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ON THE PROBLEM OF ROD HEATING BY EXTERNAL HEAT SOURCE ON THE CLASS OF ZONAL CONTROLS TAKING INTO ACCOUNT THE CURRENT STATE AT THE MEASUREMENT POINTS

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Abstract. *In the paper, we consider an approach for numerical solution to the optimal feedback control problem for an object with distributed parameters on the basis of observation of the object's phase state at its specific locations by the example of the rod heating process. The control actions are the power of the heat source, the values of which are defined on the class of "zonal" controls. The values of the parameters of zonal control actions are determined by subsets of the state space, to which belong the values of the process state at the measurement points at the current moment of time. The problem of determining zonal controls is reduced to a parametric optimal control problem on determining a finite-dimensional vector of values of the parameters of zonal control actions. We derive optimality conditions for the values of the parameters of zonal control actions. These conditions contain formulas for the gradient of the objective functional with respect to the optimizable parameters of zonal controls. They make it possible to solve the stated problem numerically with the application of efficient first-order optimization methods.*

Keywords: feedback control, zonal control, system with distributed parameters, heat conduction process, gradient of functional

Mathematics Subject Classification (2020): 93B52, 65M22, 35Q93

1. Introduction

It is known that one of the important areas in the modern automatic control theory is the theory of control of systems with distributed parameters. The problems of synthesis of distributed control systems are, in most cases, more complex than lumped systems due to the characteristics of distributed objects. Distributed control objects include many chemical-technological, radiation, aerodynamic, and hydrodynamic processes, heat conduction and diffusion processes, processes associated with the movement of elastic structures, etc. The absence of a formalized methodological approach for solving problems of

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controlling objects with distributed parameters poses certain problems for researchers that require the use of non-standard research methods and decision-making in each specific case. The main contribution to the development of the theory of distributed parameter control systems has been made by a number of fundamental results obtained in the works of Butkovskiy A.G. [8], Egorov A.I. [11], Sirazetdinov T.K. [25], Lyons J.L. [19], Moiseev N.N. [21], Lurie K.A. [20], Rapaport E.Ya. [24], Fursikov A.V. [12], Bryson A.E. and Yu-Chi Ho. [7], etc. Modern technical means of measuring and computing technology, which make it possible to carry out a large amount of measuring and computational work in real-time, have played a key role in the development of feedback control systems and their widespread practical implementation.

The paper considers the problem of synthesizing control of an object with distributed parameters on special classes of control actions. For synthesized controls, the concept of zonality is introduced, which means the constancy of the values of the synthesized control parameters in each of the subsets (zones), into which the entire set of possible states of the object is divided. The values of the control actions are also determined by the type of feedback and the class of the functional dependence of the control on the currently observed value of the state. Particularly, the case of discrete feedback is analyzed using discrete observation of the phase state of the object at its certain points. The constancy of the parameters of zonal control actions determines the robustness of the control system, as well as ensures the feasibility of synthesized control actions with sufficiently high accuracy and improves the technical performance of the equipment involved in the control loop.

We have used the principle of zonality of control parameters as the basis of numerical techniques for solving such specific optimization and inverse problems like the problem of optimal placement of production and injection wells and optimal control of their flow rates during the operation of an oil reservoir under the regime of water-driven piston displacement [1], the problem of identifying the hydraulic resistance coefficient under the unsteady flow of viscous fluids through pipelines [5], and problems of feedback control and identification of objects with lumped parameters [2], [3], [4], [13], [14], [16], [17], [18].

2. Problem Statement

To illustrate the proposed approach, we consider the problem of controlling a rod heating process in a furnace. This process can be described by the following parabolic type partial differential equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + \beta \left[v\left(t\right) - u\left(x,t\right) \right], \quad x \in (0,1)\,, \quad t \in (0,T]\,. \tag{1}$$

Here $u\left(x,t\right)$ is the temperature of the rod at the point $x \in [0,1]$ at the moment of time $t \in [0,T]$; $v\left(t\right)$ is the optimizable power of the heat source determined by the temperature of hot air supplied to the furnace; α is the thermal diffusivity coefficient; β is the heat transfer coefficient. Initial and boundary conditions are given in the following form:

$$\frac{\partial u\left(0,t\right)}{\partial x} = \frac{\partial u\left(1,t\right)}{\partial x} = 0, \quad t \in [0,T]\,; \tag{2}$$

$$u(x,0) = \varphi = const, \quad x \in [0,1]. \tag{3}$$

Note that the initial temperature, φ, constant along the entire length of the rod is not known exactly, but there is given the set, Φ, of all possible values of initial temperatures of the rod with a given density function $\rho_\Phi(\varphi)$ such that

$$\rho_\Phi(\varphi) \geq 0, \quad \int_\Phi \rho_\Phi(\varphi)\,\mathbf{d}\varphi = 1.$$

Assume that thermal sensors are installed at N points along the length of the rod with coordinates ξ_i, $i = 1, 2, \ldots, N$. These sensors are used to conduct operative observation and input to the control system of information on the state of the heating process at these points, which is determined by the vector:

$$\overline{u}(t) = (\overline{u}_1(t), \overline{u}_2(t), \ldots, \overline{u}_N(t))^* = (u(\xi_1, t), u(\xi_2, t), \ldots, u(\xi_N, t))^*,$$

where "$*$" denotes the transposition sign. Moreover, there are given discrete moments of observation time $\tau_j \in [0,T]$, $j = 0, 1, 2, \ldots, M$, at which it is possible to measure the value of the object's state at the points of the rod where the thermal sensors are installed, i.e., $\overline{u}(t)$.

To control the heat conduction process in the rod, it is required to synthesize a regulator that, based on the results of temperature measurements at the points ξ_i, $i = 1, 2, \ldots, N$, of the rod, would ensure the maintenance of the temperature $u(x,T)$ at the desired level $u^\star(x)$ by regulating the temperature, $v(t)$, in the heat source. Based on technological conditions, we have to impose certain constraints on the values that the control actions can take:

$$V = \{v(t): \quad v_{\min} \leq v(t) \leq v_{\max}, \quad t > 0\},$$

where $v_{\min}$ and $v_{\max}$ are given quantities, and V represents the set of admissible values of the control $v(t)$. The considered feedback control problem for the rod heating process consists in finding admissible values of the source's power as a function of the object's state

$$v(t) = v(\overline{u}(t)), \quad v(t) \in V, \ \ t \in [0,T],$$

at the observable points of the rod in order to minimize the objective functional. In the case of non-fixed initial conditions (3), the objective functional takes on the following form:

$$\mathcal{F}(v) = \int_\Phi \mathcal{J}(v,\varphi)\ \rho_\Phi(\varphi)\ \mathbf{d}\varphi, \ \mathcal{J}(v,\varphi) = \int_0^1 \omega(x)\left[u(x,T;v,\varphi) - u^\star(x)\right]^2 \mathbf{d}x, \tag{4}$$

where $u(x,T;v,\varphi)$ is the solution to the initial-and boundary-value problem (1)-(3) corresponding to a specific initial condition $\varphi \in \Phi$ and to admissible values of the control $v(t) \in V$; $\omega(x)$ is the given weight function; $u^\star(x)$ is the given function characterizing the desired distribution of temperature at the final moment of the heating process. The functional (4) characterizes the quality of control process on average over the set of all possible initial states Φ and the boundary conditions.

Let the phase state values of the rod satisfy the inequalities

$$u_{\min} \leq u(x, T; v, \varphi) \leq u_{\max}, \quad x \in [0,1], \quad t \in [0,T],$$

under all possible admissible values of the control, as well as initial and boundary conditions. Given the points u_k, $k = 0, 1, 2, \ldots, L$, we divide the range of all possible temperature values $[u_{\min}, u_{\max}]$ into L temperature intervals:

$$[u_{\min}, u_{\max}] = \bigcup_{k=1}^{L} [u_{k-1}, u_k), \quad u_0 = u_{\min}, \quad u_L = u_{\max}.$$

In the N-dimensional phase space $\overline{\mathrm{u}}(t) \in \mathbb{R}^N$ of the current measured temperature values at the points of the rod, we introduce the following N-dimensional parallelepipeds (zones):

$$P_{i_1,i_2,\ldots,i_N} = \{(\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_N) : u_{i_s-1} \leq u(\xi_s, t; v, \varphi) \leq u_{i_s}\}, \qquad i_s \in \{1, 2, \ldots, L\}, \quad s = 1, 2, \ldots, N, \tag{5}$$

the total number of which is L^N. Let $\mathrm{I} = (i_1, i_2, \ldots, i_N)$ denote the N-dimensional multi-index, which determines the number of the corresponding parallelepiped. The values of the control $v(t)$ constant for $t \in [\tau_l, \tau_{l+1})$ are determined depending on the last measured value of the observation vector over the current object's state, $\overline{u}(t)$, namely, depending on the number (multi-index) of the parallelepiped (5), to which the last measured (observed) object's state, $\overline{u}(t)$, belongs. To each phase parallelepiped there corresponds its constant control value, that is

$$v(t) = \vartheta_{i_1,i_2,\ldots,i_N} = \vartheta_I = const, \quad \overline{u}(t) \in P_{i_1,i_2,\ldots,i_N} = P_I, \qquad t \in [\tau_j, \tau_{j+1}), \quad j = 0, 1, 2, \ldots, M-1. \tag{6}$$

If the observed object's state belongs to the border of any zones, we use the value of the zonal control of that adjacent zone into which the trajectory has passed. The number of different values that the source's power can take is equal to the number of phase parallelepipeds, i.e., L^N. The possible configuration of phase parallelepipeds when there are only two thermal sensors is illustrated in Figure 1. The phase parallelepipeds in case of three thermal sensors installed may represent cubes, etc.

It is clear that the controls (6) assume feedback. In the case of (5), the values of the controllable source's power during the rod heating process change only at the moments when the population of states at the observable points proceeds from one phase parallelepiped (5) to another. Thus, the considered feedback control problem on the class of piecewise-constant functions consists of optimizing the L^N-dimensional vector ϑ. The considered feedback control problem (1)-(6) is a parametric optimal control problem for the system with distributed parameters. Its specific features are, firstly, the absence of prescribed initial conditions, secondly, the finite-dimensionality of the sought-for control vector, and thirdly, the control is formed depending on the values of the current state of the process at the measurement points; namely, it depends on the multi-index defining the parallelepiped (zone) of the phase space to which the current measurement values belong. The solution of the feedback control problem (1)-(6) are synthesized zonal controls provided that the feedback with the object and the choice of the values of control actions is carried out only at specified discrete moments of time. As examples of practical applications of such problems, one can cite the control of many technological processes and technical objects. The organization of continuous monitoring of the state is impossible for these objects, and each observation (feedback) requires specific measures and, therefore, costs time and material.

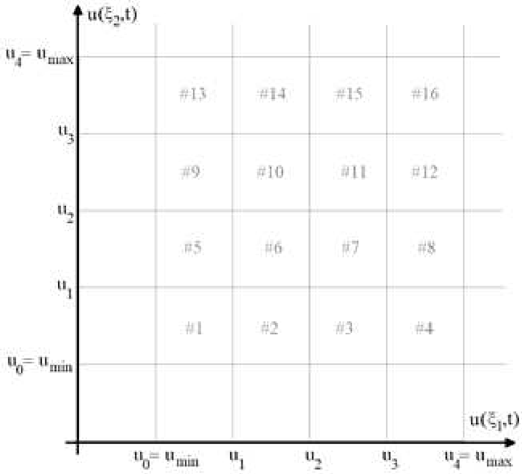

Fig. 1. Two-dimensional phase parallelepipeds (zones) in the form of rectangles

The formulated problem of synthesizing zonal controls (1)-(6) leads to a finite-dimensional optimization problem. For numerical solution to this problem, we propose to use the approach described in [16], [17]. To solve the problem in the case of a simple design of a set of admissible controls V (for example, a parallelepiped, hyper-sphere, polyhedron, etc.), it is effective use first-order numerical optimization methods such as gradient projection or conjugate gradient projection methods [23], [26]. For example, for the gradient projection method, we construct a minimizing sequence $\left\{\vartheta^k\right\}$ in this fashion:

$$\vartheta^{k+1} = \vartheta^k + P_{(V)}\left(\vartheta^k + \lambda_k \times d^k\right), \quad \lambda_k > 0, \quad k = 0, 1, 2, \ldots,$$

$$s^0 = -\nabla\mathcal{F}\left(\vartheta^0\right), \quad d^{k+1} = -\nabla\mathcal{F}\left(\vartheta^{k+1}\right) + \mu_k \times d^k, \quad \mu_k = \frac{\left\|\nabla\mathcal{F}\left(\vartheta^{k+1}\right)\right\|}{\left\|\nabla\mathcal{F}\left(\vartheta^k\right)\right\|},$$

where the index k designates the iteration number; $\vartheta^0 \in \mathbb{R}^{L^N}$ is some initial guess to the optimizable vector; $\nabla\mathcal{F}\left(\vartheta^k\right)$ is the gradient of the objective functional; λ_k the minimizing step size taken in the direction of d^k; $P_{(V)}(\cdot)$ is the projection operator onto the admissible set V. If the domain V has a complex boundary and the projection operator onto it has no constructive character, then to solve the posed problem, one can use methods of sequential unconstrained optimization (for example, methods of internal and external penalty functions) with the use of effective methods of unconstrained optimization of the first order such as quasi-Newtonian methods [6], [22]. To construct iterative procedures based on the above optimization techniques, it is essential to have exact formulas for the gradient of the objective functional in the space of optimizable parameters. To this end, we derive formulas for the gradient of the objective functional in the space of optimizable parameters. The derivation of these formulas is based on the technique for calculating the increment of the objective functional obtained by incrementing the optimizable parameters. In further calculations, the following remark is important. The initial conditions (3), i.e., the elements of the set $\varPhi$ are independent. Then the gradient of the functional obviously satisfies the formula:

$$\nabla \mathcal{F}(\vartheta) = \int_{\varPhi} \nabla \mathcal{J}(\vartheta, \varphi) \ \rho_{\varPhi}(\varphi) \ \mathbf{d}\varphi.$$

Therefore, to obtain formulas for $\nabla \mathcal{F}(\vartheta)$, we obtain formulas for the gradient of $\mathcal{J}(\vartheta, \varphi)$ with respect to an individual term φ, (i.e., assuming that the set $\varPhi$ consists of a single term). For this purpose, we obtain the formula for the increment of the functional $\mathcal{J}(\vartheta, \varphi)$, obtained by incrementing the parameter value

$$\vartheta = (\vartheta_{i_1, i_2, \ldots, i_N}) = (\vartheta_1, \vartheta_2, \ldots, \vartheta_{L^N}), \quad i_s \in \{1, 2, \ldots, L\}, \quad s = 1, 2, \ldots, N.$$

Suppose that one of the optimizable parameters of the vector ϑ is incremented, for example, ϑ_ℓ, i.e.,

$$\overline{\theta} = \vartheta + \triangle\vartheta, \quad \triangle\vartheta = (0, 0, \ldots, 0, \triangle\vartheta_\ell, 0, \ldots, 0, 0),$$

where $\overline{\vartheta}$ denotes the perturbed control vector. To express the increment of the functional $\mathcal{J}(\vartheta, \varphi)$ in terms of the increment of the control vector, $\triangle\vartheta$, we introduce the Lagrangian for the considered problem:

$$\begin{gathered} \mathcal{L}(u, \psi; \vartheta) = \int_0^1 \omega(x) \left[u(x, T) - u^\star(x)\right]^2 \mathbf{d}x + \\ + \int_0^T \int_0^1 \left\{ \frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} - \beta\left[\vartheta - u(x, t)\right] \right\} \ \psi(x, t) \, \mathbf{d}x \mathbf{d}t, \end{gathered} \tag{7}$$

where $u(x, t) = u(x, t; \vartheta, \varphi)$ designates the solution to the initial- and boundary-value problem under a concrete initial condition $u(x, 0) = \varphi$ and $\psi = \psi(x, t)$ is yet an arbitrary smooth function. Assume that at the moment of time τ_m, the measurements taken at the thermal sensors indicate that the object's state belongs to the ℓ^{th} zone, while at any previous moments of time, τ_0, τ_1,..., τ_{m-1}, the object's state does not belong to the ℓ^{th} zone. Then we can rewrite (7) as follows:

$$\begin{gathered} \mathcal{L}(u, \psi; \vartheta) = \int_0^1 \omega(x) \left[u(x, T) - u^\star(x)\right]^2 \mathbf{d}x + \\ + \int_0^{\tau_m} \int_0^1 \left\{ \frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} - \beta\left[\vartheta - u(x, t)\right] \right\} \psi(x, t) \, \mathbf{d}x \mathbf{d}t + \end{gathered}$$

$$+\int_{\tau_m}^{\tau_n}\int_0^1\left\{\frac{\partial u}{\partial t}-\alpha\frac{\partial^2 u}{\partial x^2}-\beta\left[\vartheta_\ell-u\left(x,t\right)\right]\right\}\psi\left(x,t\right)\mathbf{d}x\mathbf{d}t+$$
$$+\int_{\tau_n}^{T}\int_0^1\left\{\frac{\partial u}{\partial t}-\alpha\frac{\partial^2 u}{\partial x^2}-\beta\left[\vartheta-u\left(x,t\right)\right]\right\}\psi\left(x,t\right)\mathbf{d}x\mathbf{d}t.$$

Here τ_{n-1} denotes the last moment at which measurements taken at the thermal sensors indicate that the object's state belongs to the ℓ^{th} zone. Let's denote by $\overline{u}\left(x,t\right)=u\left(x,t\right)+\triangle u\left(x,t\right)$ the solution to the initial- and boundary-value problem w.r.t. (1) corresponding to the perturbed control $\overline{\vartheta}$. We assume that the increment $\triangle\vartheta$ of the control is so small (which is ultimately due to the definition of the derivative) that the measurements taken at the thermal sensors at the discrete moments of time τ_m, τ_{m+1},. . . ,τ_{n-1}, still indicates that the object's state belongs to the ℓ^{th} zone. Under these assumptions, the Lagrangian (7) for the perturbed solution $\overline{u}\left(x,t\right)$ takes on the following form:

$$\mathcal{L}\left(\overline{u},\psi;\overline{\vartheta}\right)=\int_0^1\omega\left(x\right)\left[\overline{u}\left(x,T\right)-u^{\star}\left(x\right)\right]^2\mathbf{d}x+$$
$$+\int_0^{\tau_m}\int_0^1\left\{\frac{\partial \overline{u}}{\partial t}-\alpha\frac{\partial^2 \overline{u}}{\partial x^2}-\beta\left[\overline{\vartheta}-\overline{u}\left(x,t\right)\right]\right\}\psi\left(x,t\right)\mathbf{d}x\mathbf{d}t+$$
$$+\int_{\tau_m}^{\tau_n}\int_0^1\left\{\frac{\partial \overline{u}}{\partial t}-\alpha\frac{\partial^2 \overline{u}}{\partial x^2}-\beta\left[\vartheta_\ell+\triangle\vartheta_\ell-\overline{u}\left(x,t\right)\right]\right\}\psi\left(x,t\right)\mathbf{d}x\mathbf{d}t+$$
$$+\int_{\tau_n}^{T}\int_0^1\left\{\frac{\partial \overline{u}}{\partial t}-\alpha\frac{\partial^2 \overline{u}}{\partial x^2}-\beta\left[\overline{\vartheta}-\overline{u}\left(x,t\right)\right]\right\}\psi\left(x,t\right)\mathbf{d}x\mathbf{d}t.$$

Subtracting $\mathcal{L}\left(u,\psi;\vartheta\right)$ from $\mathcal{L}\left(\overline{u},\psi;\overline{\vartheta}\right)$ produces the increment of the Lagrangian:

$$\triangle\mathcal{L}=\mathcal{L}\left(\overline{u},\psi;\overline{\vartheta}\right)-\mathcal{L}\left(u,\psi;\vartheta\right)=\int_0^1\omega\left(x\right)\left[u\left(x,T\right)+\triangle u\left(x,T\right)-u^{\star}\left(x\right)\right]^2\mathbf{d}x+$$
$$+\int_0^{\tau_m}\int_0^1\left\{\frac{\partial u}{\partial t}+\frac{\partial\triangle u}{\partial t}-\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2\triangle u}{\partial x^2}\right)-\beta\left[\vartheta-u-\triangle u\right]\right\}\psi\;\mathbf{d}x\mathbf{d}t+$$
$$+\int_{\tau_m}^{\tau_n}\int_0^1\left\{\frac{\partial u}{\partial t}+\frac{\partial\triangle u}{\partial t}-\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2\triangle u}{\partial x^2}\right)-\beta\left[\vartheta_\ell+\triangle\vartheta_\ell-u-\triangle u\right]\right\}\psi\;\mathbf{d}x\mathbf{d}t$$
$$+\int_{\tau_n}^{T}\int_0^1\left\{\frac{\partial u}{\partial t}+\frac{\partial\triangle u}{\partial t}-\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2\triangle u}{\partial x^2}\right)-\beta\left[\vartheta-u-\triangle u\right]\right\}\psi\;\mathbf{d}x\mathbf{d}t$$
$$-\int_0^1\omega\left(x\right)\left[u\left(x,T\right)-u^{\star}\left(x\right)\right]^2\mathbf{d}x-\int_0^{\tau_m}\int_0^1\left\{\frac{\partial u}{\partial t}-\alpha\frac{\partial^2 u}{\partial x^2}-\beta\left[\vartheta-u\right]\right\}\psi\;\mathbf{d}x\mathbf{d}t-$$
$$-\int_{\tau_m}^{\tau_n}\int_0^1\left\{\frac{\partial u}{\partial t}-\alpha\frac{\partial^2 u}{\partial x^2}-\beta\left[\vartheta_\ell-u\right]\right\}\psi\;\mathbf{d}x\mathbf{d}t-$$
$$-\int_{\tau_n}^{T}\int_0^1\left\{\frac{\partial u}{\partial t}-\alpha\frac{\partial^2 u}{\partial x^2}-\beta\left[\vartheta-u\right]\right\}\psi\;\mathbf{d}x\mathbf{d}t.$$

After canceling out the similar terms, we obtain:

$$\triangle\mathcal{L} = \int_0^1 2\omega(x)\left[u(x,T) - u^\star(x)\right]\triangle u(x,T)\,\mathbf{d}x + \int_0^1 \omega(x)\left[\triangle u(x,T)\right]^2\mathbf{d}x +$$
$$+\int_0^{\tau_m}\int_0^1 \left\{\frac{\partial\triangle u}{\partial t} - \alpha\frac{\partial^2\triangle u}{\partial x^2} + \beta\triangle u\right\}\psi\ \mathbf{d}x\mathbf{d}t +$$
$$+\int_{\tau_m}^{\tau_n}\int_0^1 \left\{\frac{\partial\triangle u}{\partial t} - \alpha\frac{\partial^2\triangle u}{\partial x^2} - \beta\left[\triangle\vartheta_\ell - \triangle u\right]\right\}\psi\ \mathbf{d}x\mathbf{d}t +$$
$$+\int_{\tau_n}^{T}\int_0^1 \left\{\frac{\partial\triangle u}{\partial t} - \alpha\frac{\partial^2\triangle u}{\partial x^2} + \beta\triangle u\right\}\psi\ \mathbf{d}x\mathbf{d}t.$$

Combining the 3rd, 4th, and the 5th terms of the last expression, we obtain:

$$\triangle\mathcal{L} = \int_0^1 2\omega(x)\left[u(x,T) - u^\star(x)\right]\triangle u(x,T)\,\mathbf{d}x + \int_0^1 \omega(x)\left[\triangle u(x,T)\right]^2\mathbf{d}x +$$
$$+\int_0^{T}\int_0^1 \left\{\frac{\partial\triangle u}{\partial t} - \alpha\frac{\partial^2\triangle u}{\partial x^2} + \beta\triangle u\right\}\psi\ \mathbf{d}x\mathbf{d}t + \int_{\tau_m}^{\tau_n}\int_0^1 -\beta\ \triangle\vartheta_\ell\ \psi\ \mathbf{d}x\mathbf{d}t.$$

We have managed to extract the linear with respect to $\triangle\vartheta_\ell$ part of the Lagrangian increment. In order to obtain the expression for the derivative of $\mathcal{L}(\cdot)$ with respect to ϑ_ℓ, we have to eliminate the 1st through the 3rd terms of $\triangle\mathcal{L}$. In order to do that, we modify the 3rd term of $\triangle\mathcal{L}$ by breaking it down into two terms:

$$\int_0^{T}\int_0^1 \left\{\frac{\partial\triangle u}{\partial t} - \alpha\frac{\partial^2\triangle u}{\partial x^2} + \beta\triangle u\right\}\psi\ \mathbf{d}x\mathbf{d}t =$$
$$= \int_0^{T}\int_0^1 \beta\ \triangle u\ \psi\mathbf{d}x\mathbf{d}t + \int_0^{T}\int_0^1 \left\{\frac{\partial\triangle u}{\partial t} - \alpha\frac{\partial^2\triangle u}{\partial x^2}\right\}\psi\ \mathbf{d}x\mathbf{d}t.$$

Applying integration by parts to the second integral, we obtain:

$$\int_0^{T}\int_0^1 \frac{\partial\triangle u}{\partial t}\ \psi\ \mathbf{d}x\mathbf{d}t = \int_0^1 \left\{\int_0^T \frac{\partial\triangle u}{\partial t}\ \psi\ \mathbf{d}t\right\}\mathbf{d}x = \int_0^1 \left\{\int_0^T \psi\ \mathbf{d}_t\triangle u\right\}\mathbf{d}x =$$
$$= \int_0^1 \left\{\psi(x,T)\ \ \triangle u(x,T) - \psi(x,0)\ \ \triangle u(x,0) - \int_0^T \frac{\partial\psi}{\partial t}\ \triangle u\ \mathbf{d}t\right\}\mathbf{d}x.$$

According to the initial condition (3), we have:

$$\triangle u(x,0) = \overline{u}(x,0) - u(x,0) = \varphi - \varphi \equiv 0, \quad x \in [0,1].$$

Therefore, the last equation reduces to

$$\int_0^{T}\int_0^1 \frac{\partial\triangle u}{\partial t}\psi\mathbf{d}x\mathbf{d}t = \int_0^1 \psi(x,T)\ \ \triangle u(x,T)\,\mathbf{d}x - \int_0^{T}\int_0^1 \frac{\partial\psi}{\partial t}\ \triangle u\ \mathbf{d}x\mathbf{d}t. \qquad (8)$$

As for the second term, we have:

$$\int_0^T \int_0^1 -\alpha \frac{\partial^2 \triangle u}{\partial x^2} \ \psi \ \mathbf{d}x\mathbf{d}t = -\alpha \int_0^T \left\{ \int_0^1 \psi \ \mathbf{d}_x \frac{\partial \triangle u}{\partial x} \right\} \mathbf{d}t =$$

$$= -\alpha \int_0^T \left\{ \psi\left(1,t\right) \frac{\partial \triangle u\left(1,t\right)}{\partial x} - \psi\left(0,t\right) \frac{\partial \triangle u\left(0,t\right)}{\partial x} - \int_0^1 \frac{\partial \triangle u}{\partial x} \ \frac{\partial \psi}{\partial x} \mathbf{d}x \right\} \mathbf{d}t.$$

According to the boundary conditions (2), we have:

$$\frac{\partial \triangle u\left(0,t\right)}{\partial x} = \frac{\partial \overline{u}\left(0,t\right)}{\partial x} - \frac{\partial u\left(0,t\right)}{\partial x} = 0 - 0 = 0,$$

$$\frac{\partial \triangle u\left(1,t\right)}{\partial x} = \frac{\partial \overline{u}\left(1,t\right)}{\partial x} - \frac{\partial u\left(1,t\right)}{\partial x} = 0 - 0 = 0.$$

Therefore,

$$\int_0^T \int_0^1 -\alpha \frac{\partial^2 \triangle u}{\partial x^2} \ \psi \ \mathbf{d}x\mathbf{d}t = \alpha \int_0^T \left\{ \int_0^1 \frac{\partial \triangle u}{\partial x} \ \frac{\partial \psi}{\partial x} \mathbf{d}x \right\} \mathbf{d}t.$$

Applying integration by parts the second time, we obtain:

$$\int_0^T \int_0^1 -\alpha \frac{\partial^2 \triangle u}{\partial x^2} \ \psi \ \mathbf{d}x\mathbf{d}t = \alpha \int_0^T \left\{ \int_0^1 \frac{\partial \psi}{\partial x} \mathbf{d}_x \triangle u \right\} \mathbf{d}t =$$

$$= \alpha \int_0^T \left\{ \frac{\partial \psi\left(1,t\right)}{\partial x} \triangle u\left(1,t\right) - \frac{\partial \psi\left(0,t\right)}{\partial x} \triangle u\left(0,t\right) - \int_0^1 \triangle u \ \frac{\partial^2 \psi}{\partial x^2} \mathbf{d}x \right\} \mathbf{d}t.$$

This time, due to arbitrariness of the function $\psi\left(\cdot\right)$, we require that

$$\frac{\partial \psi\left(1,t\right)}{\partial x} = \frac{\partial \psi\left(0,t\right)}{\partial x} = 0, \quad t \in \left[0,T\right]. \tag{9}$$

We, thus, obtain:

$$\int_0^T \int_0^1 -\alpha \frac{\partial^2 \triangle u}{\partial x^2} \ \psi \mathbf{d}x\mathbf{d}t = -\alpha \int_0^T \left\{ \int_0^1 \frac{\partial^2 \psi}{\partial x^2} \ \triangle u \ \mathbf{d}x \right\} \mathbf{d}t. \tag{10}$$

Taking (8) and (10) into account in $\triangle \mathcal{L}$ produces:

$$\triangle \mathcal{L} = \int_0^1 2\omega\left(x\right)\left[u\left(x,T\right) - u^\star\left(x\right)\right] \triangle u\left(x,T\right) \mathbf{d}x + \int_0^1 \omega\left(x\right)\left[\triangle u\left(x,T\right)\right]^2 \mathbf{d}x +$$

$$+ \int_{\tau_k}^{\tau_l} \int_0^1 -\beta \ \psi \ \triangle \vartheta_\ell \ \mathbf{d}x\mathbf{d}t + \int_0^1 \psi\left(x,T\right) \ \triangle u\left(x,T\right) \mathbf{d}x - \int_0^T \int_0^1 \frac{\partial \psi}{\partial t} \ \triangle u \ \mathbf{d}x\mathbf{d}t +$$

$$+ \int_0^T \int_0^1 \left[\beta \ \psi - \frac{\partial \psi}{\partial t} - \alpha \frac{\partial^2 \psi}{\partial x^2}\right] \triangle u \ \mathbf{d}x\mathbf{d}t.$$

By combining the 1st with the 4th terms, and the 5th with the 6th terms, and requiring that

$$\psi\left(x,T\right) = -2\omega\left(x\right)\left[u\left(x,T\right) - u^\star\left(x\right)\right], \quad x \in \left[0,1\right], \tag{11}$$

$$\beta\psi - \frac{\partial\psi}{\partial t} - \alpha\frac{\partial^2\psi}{\partial x^2} = 0, \quad x \in [0,1], \quad t \in [0,T), \tag{12}$$

and neglecting the higher-order term

$$\int_0^1 \omega(x) \left[\triangle u(x,T)\right]^2 \mathbf{d}x,$$

we obtain the following increment of the Lagrangian:

$$\triangle\mathcal{L} = -\int_{\tau_m}^{\tau_n}\int_0^1 \beta\ \psi(x,t)\ \ \triangle\vartheta_\ell\ \mathbf{d}x\mathbf{d}t.$$

Because $\triangle\vartheta_\ell$ is constant, we can take it out of the integral sign, thus obtaining:

$$\triangle\mathcal{L} = \left\{-\int_{\tau_m}^{\tau_n}\int_0^1 \beta\ \psi(x,t)\ \ \mathbf{d}x\mathbf{d}t\right\}\triangle\vartheta_\ell.$$

Dividing the left- and the right-hand sides of the last equation by $\triangle\vartheta_\ell$ and proceeding to the limit as $\triangle\vartheta_\ell \to 0$, we obtain the expression for the derivative of $\mathcal{L}(u,\psi;\vartheta)$ w.r.t. ϑ_ℓ:

$$\frac{\partial\mathcal{L}(u,\psi;\vartheta)}{\partial\vartheta_\ell} = -\int_{\tau_m}^{\tau_n}\int_0^1 \beta\ \psi(x,t)\ \ \mathbf{d}x\mathbf{d}t. \tag{13}$$

Remark 1. In general, the object's state at the observed points can belong to the same zone in disparate time intervals $[\tau_j, \tau_{j+1})$, $j \in \{0,1,2,\ldots,M-1\}$, i.e., the trajectory of the system may enter and leave the same zone multiple times. In this case, we have to modify the formula (13) by calculating the definite integral over the combined time interval

$$\prod_{i_1,i_2,\ldots,i_N}(\vartheta_\ell) = \bigcup_{\overline{u}(\tau_j)\in P_{i_1,i_2,\ldots,i_N}}[\tau_j,\tau_{j+1}).$$

During which the vector $\overline{u}(\tau_j)$ belongs to the $(i_1,i_2,\ldots,i_N)$th phase parallelepiped under the current values of the control ϑ and initial condition φ. The formula (13) should be modified accordingly:

$$\frac{\partial\mathcal{L}(u,\psi;\vartheta)}{\partial\vartheta_\ell} = -\int_{\prod_{i_1,i_2,\ldots,i_N}(\vartheta_\ell)}\int_0^1 \beta\ \psi(x,t)\ \ \mathbf{d}x\mathbf{d}t.$$

Generalizing this formula to all possible states of the initial condition, i.e., covering the entire set $\varPhi$, we thus prove the following theorem.

Theorem. *The components of the gradient of the functional in the problem* (1) – (5)*, in the space of piecewise constant controls* (6) *for an arbitrary control* $\vartheta \in V$ *are determined by the formula:*

$$\frac{\partial\mathcal{F}(\vartheta)}{\partial\vartheta_\ell} = \int_{\varPhi}\left\{-\int_{\prod_{i_1,i_2,\ldots,i_N}(\vartheta_\ell)}\int_0^1 \beta\ \psi(x,t;\vartheta,\varphi)\ \ \mathbf{d}x\mathbf{d}t\right\}\ \rho_{\varPhi}(\varphi)\,\mathbf{d}_{\varphi}, \tag{14}$$

where $\psi(x, t; \vartheta, \varphi)$ *is the solution of the adjoint problem* (9), (11), *and* (12), *corresponding to the current zonal control.*

Using the formulas for the objective functional gradient (14), we can propose the following iterative algorithm for determining piecewise constant synthesizing controls based on first-order numerical optimization methods.

Step 1. For the current admissible value of the vector $\vartheta \in V$ and all possible initial conditions $\phi \in \Phi$, we solve the direct initial-and boundary-value problem w.r.t. (1) by some numerical scheme and the trajectory $u(x, t; \vartheta, \varphi)$ is calculated.

Step 2. We find the solution to the adjoint initial-and boundary-value problem (9), (11), and (12), corresponding to the direct problem's solution, and the trajectory $\psi(x, t; \vartheta, \varphi)$ is calculated.

Step 3. We evaluate the components of the gradient of the objective functional by formulas (14) using any quadrature formula.

Step 4. The new approximation of the control vector is calculated using first-order finite-dimensional optimization numerical procedures using, for example, the iterative gradient projection method.

Step 5. If the optimality condition is not met or the iterative process ends (for example, when $\lambda_k \approx 0$ or $\left|\mathcal{F}\left(\vartheta^{k+1}\right) - \mathcal{F}\left(\vartheta^{k}\right)\right| / \left(1 + \left|\mathcal{F}\left(\vartheta^{k+1}\right)\right|\right) < \epsilon$, where ϵ is a relatively small positive quantity), steps $1 - 4$ are repeated.

The quality of the control system based on zonal control actions described above is significantly affected by choice of both the number of zones (5) and their structure. Namely, an increase in the number of zones due to their refinement can only decrease the objective functional's value. So, an increase in the number of zones leads to a situation when control actions can change their values more often in time, and, therefore, on the one hand, the robustness of the control system deteriorates, and, on the other hand, this leads to rapid wear and failure of the actuating mechanisms. Conversely, an increase in the size of the zones, i.e., a decrease in their number, on the one hand, deteriorates the controllability of the object, and with a small number of them, the object may become completely uncontrollable. On the other hand, this increases the objective functional's value, i.e., the quality of control deteriorates. Taking these issues into account, the following approach is recommended, in which at first an initial value of L is arbitrarily selected and some zones (5) are assigned. Having solved the feedback control problem, we can analyze the computed optimal zonal values of the controls for all neighboring zones. If the optimizable parameters in any two adjacent zones differ by a sufficiently small amount, then these adjacent zones can be combined into one, thus reducing the number L, the number of switchings of the control. If the optimizable parameters in any two adjacent zones differ significantly, then, on the contrary, each of these adjacent zones should be divided, for example, into two zones, i.e., increase the number L, and again solve the feedback control problem. An increase in the number of zones should be carried out until the objective functional's value ceases to change (decrease) significantly.

Remark 2. The frequency of observation times τ_{j}, $\mathrm{j} = 0, 1, 2, \ldots, M$, should be such that while the object's state belongs to any zone, at least one observation is made. If this condition is not met, the zones through which the system's trajectory did not pass under all possible initial conditions, as well as the zones through which no state measurements were carried out, will not be assigned the values of the zonal control parameters.

Remark 3. The main issue with the proposed approach to feedback is the high dimensionality of the optimizable control vector. The optimizable control vector's dimension represents a power function with respect to the number L of temperature intervals within the range $[u_{\min}, u_{\max}]$ of all possible temperature values of the object, and an exponential function with respect to the number of thermal sensors installed along the length of the rod. Besides, the number of thermal sources also affects the optimizable control vector (as a multiplication factor of the term L^N). It is known that one of the basic problems of numerical optimization techniques (of any order) is the computation of optimal solutions of high-dimensional objective functions. This is because the optimization of high-dimensional objective functions is computationally expensive and cost involved, especially when seeking the global optimal solution. Many parameters characterize these kinds of problems, and many iterations and arithmetic operations are usually needed for evaluations of these objective functions. Moreover, the use of simulation models in the practice of calculating complex computational problems is based on the exact formulation of the optimization problem, because the quality of the employed numerical method is characterized by many factors: the domain of convergence, the rate of convergence, the execution time of one iteration, the amount of machine memory required to implement the method, the class of the problem being solved, etc. Optimization problems also have a great variety: among them there are smooth and non-smooth problems, problems of low and high dimensions, ravine type, unimodal and multimodal, etc. It is quite clear that not the search for a universal method, but a reasonable combination of various methods will make it possible to solve the set optimization problem with the greatest efficiency. It is more expedient to manage calculations in an interactive mode, when the user receives information about the current results in the process of calculations, changes the parameters of the method, and makes a purposeful and conscious transition from one optimization method to another [15].

Remark 4. In order to speed up the evaluation of the objective functional in the posed feedback control problem, under the given value of the control vector, we can make use of the inherent concurrency present in the form of the objective functional. Namely, because the evaluation of the objective functional involves the computation of the definite integral, knowing that the elements of the set $\varPhi$ are independent, we can efficiently parallelize its computation by assigning to each thread (or process) a specific element of the set $\varPhi$, and computing the innermost definite integral in (4) with sufficiently high accuracy. The innermost integration can also be parallelized if we preliminarily slice the interval $[0, 1]$ into several non-overlapping subintervals and computing the definite integral over all these subintervals concurrently [9]. The same concurrency pattern also applies to evaluating the gradient of the objective functional by the formula (14). Note that the solution to both the direct and adjoint initial- and boundary-value problems with respect to the parabolic type differential equation can be easily parallelized, too, if we employ an explicit finite difference scheme to their solution [9], [10]. Even if we employ implicit finite difference schemes to their solution, we will still be able to parallelize computations using efficient iterative methods of solution to linear systems with banded coefficient matrices.

3. Results of Numerical Experiments

Consider the following initial- and boundary-value problem for the parabolic type partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + [v(t) - u(x,t)], \quad x \in (0,1), \quad t \in (0,10],$$

satisfying the boundary conditions:

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(1,t)}{\partial x} = 0, \quad t \in [0,10],$$

and initial conditions:

$$u(x,0) = const \in \Phi = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9.1.0\}, \quad x \in [0,1].$$

There are two thermal sensors installed at the points $\xi_1 = 0.3$ and $\xi_2 = 0.7$ of the rod. These sensors are used to conduct operative observation and input to the control system of information on the state of the heating process at these points, which is determined by the vector:

$$\overline{\mathrm{u}}(t) = (\overline{u}_1(t), \overline{u}_2(t))^* = (u(0.3,t), u(0.7,t))^*.$$

We assume that it is possible to measure the value of the object's state at the points ξ_1 and ξ_2 continuously in time. To control the heat conduction process in the rod, we synthesize a regulator that, based on the results of temperature measurements at the points ξ_1 and ξ_2, would ensure the maintenance of the temperature $u(x,T)$ at the desired level $u^\star(x) = 9.0$ by regulating the temperature, $v(t)$, in the heat source. We impose constraints on the values that the control actions can take:

$$V = \{v(t): \quad 0 \le v(t) \le 10, \ t > 0\};$$

V represents the set of admissible values of the control $v(t)$. For the considered feedback control problem, the objective is to minimize the functional

$$\mathcal{F}(v) = \int_\Phi \mathcal{J}(v,\varphi) \cdot \rho_\Phi(\varphi)\,\mathbf{d}\varphi, \ \mathcal{J}(v,\varphi) = \int_0^1 e^{-(x-0.5)^2} [u(x,T;v,\varphi) - u^\star(x)]^2 \mathbf{d}x.$$

Based on the results of numerical experiments, the phase state values of the rod were found to satisfy the inequalities

$$0 \le u(x,T;v,\varphi) \le 10, \quad x \in [0,1], \quad t \in [0,10],$$

under all possible admissible values of the control, as well as prescribed initial and boundary conditions. Given the partition points

$$u_k = \{0,1,2,3,4,5,6,7,8,9,10\}, \quad k = 1,2,\ldots,11,$$

we divide the range of all possible temperature values $[0, 10]$ into 12 temperature intervals:

$$[u_{\min}, u_{\max}] = \bigcup_{k=1}^{12} (u_{k-1}, u_k), \quad u_0 = -\infty, \quad u_L = +\infty.$$

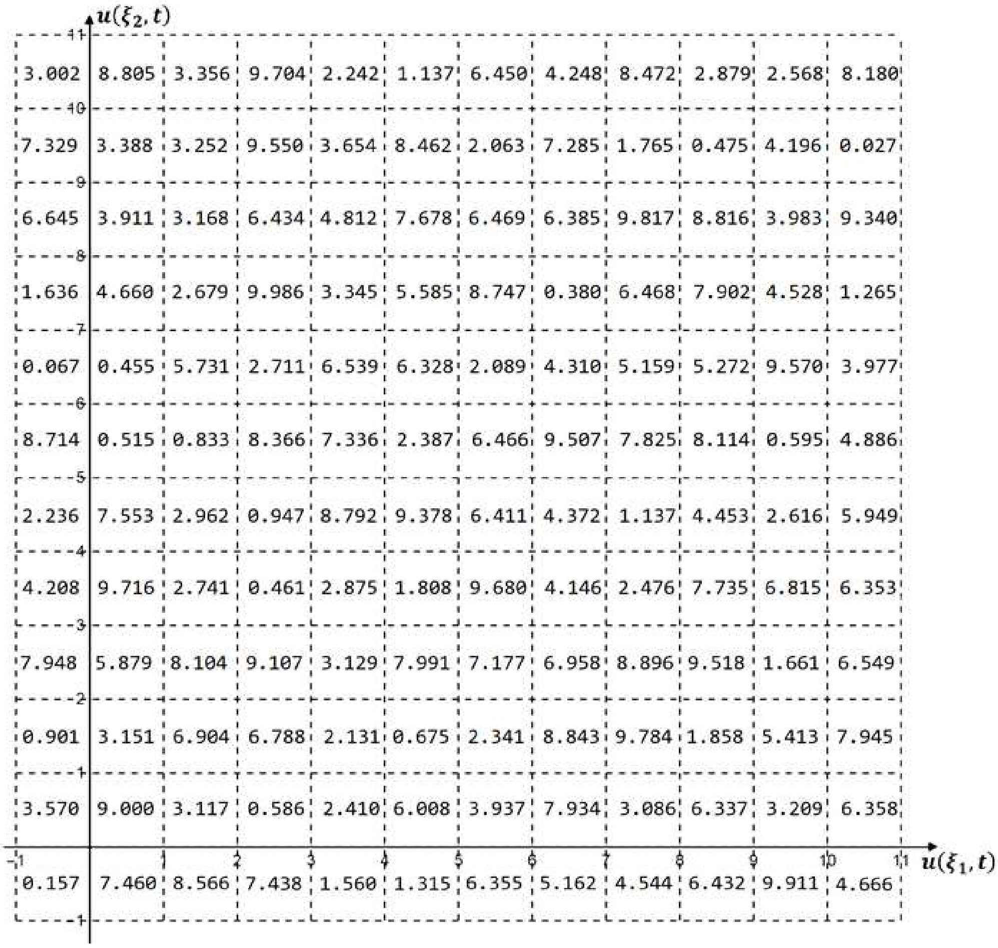

Fig. 2. Final values of the control actions for all the 144 zones

In the 2-dimensional phase space $\overline{u}(t) \in \mathbb{R}^2$ of the current measured temperature values at the points ξ_1 and ξ_2 of the rod, we introduce the following 2-dimensional (zones):

$$\mathrm{P}_{i_1,i_2} = \{(\overline{u}_1, \overline{u}_2) : u_{i_s-1} \le u(\xi_s, t; v, \varphi) \le u_{i_s}\}, \quad i_s \in \{1, 2, \ldots, 12\}, \quad s = 1, 2,$$

the total number of which is 144. Let $\mathrm{I} = (i_1, i_2)$ denote the 2-dimensional multi-index, which determines the number of the corresponding parallelepiped. To each phase zone there corresponds its constant control value, that is

$$v(t) = \vartheta_{i_1,i_2} = \vartheta_{\mathrm{I}} = const, \quad \overline{u}(t) \in P_{i_1,i_2} = P_{\mathrm{I}}.$$

For the minimization procedure, initial guess was taken randomly within the range $[0, 10]$. The initial value of the objective functional is approximately equal to 8.47. After running the minimization procedure, we obtained the following results: the final value of the objective functional is approximately equal to 2.61×10^{-8}; the final values of the control actions in all zones are given in the Figure 2.

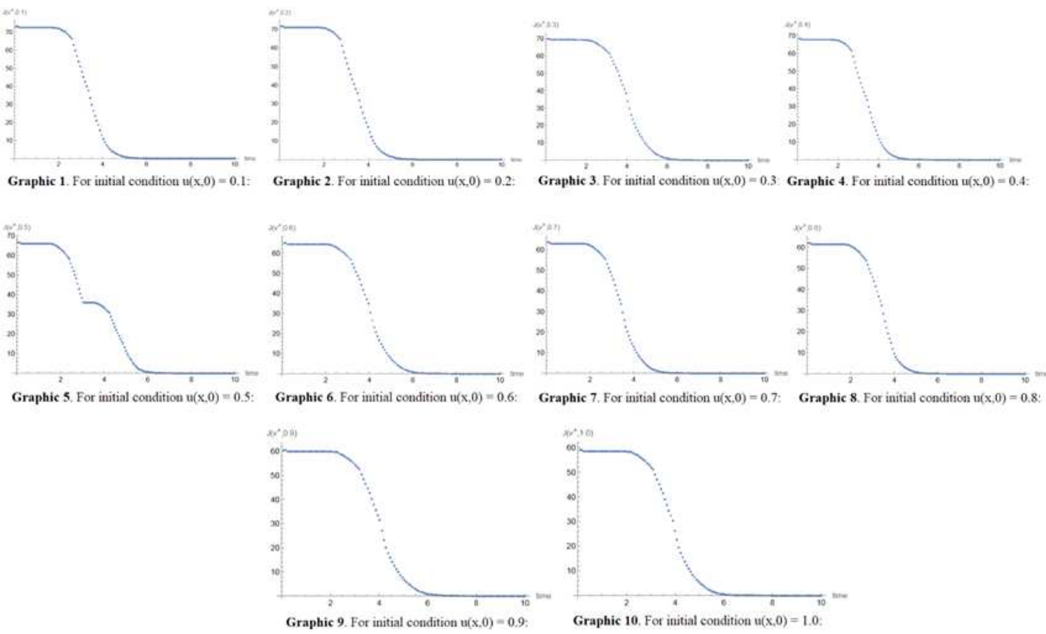

Fig. 3. Progression of the optimality criterion under the optimal control vector for different initial states

Figure 3 shows the behavior of the optimality criterion:

$$\mathcal{J}(v^*, \varphi, t) = \int_0^1 \mathbf{e}^{-(x-0.5)^2} [u(x, t; v^*, \varphi) - u^\star(x)]^2 \mathbf{d}x,$$

under the optimal vector $v^*(t)$ for different initial states $u(x, 0) = \varphi = const$.

4. Conclusion

In this work, we have obtained formulas for the gradient of the objective function for the problem of optimal feedback zonal control of an object described by a system of differential equations with partial derivatives given inaccurate information on the values of the initial conditions of the object. The formulas obtained made it possible to apply first-order finite-dimensional optimization methods for the numerical solution of the problems under consideration. It is known that the technical implementation of piecewise-constant synthesizing functions with sufficiently high accuracy is relatively simple. Therefore, the proposed approach to solving optimal control synthesis problems can find wide application in automated systems and automatic control of systems in the presence of inaccurate information on the object's state. The application objects can be many controlled mechanical systems, technological processes described by systems of nonlinear differential equations. Note that the proposed approach can easily be extended to two- and three-dimensional heat conduction processes.

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