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OPTIMAL CONTROL OF A DYNAMIC SYSTEM WITH NON-SEPARADED POINT AND INTEGRAL CONDITIONS

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Abstract. *In the work, a numerical method is proposed to solve a system of linear non-autonomous ordinary differential equations with non-separated multipoint and integral conditions. The proposed approach is based on the introduced operation of convolution of integral conditions into local ones. The method allows to reduce the solution of the original problem to the solution of Cauchy problem with respect to a system of ordinary differential equations and of linear algebraic equations. The stability of the computational schemes is shown.*

Keywords: optimal control, gradient of functional, integral conditions, intermediate conditions, non-local multipoint conditions

Mathematics Subject Classification (2020): 49J15, 65L10

1. Introduction

Much research activity in the past years has been directed at solving boundary problems involving non-local multipoint and integral conditions, and the corresponding optimal control problems. This is connected with non-local character of information provided by measurement equipment. Namely, the measurements are not taken instantly, but during some time interval and the measurements at a separate point actually characterize the state of the object in some domain which contains the measurement point. Problems of this kind arise when controlling an object if it is otherwise impossible to affect the object instantly at time and locally at its separate points [5], [8]-[13].

In the present work, we derive formulas for the gradient of the target functional in optimal control problems with non-separated multipoint and integral conditions. To solve the corresponding boundary value problems, we use the approach proposed by the author

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in the works [1]-[4], [6], [7]. Results of some numerical experiments and their analysis will be given at the presentation.

2. Problem Statement

Consider the following optimal control problem:

$$\dot{x}(t) = A(t,u)x(t) + B(t,u)\,,\ t \in [t_0, T]\,, \tag{1}$$

$$\sum_{i=1}^{l_1} \int_{\bar{t}_{2i-1}}^{\bar{t}_{2i}} \bar{D}_i(\tau)x(\tau)d\tau + \sum_{j=1}^{l_2} \tilde{D}_j x(\tilde{t}_j) = C_0, \tag{2}$$

$$J(u) = \Phi(x(\hat{t})) + \int_{t_0}^{T} f^0(x,u,t)dt \to \min_{u(t)\in U}, \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the phase variable; $u(t) \in U \subset \mathbb{R}^r$ is the control function from the class of piecewise continuous functions, admissible values of $u(t)$ belong to a given compact set U; $A(t,u) \neq const$ is $(n\times n)$ matrix function, $B(t,u)$ is $n-$dimensional vector function, $A(t,u)$, $B(t,u)$ are continuous with respect to t and continuously differentiable with respect to u; $\bar{D}_i\,(\tau)$ is the continuously differentiable $(n\times n)$ matrix function; $\tilde{D}_j$ is the $(n\times n)$ scalar matrix; C_0 is the n-dimensional vector; $\bar{t}_i$,$\tilde{t}_j$ are given time instances from $[t_0, T]$; the function Φ and its partial derivatives are continuous with respect to its arguments, and $f^0(x,u,t)$ is continuously differentiable with respect to (x,u) and continuous with respect to t; $\hat{t} = (\hat{t}_1, \hat{t}_2, ..., \hat{t}_{2l_1+l_2})$ is the ordered union of the sets $\tilde{t} = (\tilde{t}_1, \tilde{t}_2, ..., \tilde{t}_{l_2})$ and $\bar{t} = (\bar{t}_1, \bar{t}_2, ..., \bar{t}_{2l_1})$, i.e. $\hat{t}_j < \hat{t}_{j+1}$, $\quad j = 1,2,..,2l_1+l_2-1$, $\hat{x}(\hat{t}) = (x(\hat{t}_1), ..., x(\hat{t}_{2l_1+l_2}))$.

Without loss of generality, assume that

$$\min\left(\bar{t}_1,\ \tilde{t}_1\right) = t_0, \quad \max\left(\bar{t}_{2l_1},\ \tilde{t}_{l_2}\right) = T, \tag{4}$$

$$\tilde{t}_j \bar{\in} \left[\bar{t}_{2i-1}, \bar{t}_{2i}\right], \quad i = 1,2,..,2l_1,\ \ j = 1,2,..,l_2\,. \tag{5}$$

3. Formula for the Gradient of the Functional of Problem

To solve optimal control problem (1)–(5) numerically with the application of first order optimization methods, we obtain formulas for the gradient of the functional.

With respect to an arbitrary admissible process $(u(t), x(t;u))$, we define problem (1), (2) in increments, corresponding to an admissible control $\tilde{u} = u + \Delta u$:

$$\Delta\dot{x}(t) = A(t,u)\Delta x(t) + \Delta_u A(t,u)x(t) + \Delta_u B(t,u)\,,$$

$$\sum_{i=1}^{l_1}\int\limits_{\bar t_{2i-1}}^{\bar t_{2i}}\bar D_i(\tau)\Delta x(\tau)d\tau+\sum_{j=1}^{l_2}\tilde D_j\Delta x(\tilde t_j)=0.$$

Here the following notations are used:

$$\Delta x(t)=x(t,\tilde u)-x(t,u),\quad \Delta_u A(t,u)=A(t,\tilde u)-A(t,u),$$
$$\Delta_u B(t,u)=B(t,\tilde u)-B(t,u).$$

Let $\psi(t)\in\mathbb{R}^n$- be an almost everywhere continuously differentiable vector function and let $\lambda\in\mathbb{R}^n$ be as yet arbitrary numerical vector. Taking into account that $x(t)$ and $x(t)+\Delta x(t)$ are the solutions to problem (1)–(2) under corresponding values of the controls, we can write:

$$J(u)=\Phi(x(\hat t))+\int\limits_{t_0}^{T}f^0(x,u,t)dt+\int\limits_{t_0}^{T}\psi^*(t)\left[\dot x(t)-A(t,u)x(t)-B(t,u)\right]dt+$$
$$+\lambda^*\left[\sum_{i=1}^{l_1}\int\limits_{\bar t_{2i-1}}^{\bar t_{2i}}\bar D_i(\tau)x(\tau)d\tau+\sum_{j=1}^{l_2}\tilde D_j x(\tilde t_j)-C_0\right],$$

$$J(u+\Delta u)=\Phi(x(\hat t)+\Delta x(\hat t))+\int\limits_{t_0}^{T}f^0(x+\Delta x,u+\Delta u,t)dt+$$
$$+\int\limits_{t_0}^{T}\psi^*(t)\left[\dot x+\Delta\dot x)-A(t,u+\Delta u)(x+\Delta x)-B(t,u+\Delta u)dt+\right.$$
$$+\lambda^*\left[\sum_{i=1}^{l_1}\int\limits_{\bar t_{2i-1}}^{\bar t_{2i}}\bar D_i(\tau)(x(\tau)+\Delta x(\tau))d\tau+\sum_{j=1}^{l_2}\tilde D_j(x(\tilde t_j)+\Delta x(\tilde t_j))-C_0\right].$$

where "*" is the transposition sign. Then for the increment of the target functional, after performing partial integration and grouping of the corresponding terms, with an accuracy up to the terms of the first order of smallness, we shall have:

$$\Delta J(u)=\int\limits_{t_0}^{T}\left[-\dot\psi^*(t)-\psi^*(t)A(t,u)+\lambda^*\sum_{i=1}^{l_1}\left[\chi(\bar t_{2i})-\chi(\bar t_{2i-1})\right]\bar D_i(t)+\right.$$
$$\left.+f_x^0(x,u,t)\right]\Delta x(t)dt+\int\limits_{t_0}^{T}\left\{f_u^0(x,u,t)+\psi^*(t)\left[-A_u^*(t,u)x(t)-\right.\right.$$

$$-B_u(t,u)]\,\Delta u(t)dt+\sum_{k=2}^{2l_1+l_2-1}\left[\psi^{*-}(\hat t_k)-\psi^{*+}(\hat t_k)+\frac{\partial\Phi(\hat x(\hat t))}{\partial x(\hat t_k)}\right]\Delta x(\hat t_k)+$$

$$+\sum_{j=1}^{l_2}\lambda^*\tilde D_j\Delta x(\tilde t_j)+\psi^*(T)\Delta x(T)-\psi^*(t_0)\Delta x(t_0)+$$

$$+\int_{t_0}^{T}o_1(\|\Delta x(t)\|)dt+\int_{t_0}^{T}o_2(\|\Delta u(t)\|)dt+o_3(\|\Delta\hat x(\hat t_k)\|),$$

where $\psi^+(\hat t_k)=\psi(\hat t_k+0)$, $\psi^-(\hat t_k)=\psi(\hat t_k-0)$, $k=1,2,..,(2l_1+l_2)$.

Here and from now onwards, the norms of the vector-functions $\|x(t)\|$ and $\|u(t)\|$ will be thought of as $\|x(t)\|_{L_2^r[t_0,T]}$ and $\|u(t)\|_{L_2^r[t_0,T]}$, respectively; $A_u(t,u)=\left(\left(\frac{\partial A_{ij}(t,u)}{\partial u_s}\right)\right)$ and $B_u(t,u)=\left(\left(\frac{\partial B_i(t,u)}{\partial u_s}\right)\right)$ are regarded as $(n\times n\times r)$ and $(n\times r)$ matrices, respectively, the result of their transposition being $(n\times r\times n)$and $(r\times n)$ matrices $A_u^*(t,u)$ and $B_u^*(t,u)$, respectively.

Let $\psi(t)$ be the solution to the following system of equations

$$\dot\psi(t)=-A^*(t,u)\psi(t)+\sum_{i=1}^{l_1}\left[\chi(\bar t_{2i})-\chi(\bar t_{2i-1})\right]\bar D^*(t)\lambda+f_x^{0*}(x,u,t), \tag{6}$$

with the following boundary conditions

$$\psi(t_0)=\begin{cases}\left(\frac{\partial\Phi(\hat x(\hat t))}{\partial x(\tilde t_1)}\right)^*+\tilde D_1^*\lambda, & if\ t_0=\tilde t_1\,,\\[2ex] \left(\frac{\partial\Phi(\hat x(\hat t))}{\partial x(\bar t_1)}\right)^*, & if\ t_0=\bar t_1\,,\end{cases} \tag{7}$$

$$\psi(T)=\begin{cases}-\left(\frac{\partial\Phi(\hat x(\hat t))}{\partial x(\tilde t_{l_2})}\right)^*-\tilde D_{l_2}^*\lambda, & if\ \tilde t_{l_2}=T\,,\\[2ex] -\left(\frac{\partial\Phi(\hat x(\hat t))}{\partial x(\bar t_{2l_1})}\right)^*, & if\ \bar t_{2l_1}=T,\end{cases} \tag{8}$$

as well as jump conditions at the intermediate $\tilde t_j$ such that $t_0<\tilde t_j<T$,

$$\psi^+(\tilde t_j)-\psi^-(\tilde t_j)=\left(\frac{\partial\Phi(\hat x(\hat t))}{\partial x(\tilde t_j)}\right)^*+\tilde D_j^*\lambda,\ \ j=1,...\,l_2, \tag{9}$$

and at the points $\bar t_i$, $i=1,2,...,2\,l_1$ such that $t_0<\bar t_i<T$,

$$\psi^+(\bar t_i)-\psi^-(\bar t_i)=\left(\frac{\partial\Phi(\hat x(\hat t))}{\partial x(\bar t_i)}\right)^*,\ \ i=1,2,...\,2l_1. \tag{10}$$

Thus the gradient of the target functional under the admissible control $u(t)$ in problem (1)-(3) is determined as follows:

$$(grad\,J(u))^* = f_u^0(x,u,t) + \psi^*(t)\left[-A_u^*(t,u)x(t) - B_u(t,u)\right], \tag{11}$$

where $x(t)$ and $\psi(t)$ are the solutions to direct system (1), (2) and to adjoint system (6)-(10), respectively, corresponding to this control.

On condition that there is a constructive algorithm for computing the value of the gradient of functional (11), it is not difficult to implement iterative techniques of first order minimization, particularly, of gradient projection method [14]:

$$u^{k+1}(t) = P_U(u^k(t) - \alpha_k\; grad\; J(u^k(t))), \quad k = 0, 1, \dots, \tag{12}$$

$$\alpha_k = \arg\min_{\alpha \geq 0} J(P_U(u^k(t) - \alpha\; gradJ(u^k(t)))),$$

where $P_U(\upsilon)$ is the projection operator of the element $\upsilon \in \mathbb{R}^r$ on the admissible set U; α_k is the one-dimensional minimization step.

On every iteration (12), the calculation of the gradient of the functional under given control confronts with two the most essential difficulties associated with the specific character of the problem, namely, with the problem of solution to non-autonomous differential equations system involving non-separated multipoint and integral conditions (1), (2), and with the problem of solution to adjoint boundary problem (6)–(10), the non-local conditions of which contain an unknown n-dimensional vector of parameters λ. As a whole, system of relations (1), (2), (6)–(10) for determining the gradient of the functional under given control $u(t)$ is closed: to determine unknown $2n$ functions $x(t), \psi(t)$, their $2n$ initial conditions, and n-dimensional vector λ, we have $2n$-dimensional differential equations system, n conditions in (2) and $2n$ conditions in (7), (8).

4. Numerical Scheme of Solution to the Problem

We shall give below an algorithm of computing the gradient of the target functional under given control. To solve the problem (1), (2) under given admissible function $u(t)$, we can make use of, for example, the numerical method proposed in [3], [4], [6].

Solve the adjoint boundary value problem (6)–(10) on the condition that the phase variable $x(t)$, the solution to the problem (1) and (2), is already determined for given $u(t)$ by application of the procedure described above. To avoid cumbersome expressions in the formulas and in the description of numerical scheme of solution given below, we assume that $\tilde{t}_1 = t_0$ and $\tilde{t}_{l_2} = T$, and rewrite adjoint problem (6)–(10) in the following form:

$$\dot{\psi}(t) = A_1(t)\psi(t) + \sum_{i=1}^{l_1}\left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right]\bar{D}_i(t)\lambda + C(t), \tag{13}$$

with the following boundary conditions:

$$\tilde{G}_1\psi(t_0) = \tilde{K}_1 + \tilde{D}_1\lambda, \tag{14}$$

$$\psi(T) = -\tilde{K}_{l_2} - \tilde{D}_{l_2}\lambda, \tag{15}$$

and jump conditions at the intermediate points $\tilde{t}_j$, for which $t_0 < \tilde{t}_j < T$:

$$\psi^+(\tilde{t}_j) - \psi^-(\tilde{t}_j) = \tilde{K}_j + \tilde{D}_j\lambda, \quad j = 2, 3, ..., l_2 - 1, \tag{16}$$

and at the points $\bar{t}_i$, for which $t_0 < \bar{t}_i < T$, $i = 1, 2, ... 2l_1$:

$$\psi^+(\bar{t}_i) - \psi^-(\bar{t}_i) = \bar{K}_i, \quad i = 1, 2, ..., 2\, l_1. \tag{17}$$

Here $\bar{G}_1 = I_n$ is the n-dimensional identity matrix, and the following notations are introduced for the matrices and vectors:

$$A_1(t) = -A^*(t, u),\;\; C(t) = \partial f^0(x, u, t)/\partial x,$$
$$\tilde{K}_j = \partial\Phi(x(\hat{t}))/\partial x(\tilde{t}_j),\;\; \tilde{D}^*_j = \tilde{D}_j,\;\; j = 1, 2, ..., l_2,$$
$$\bar{D}^*_i(t) = \bar{D}_i(t), \quad \bar{K}_i = \partial\Phi(x(\hat{t}))/\partial x(\bar{t}_i), \quad i = 1, 2, ..., 2l_1,$$

that were already calculated when solving the direct problem.

In problem (13)–(17), defined by system of n differential equations (13), in general case, we have $2n$ boundary conditions that include the unknown n-dimensional vector λ. Thus the conditions of problem (13)–(17) are closed, but there is a specific character which lies in the presence of discontinuities of the function $\psi(t)$ defined by jumps (16).

Condition (14) is called a condition shifted to the right in the semi-interval $t \in [\tilde{t}_1, \tilde{t}_2)$ by the matrix and vector functions $G_1(t),\;\; D_1(t) \in \mathbb{R}^{n\times n}$, $K_1(t) \in \mathbb{R}^n$ such that

$$G_1(\tilde{t}_1) = \tilde{G}_1, \;\; K_1(\tilde{t}_1) = \tilde{K}_1,\;\; \tilde{D}_1(\tilde{t}_1) = \tilde{D}_1\,, \tag{18}$$

if for the solution $\psi(t)$ to (20), the following relation holds:

$$G_1(t)\psi(t) = K_1(t) + D_1(t)\lambda\,, \quad t \in [\tilde{t}_1, \tilde{t}_2). \tag{19}$$

Next, using the results of [8], we give the techniques to find the shifting functions $G_1(t),\;\; D_1(t), K_1(t)$. By using formula (19), we shift initial conditions (18) to the point $t = \tilde{t}_2 - 0$ and, taking shift condition (16) into account at the point $t = \tilde{t}_2 + 0$, we obtain

$$G_1(\tilde{t}_2)\psi(\tilde{t}_2 + 0) = \Big[\; K_1(\tilde{t}_2) + G(\tilde{t}_2)\tilde{K}_2 \Big] + \Big[\; D_1(\tilde{t}_2) + G_1(\tilde{t}_2)\tilde{D}_2 \Big]\,\lambda.$$

Introducing the notations

$$\tilde{t}_2 = \tilde{t}_2 + 0,\, \tilde{G}^1_1 = G_1(\tilde{t}_2),\, \tilde{K}^1_1\, = K_1(\tilde{t}_2) + G(\tilde{t}_2)\tilde{K}_2,\, \tilde{D}^1_1 = D_1(\tilde{t}_2) + G_1(\tilde{t}_2)\tilde{D}_2,$$

we obtain the conditions similar to (14) and defined at the point $\tilde{t}_2$:

$$\tilde{G}^1_1\psi(\tilde{t}_2) = \tilde{K}^1_1 + \tilde{D}^1_1\lambda\,.$$

By shifting condition (14) $l_2 - 1$ times and taking (18) into account, we obtain a linear system of $2n$ algebraic equations with respect to $\psi(\tilde{t}_{l_2}) = \psi(T)$ and λ. After solving this system, we determine the vector function $\psi(t)$ from right to left from Cauchy problem with respect to (13).

Illustrate the stages of the shift process applied to condition (14). To be specific we assume that $[\bar{t}_1, \bar{t}_2] \subset [\tilde{t}_1, \tilde{t}_2)$ and $\tilde{t}_1 = t_0$. Shift of condition is carried out successively in the intervals $[\tilde{t}_1, \bar{t}_1)$, $[\bar{t}_1, \bar{t}_2)$, $[\bar{t}_2, \tilde{t}_2)$, by using formulas (19).

1) For $t \in [\tilde{t}_1, \bar{t}_1)$, we shift initial conditions (14) to the point $t = \bar{t}_1 - 0$ and, taking shift condition (17) into account at the point $t = \bar{t}_1$, we obtain

$$G_1(\bar{t}_1)\psi(\bar{t}_1 + 0) = \left[\; K_1(\bar{t}_1) + G_1(\bar{t}_1)\bar{K}_1 \right] + D_1(\bar{t}_1)\,\lambda.$$

Assuming $\bar{t}_1 = \bar{t}_1 + 0$, introduce the notations

$$\tilde{G}_1^1 = G_1(\bar{t}_1), \tilde{K}_1^1 = K_1(\bar{t}_1) + G_1(\bar{t}_1)\bar{K}_1, \tilde{D}_1^1 = D_1(\bar{t}_1),$$

following which, we obtain the initial conditions similar to (14) and defined at the point $\bar{t}_1$,

$$\tilde{G}_1^1\psi(\bar{t}_1) = \tilde{K}_1^1 + \tilde{D}_1^1\lambda\,. \tag{20}$$

2) For $t \in [\bar{t}_1, \bar{t}_2)$, we shift conditions (20) to the point $t = \bar{t}_2 - 0$ and, taking shift condition (17) into account at the point $t = \bar{t}_2$, we obtain

$$G_1(\bar{t}_2)\psi(\bar{t}_2 + 0) = \left[\; K_1(\bar{t}_2) + G_1(\bar{t}_2)\bar{K}_2 \right] + D_1(\bar{t}_2)\lambda.$$

Assuming $\bar{t}_2 = \bar{t}_2 + 0$, introduce the notations

$$\tilde{G}_1^2 = G_1(\bar{t}_2), \quad \tilde{K}_1^2 = K_1(\bar{t}_2) + G_1(\bar{t}_2)\bar{K}_2, \quad \tilde{D}_1^2 = D_1(\bar{t}_2),$$

and obtain initial conditions equivalent to (20) and defined at the point $\bar{t}_2$

$$\tilde{G}_1^2\psi(\bar{t}_2) = \tilde{K}_1^2 + \tilde{D}_1^2\lambda\,. \tag{21}$$

3) For $t \in [\bar{t}_2, \tilde{t}_2)$ we shift conditions (21) to the point $t = \tilde{t}_2 - 0$ and, taking shift condition (16) into account at the point $t = \tilde{t}_2$, we obtain

$$G_1(\tilde{t}_2)\psi(\tilde{t}_2 + 0) = \left[K_1(\tilde{t}_2) + G_1(\tilde{t}_2)\bar{K}_2 \right] + \left[D_1(\tilde{t}_2) + G_1(\tilde{t}_2)\tilde{D}_2 \right] \lambda.$$

Assuming $\tilde{t}_2 = \tilde{t}_2 + 0$, introduce the notations

$$\tilde{G}_1^3 = G_1(\tilde{t}_2),\, \tilde{K}_1^3 = K_1(\tilde{t}_2) + G_1(\tilde{t}_2)\bar{K}_2,\, \tilde{D}_1^3 = D_1(\tilde{t}_2) + G_1(\tilde{t}_2)\tilde{D}_2,$$

and obtain the conditions equivalent to (21) and defined at the point $\tilde{t}_2$

$$\tilde{G}_1^3\psi(\tilde{t}_2) = \tilde{K}_1^3 + \tilde{D}_1^3\lambda.$$

The functions $G_j(t),\ K_j(t),\ D_j(t),\ \ j = 1, 2, ..., l_2$ that shift conditions (14) successively to the right (i.e. the functions $G_j(t),\ K_j(t),\ D_j(t),\ \ j = 1, 2, ..., l_2$ must satisfy (18), (19), are not determined uniquely. For example, it is possible to use functions proposed in the following theorem.

Theorem 1. *Let the functions $G_1(t)$, $K_1(t)$, $D_1(t)$ be the solution to the following Cauchy problems for $t \in (\tilde{t}_1, \tilde{t}_2]$:*

$$
\begin{aligned}
&\dot{G}_1(t) = Q^0(t)G_1(t) - G_1(t)A_1(t), \quad G_1(\tilde{t}_1) = \tilde{G}_1,\\
&\dot{D}_1(t) = Q^0(t)D_1(t) + G_1(t)\sum_{i=1}^{l_1}\left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right]\bar{D}_i(t), \quad D_1(\tilde{t}_1) = \tilde{D}_1,\\
&\dot{K}_1(t) = Q^0(t)K_1(t) + G_1(t)C(t), \quad K_1(\tilde{t}_1) = \tilde{K}_1,\\
&\dot{Q}(t) = Q^0(t)Q(t), \qquad Q(\tilde{t}_1) = I_n,\\
&Q^0(t) = \left[G_1(t)A_1(t)G_1^*(t) - G_1(t)\sum_{i=1}^{l_1}\left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right]\bar{D}_i(t)D_1^*(t) - \right.\\
&\left. - G_1(t)C(t)K_1^*(t)\right] \times \left[G_1(t)G_1^*(t) + D_1(t)D_1^*(t) + K_1(t)K_1^*(t)\right]^{-1}.
\end{aligned}
\tag{22}
$$

Then these functions shift condition (14) to the right on the semi-interval $t \in [\tilde{t}_1, \tilde{t}_2)$, and relation (19) holds true for them. The following condition

$$
\begin{aligned}
&\|G_1(t)\|^2_{\mathbb{R}^{n\times n}} + \|D_1(t)\|^2_{\mathbb{R}^{n\times n}} + \|K_1(t)\|^2_{\mathbb{R}^n} =\\
&= \left\|\tilde{G}_1\right\|^2_{R^{n\times n}} + \left\|\tilde{D}_1\right\|^2_{\mathbb{R}^{n\times n}} + \left\|\tilde{K}_1\right\|^2_{\mathbb{R}^n} = const, t \in [\tilde{t}_1, \tilde{t}_2),
\end{aligned}
\tag{23}
$$

also holds. Condition (23) provides stability for the solution to Cauchy problem (22).

It is not difficult to carry on similar considerations and to obtain formulas for shifting condition (15) successively to the left.

Thus to implement iterative procedure (12), it is necessary to go through the following steps on each iteration for given $u(t) = u^k(t), t \in [t_0, T], \quad k = 0, 1, ...$:

1) to solve the problem (1), (2) by using the numerical scheme proposed in [7], and to determine the phase trajectory $x(t), \quad t \in [t_0, T]$;

2) to solve the problem (6)–(10) by using the shift procedure (19) applied to the boundary conditions, and to determine the adjoint vector function $\psi(t), \quad t \in [t_0, T]$ and the vector of dual variables λ;

3) to substitute the values of $x(t), \ \psi(t), \quad t \in [t_0, T]$, obtained into the formula (11), and to determine the value of the gradient of the target functional.

5. Conclusion

In the work, we propose numerical method for optimal control problems for ordinary differential equations with non-separated integrals and multipoint conditions . Note that the numerical solution of such differential systems is difficult. The adjoint system also has some specific features determined by the equation itself and by the fact that the unknown vector of the Lagrange multipliers is involved in the problem conditions. The formulas and the calculation schemes proposed in this paper make it possible to take into account all the specific features encountered in the calculation of the gradient of the objective functional. On the whole, the proposed approach makes it possible to use of numerical first-order optimization methods and the corresponding standard software for solving optimal control problems.

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