),\ f_{xx}(x,t) \in C(D_T),\ f_{xxx}(x,t) \in L_2(D_T),$
$f_x(0,t) = f(1,t) = f_{xx}(1,t) = 0,\ 0 \le t \le T;$
$H_3)$ $g(x,t),\ g_x(x,t),\ g_{xx}(x,t) \in C(D_T),\ g_{xxx}(x,t) \in L_2(D_T),$
$g_x(0,t) = g(1,t) = g_{xx}(1,t) = 0,\ 0 \le t \le T;$
$H_4)$ $\delta \ge 0,\ 0 < c(t) \in C[0,T],\ 0 \le p(t) \in C[0,T],\ H(x) \in C[0,1],$
$h_i(t) \in C^1[0,T]\ (i = 1,2),\ h(t) \equiv h_1(t) \int\limits_0^1 H(x)g(x,t)dx - g(0,t)h_2(t) \ne 0,\ 0 \le t \le T.$

Then, using simple transformations from (21), (25) and (26), respectively, we find that the inequalities

$$\left(\sum_{k=1}^{\infty} (\lambda_k^3 \|\tilde{u}_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le 2\sqrt{2}\left(\sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_k|)^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{2}T\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+2(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{2T}\left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^3 |f_k(\tau)|)^2 d\tau\right)^{\frac{1}{2}}$$

$$+2(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{2}T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+2(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{2T}\|a(t)\|_{C[0,T]}\left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^3 |g_k(\tau)|)^2 d\tau\right)^{\frac{1}{2}}, \tag{27}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|(c(t)h_1'(t) - f(0,t))\int_0^1 H(x)g(x,t)dx\right\|_{C[0,T]}\right.$$

$$+\left\|(c(t)h_2'(t) - \int_0^1 H(x)f(x,t)dx)g(0,t))\right\|_{C[0,T]}$$

$$+\left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{\frac{1}{2}}\|H(x)\|_{C[0,1]}(\|g(x,t)\|_{C(D_T)} + \|g(0,t)\|_{C[0,T]})\left[\left(\sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_k|)^2\right)^{\frac{1}{2}}\right.$$

$$+T\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+(1+\delta)\sqrt{T}\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\left|f_k(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}$$

$$+(1+\delta)T\left\|\frac{1}{c(t)}\right\|_{C[0,T]}T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+(1+\delta)\sqrt{T}\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\left\|a(t)\right\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\left|g_k(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\Bigg]\Bigg\}, \tag{28}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\Bigg\{\left\|((c(t)h_2'(t)-\int_0^1H(x)f(x,t)dx)h_1(t))\right\|_{C[0,T]}$$

$$+\left\|(c(t)h_1'(t)-f(0,t)h_2(t))\right\|_{C[0,T]}$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(\left\|H(x)\right\|_{C[0,1]}\left\|h_1(t)\right\|_{C[0,T]}+\left\|h_2(t)\right\|_{C[0,T]})\Bigg[\left(\sum_{k=1}^{\infty}(\lambda_k^3\left|\varphi_k\right|)^2\right)^{\frac{1}{2}}$$

$$+T\left\|p(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+(1+\delta)\sqrt{T}\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\left|f_k(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}$$

$$+(1+\delta)T\left\|\frac{1}{c(t)}\right\|_{C[0,T]}T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+(1+\delta)\sqrt{T}\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\left\|a(t)\right\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\left|g_k(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\Bigg]\Bigg\}, \tag{29}$$

holds true.

Now, taking into account H_1) - H_4), the estimates (27)–(29) can be written in the form:

$$\left\|\tilde{u}(x,t)\right\|_{B_{2,T}^3}\leq A_1(T)$$

$$+B_1(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B_{2,T}^3}+C_1(T)\left\|u(x,t)\right\|_{B_{2,T}^3}+D_1(T)\left\|b(t)\right\|_{C[0,T]}, \tag{30}$$

$$\left\|\tilde{a}(t)\right\|_{C[0,T]}\leq A_2(T)$$

$$+B_2(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B_{2,T}^3}+C_2(T)\left\|u(x,t)\right\|_{B_{2,T}^3}+D_2(T)\left\|b(t)\right\|_{C[0,T]}, \tag{31}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]} \leq A_3(T)$$

$$+B_3(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} + C_3(T)\,\|u(x,t)\|_{B^3_{2,T}} + D_3(T)\,\|b(t)\|_{C[0,T]}\,, \qquad (32)$$

where

$$A_1(T) = 2\sqrt{2}\left\|\varphi^{(3)}(x)\right\|_{L_2(0,1)} + 2\sqrt{2T}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\|f_{xxx}(x,t)\|_{L_2(D_T)}\,,$$

$$B_1(T) = 2\sqrt{2}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]} T,$$

$$C_1(T) = \sqrt{6}T\,\|p(t)\|_{C[0,T]}\,,$$

$$D_1(T) = 2\sqrt{2T}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\|g_{xxx}(x,t)\|_{L_2(D_T)}\,,$$

$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|(c(t)h_1'(t) - f(0,t))\int_0^1 H(x)g(x,t)dx\right\|_{C[0,T]}\right.$$

$$+\left\|(c(t)h_2'(t) - \int_0^1 H(x)f(x,t)dx)g(0,t))\right\|_{C[0,T]}$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\|H(x)\|_{C[0,1]}\left(\|g(x,t)\|_{C(D_T)} + \|g(0,t)\|_{C[0,T]}\right)$$

$$\left.\times\left[\left\|\varphi^{(3)}(x)\right\|_{L_2(0,1)} + (1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)}\right]\right\},$$

$$B_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\|H(x)\|_{C[0,1]}$$

$$\times(\|g(x,t)\|_{C(D_T)} + \|g(0,t)\|_{C[0,T]})(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]} T,$$

$$C_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\|H(x)\|_{C[0,1]}$$

$$\times(\|g(x,t)\|_{C(D_T)} + \|g(0,t)\|_{C[0,T]})\,\|p(t)\|_{C[0,T]}\,T,$$

$$D_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\|H(x)\|_{C[0,1]}$$

$$\times(\|g(x,t)\|_{C(D_T)}+\|g(0,t)\|_{C[0,T]})(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{T}\,\|g_{xxx}(x,t)\|_{L_2(D_T)},$$

$$A_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|(c(t)h_2'(t)-\int\limits_0^1 H(x)f(x,t)dx)h_1(t))\right\|_{C[0,T]}\right.$$

$$+\left\|(c(t)h_1'(t)-f(0,t)h_2(t))\right\|_{C[0,T]}$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(\|H(x)\|_{C[0,1]}\,\|h_1(t)\|_{C[0,T]}+\|h_2(t)\|_{C[0,T]})$$

$$\left.\times\left[\left\|\varphi^{(3)}(x)\right\|_{L_2(0,1)}+(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)}\right]\right\},$$

$$B_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}$$

$$\times(\|H(x)\|_{C[0,1]}\,\|h_1(t)\|_{C[0,T]}+\|h_2(t)\|_{C[0,T]})(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}T,$$

$$C_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}$$

$$\times(\|H(x)\|_{C[0,1]}\,\|h_1(t)\|_{C[0,T]}+\|h_2(t)\|_{C[0,T]})\,\|p(t)\|_{C[0,T]}\,T,$$

$$D_3(T)=(1+\delta)\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}$$

$$\times(\|H(x)\|_{C[0,1]}\,\|h_1(t)\|_{C[0,T]}+\|h_2(t)\|_{C[0,T]})\sqrt{T}\,\|g_{xxx}(x,t)\|_{L_2(D_T)}.$$

Further, estimates (30)–(32) imply that

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}}+\|\tilde{a}(t)\|_{C[0,T]}+\left\|\tilde{b}(t)\right\|_{C[0,T]}\leq A(T)$$

$$+B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}+C(T)\,\|u(x,t)\|_{B^3_{2,T}}+D(T)\,\|b(t)\|_{C[0,T]},\tag{33}$$

where

$$A(T)=A_1(T)+A_2(T)+A_3(T),\ B(T)=B_1(T)+B_2(T)+B_3(T),$$

$$C(T)=C_1(T)+C_2(T)+C_3(T),\ D(T)=D_1(T)+D_2(T)+D_3(T).$$

Let us prove the following theorem.

Theorem 2. *Let the conditions $H_1)$ - $H_4)$ and the condition*

$$((A(T)+2)B(T)+C(T)+D(T))(A(T)+2)<1,\tag{34}$$

be fulfilled. Then, problem (1)–(3), (8), (9) *has a unique solution in the ball* $K=K_R$ $(\|z\|_{E^3_T}\leq A(T)+2)$.

Proof. First, we write the system of equations (22), (25), (26) in the operator form

$$z = \Phi z, \tag{35}$$

where $z = \{u, a, b\}$. The components $\Phi_i(u, a, b),\ i = 1, 2, 3$ of operator $\Phi(u, a, b)$ defined by the right side of equations (22), (25), and (26), respectively.

Now, consider the operator $\Phi(u, a, b)$ in the ball $K = K_R$ of the space E_T^3 and show that the operator Φ takes the elements of the ball $K = K_R$ into itself. Similar to (33), we obtain that for any $z \in K_R$ the following inequality holds

$$\|\Phi z\|_{E_T^3} \le A(T)$$

$$+B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^3} + C(T)\,\|u(x,t)\|_{B_{2,T}^3} + D(T)\,\|b(t)\|_{C[0,T]}$$

$$\le A(T) + B(T)(A(T)+2)^2 + C(T)(A(T)+2) + D(T)(A(T)+2).$$

Then by (34), from last estimate it is clear that the operator Φ acts in a ball $K = K_R$ and it can be show that the operator Φ is contractive. Indeed, it is clear that for any $z_1, z_2 \in K_R$ the inequality

$$\|\Phi z_1 - \Phi z_2\|_{E_T^3} \le (B(T)(A(T)+2) + C(T) + D(T))$$

$$\times(\|u_1(x,t) - u_2(x,t)\|_{B_{2,T}^3} + \|a_1(t) - a_2(t)\|_{C[0,T]} + \|b_1(t) - b_2(t)\|_{C[0,T]})$$

is satisfied. In turn, this means that, by virtue of (34), the operator Φ is contractive.

Therefore, the operator Φ satisfies assertions of the contraction mapping principle in the ball $K = K_R$. That is why the operator Φ has a unique fixed point $\{z\} = \{u, a, b\}$ in the ball $K = K_R$, which is a unique solution of equation (35); i.e. $\{z\} = \{u, a, b\}$ is a unique solution of the system (22), (25), (26) in the ball $K = K_R$.

Thus, the function $u(x,t)$ as an element of space $B_{2,T}^3$ is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in D_T.

On the other hand, from equation (19) it is easy to see that

$$\left(\sum_{k=1}^{\infty}(\lambda_k\,\|u_k'(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{2}\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\sqrt{2}\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\left\|\|f_x(x,t) + a(t)u_x(x,t) + b(t)g_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)} < +\infty.$$

Thus, the function $u_t(x,t)$ is continuous in D_T.

It can be verified that Eq. (1) and conditions (2), (3), (8), (9) are satisfied in the usual sense. Therefore, $\{u(x,t), a(t), b(t)\}$ is a solution to problem (1)–(3), (8), (9) and by the corollary of Lemma 2, it is unique in the ball $K = K_R$. The theorem is proved.

Theorem 3. *Let all assertions of Theorem 2 and compatibility conditions* (6), (7) *be satisfied, then problem* (1)–(5) *has a unique classical solution in the ball* $K = K_R$ $(\|z\|_{E_T^3} \le A(T) + 2)$ *of the space* E_T^3.

3. Conclusions

The goal of this paper was to investigate the unique solvability of an inverse boundary value problem for a second order parabolic equation with nonlocal boundary conditions. First, the original problem was reduced to an auxiliary problem with trivial data. Then using the Fourier method and contraction mappings principle, the existence and uniqueness theorem for auxiliary problem is proved. Further, on the basis of the equivalency of these problems, the existence and uniqueness theorem for the classical solution of the original inverse boundary value problem is established.

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