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TWO-WEIGHTED INEQUALITIES FOR MAXIMAL OPERATOR IN GENERALIZED WEIGHTED MORREY SPACES ON SPACES OF HOMOGENEOUS TYPE

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Abstract. *In this paper we give a characterization of two-weighted inequalities for maximal operators in generalized weighted Morrey spaces on spaces of homogeneous type* $\mathcal{M}^{p,\varphi}_{\omega}(X)$*. We prove the boundedness of maximal operator* M *from the spaces* $\mathcal{M}^{p,\varphi_1}_{\omega_1^{\delta}}(X)$ *to the spaces* $\mathcal{M}^{p,\varphi_2}_{\omega_2^{\delta}}(X)$*, where* $1 < p < \infty$*,* $0 < \delta < 1$ *and* $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$*.*

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1. Introduction

In the 1970s, in order to extend the theory of Calderón-Zygmund singular integrals to a more general setting, R.R. Coifman and G. Weiss introduced certain topological measure spaces which are equipped with a metric which is compatible with the given measure in a sense. These spaces are called spaces of homogeneous type. In this work, we find necessary and sufficient conditions for the boundedness of Hardy-Littlewood operators in generalized weighted Morrey spaces on spaces of homogeneous type. As a generalization of $L_p(\mathbb{R}^n)$, the classical Morrey spaces were introduced by Ch.B. Morrey [18] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. V.S. Guliyev, T. Mizuhara and E. Nakai [7], [17], [20] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see, also [8], [24]).

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Recently, Y. Komori and S. Shirai [15] defined the weighted Morrey spaces $L_w^{p,\kappa}(\mathbb{R}^n)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Also, V.S. Guliyev in [9] first introduced the generalized weighted Morrey spaces $M_w^{p,\varphi}(\mathbb{R}^n)$ and studied the boundedness of the sublinear operators and their higher order commutators generated by Calderón-Zygmund operators and Riesz potentials in these spaces (see, also [1], [12], [13]). Note that, V.S. Guliyev [9] gave the concept of generalized weighted Morrey space which could be viewed as an extension of both $M_w^{p,\varphi}(\mathbb{R}^n)$ and $L_w^{p,\kappa}(\mathbb{R}^n)$.

In order to extend the traditional Euclidean space to build a general underlying structure for the real harmonic analysis, the notion of spaces of homogeneous type was introduced by R.R. Coifman and G. Weiss [3].

We say that $X = (X, d, \mu)$ is a space of homogeneous type if d is a quasi-metric on X and μ is a positive measure satisfying the doubling condition, i.e. X is a topological space endowed with a quasi-metric d and a positive measure μ such that

$$\begin{aligned} &d\,(x,y) = d\,(y,x) \geq 0 \text{ for all } x,y \in X, \\ &d\,(x,y) = 0 \text{ if and only if } x = y\ , \\ &d\,(x,y) \leq K_1\,(d\,(x,z) + d\,(z,y)) \text{ for all } x,y,z \in X, \end{aligned}$$

the balls $B(x,r) = \{y \in X : d\,(x,y) < r\}\,,\ r > 0$, form a basis of neighborhoods of the point x, μ is defined on a σ-algebra of subsets of X which contains the balls, and

$$0 < \mu\,(B(x,2r)) < K_2 \mu\,(B(x,r)) < \infty, \tag{1}$$

where $K_i \geq 1\;(i = 1,2)$ are constants independent of $x,y,z \in X$ and $r > 0$. As usual, the dilation of a ball $B = B(x,r)$ will be denoted by $\lambda B = B(x,\lambda r)$ for every $\lambda > 0$.

Throughout this paper we always assume that $\mu(X) = \infty$, the space of compactly supported continuous function is dense in $L_1(X,\mu)$ and that X is Q-homogeneous ($Q > 0$), i.e.

$$K_3^{-1} r^Q \leq \mu\,(B(x,r)) \leq K_3 r^Q,$$

where $K_3 \geq 1$ is a constant independent of x and r. The n-dimensional Euclidean space $\mathbb{R}^n$ is n-homogeneous.

Let (X,d,μ) be Q-homogeneous, $1 \leq p < \infty$, φ be a positive measurable function on $(0,\infty)$ and ω be a non-negative measurable function on X. We denote by $\mathcal{M}_\omega^{p,\varphi}$ the generalized weighted Morrey space on spaces of homogeneous type, the space of all functions $f \in L_{p,\omega}^{loc}(X)$ with finite norm

$$\|f\|_{\mathcal{M}_\omega^{p,\varphi}} = \sup_{x\in X, r>0} \frac{1}{\varphi(x,r)\|\omega\|_{L_p(B(x,r))}} \|f\|_{L_{p,\omega}(B(x,r))},$$

where the supremum is taken over all balls $B(x,r)$ in X and $L_{p,\omega}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,\omega}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{L_{p,\omega}(X)} = \left(\int_{B(x,r)} |f(y)|^p \omega(y) d\mu\,(y)\right)^{\frac{1}{p}}.$$

Moreover, by $W\mathcal{M}_{\omega}^{p,\varphi}$ we denote the weak generalized weighted Morrey space on spaces of homogeneous type of all functions $f \in WL_{p,\omega}^{loc}(X)$ with finite norm

$$\|f\|_{W\mathcal{M}_{\omega}^{p,\varphi}} = \sup_{x\in X, r>0} \frac{1}{\varphi(x,r)\|\omega\|_{L_p(B(x,r))}} \|f\|_{WL_{p,\omega}(B(x,r))},$$

where $WL_{p,\omega}(B(x,r))$ denotes the weak weighted L_p-space of measurable functions f for which

$$\|f\|_{WL_{p,\omega}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,\omega}(X)} = \sup_{t>0} t \left(\int_{\{y\in B(x,r): |f(y)|>t\}} |f(y)|^p \omega(y) d\mu(y) \right)^{\frac{1}{p}}.$$

Note that if $\omega(x) = \chi_{B(x,r)}$, then $\mathcal{M}_{\omega}^{p,\varphi}(X) = \mathcal{M}^{p,\varphi}(X)$ is the generalized Morrey space and if $\varphi(x,r) = \left(\frac{r^{\lambda}}{\mu(B(x,r))}\right)^{\frac{1}{p}}$, then $\mathcal{M}_{\omega}^{p,\varphi}(X) = L_{p,\lambda}(X)$ is the classic Morrey space. A characterization for fractional integral and its commutators in Orlicz and generalized Orlicz-Morrey space on space of homogeneous type were investigated in [11].

Let f be a locally integrable function on X. The so-called of Hardy-Littlewood maximal function is defined by the formula

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| d\mu(y),$$

where $\mu(B(x,r))$ is measure of the ball $B(x,r)$.

In this paper we aim to give a characterization of two-weighted inequalities for maximal operator in generalized weighted Morrey spaces on spaces of homogeneous type. Two-weight norm inequalities for fractional maximal operators and singular integrals on Lebesgue spaces were widely studied (see, for example [4], [5], [6], [14], [16]). The weighted norm inequalities with different types of weights on Morrey spaces were also studied (see, for example [21]). The two-weight norm inequality for the Hardy-Littlewood maximal function on Morrey spaces was obtained in [25]. Two-weight norm inequalities on weighted Morrey spaces for fractional maximal operators and fractional integral operators were obtained in [23].

In the sequel we use the letter C for a positive constant, independent of appropriate parameters and not necessary the same at each occurrence. For every $p \in (1,\infty)$,we denote p' the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. $\mathfrak{M}(\mathbb{R}_+)$, $\mathfrak{M}^+(\mathbb{R}_+)$ and $\mathfrak{M}^+(\mathbb{R}_+;\uparrow)$ stand for the set of Lebesgue-measurable functions on $\mathbb{R}_+$, and its subspaces of nonnegative and nonnegative non-decreasing functions, respectively.

2. Preliminaries

Let (X,d,μ) be space of Q-homogeneous type as mentioned in Section 1. We now recall the definition of A_p weight functions.

Definition. *The weight function* ω *belongs to the class* $A_p(X)$ *for* $1 \le p < \infty$, *if*

$$\sup_{x\in X, r>0}\left(\frac{1}{\mu\left(B\left(x,r\right)\right)}\int\limits_{B(x,r)}\omega^p(y)d\mu\left(y\right)\right)^{\frac{1}{p}}\left(\frac{1}{\mu\left(B\left(x,r\right)\right)}\int\limits_{B(x,r)}\omega^{-p'}(y)d\mu\left(y\right)\right)^{\frac{1}{p'}}$$

is finite and ω *belongs to* $A_1(X)$, *if there exists a positive constant* C *such that for any* $x \in X$ *and* $r > 0$

$$\mu\left(B\left(x,r\right)\right)^{-1}\int\limits_{B(x,r)}\omega(y)d\mu\left(y\right) \le C \operatorname*{ess\,sup}_{y\in B(x,r)}\frac{1}{\omega(y)}.$$

The weight function (ω_1,ω_2) *belongs to the class* $\widetilde{A}_p(X)$ *for* $1 < p < \infty$, *if*

$$\sup_{x\in X, r>0}\left(\frac{1}{\mu\left(B\left(x,r\right)\right)}\int\limits_{B(x,r)}\omega_2^p(y)d\mu\left(y\right)\right)^{\frac{1}{p}}\left(\frac{1}{\mu\left(B\left(x,r\right)\right)}\int\limits_{B(x,r)}\omega_1^{-p'}(y)d\mu\left(y\right)\right)^{\frac{1}{p'}}$$

is finite.

The following theorem was proved in [19].

Theorem 1. *Let* $1 \le p < \infty$.

1) Then the operator M *is bounded in* $L_{p,\omega}(X)$ *if and only if* $\omega \in A_p(X)$.

2) Then the operator M *is bounded from* $L_{1,\omega}(X)$ *to* $WL_{1,\omega}(X)$ *if and only if* $\omega \in A_1(X)$.

Lemma 1. *[22] Let* $1 < p < \infty$ *and* $(\omega_1,\omega_2) \in \widetilde{A}_p(X)$, *then* $(\omega_2^{-1},\omega_1^{-1}) \in \widetilde{A}_{p'}(X)$, *with* $\frac{1}{p}+\frac{1}{p'}=1$.

Lemma 2. *[22] Let* $1 < p < \infty$, $0 < \delta < 1$ *and* $(\omega_1,\omega_2) \in \widetilde{A}_p(X)$. *If* $\frac{q-1}{p-1}=\delta$, *then* $(\omega_1,\omega_2) \in \widetilde{A}_{q'}(X)$, *with* $\frac{1}{p}+\frac{1}{p'}=1$.

Theorem 2. *[22] Let* $1 < p < \infty$, $0 < \delta < 1$ *and* $(\omega_1,\omega_2) \in \widetilde{A}_p(X)$, *then the operator* M *is bounded from* $L_{p,\omega_1^\delta}(X)$ *to* $L_{p,\omega_2^\delta}(X)$.

Let $L_{\infty,\omega}(\mathbb{R}_+)$ be the weighted L_∞-space with the norm

$$\|g\|_{L_{\infty,\omega}(\mathbb{R}_+)} = \operatorname*{ess\,sup}_{t>0}\omega(t)g(t).$$

We denote

$$\mathbb{A} = \left\{\varphi \in \mathfrak{M}^+(\mathbb{R}_+;\uparrow) : \lim_{t\to 0+}\varphi(t) = 0\right\}.$$

Let u be a continuous and non-negative function on $\mathbb{R}_+$. We define the supremal operator $\overline{S}_u$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(0,t)}, \quad t \in (0,\infty).$$

The following theorem was proved in [2].

Theorem 3. *[2] Suppose that v_1 and v_2 are nonnegative measurable functions such that $0 < \|v_1\|_{L_\infty(0,t)} < \infty$ for every $t > 0$. Let u be a continuous nonnegative function on $\mathbb{R}$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(\mathbb{R}_+)$ to $L_{\infty,v_2}(\mathbb{R}_+)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(0,\cdot)} \right) \right\|_{L_\infty(\mathbb{R}_+)} < \infty.$$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_0^t g(s)w(s)ds, \;\; H_w^* g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorems was proved in [10].

Theorem 4. *[10] Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w^* g(t) \le C \sup_{t>0} v_1(t) g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Theorem 5. *[10] Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w g(t) \le C \sup_{t>0} v_1(t) g(t) \tag{2}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_0^t \frac{w(s)ds}{\sup_{0<\tau<s} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (2).

3. Two-Weighted Inequalities for Maximal Operators in Generalized Weighted Morrey Spaces on Spaces of Homogeneous Type

In this section we prove two-weighted inequalities for maximal operators in generalized weighted Morrey spaces on spaces of homogeneous type. The following two-weighted local estimates are valid.

Theorem 6. *Let $1 < p < \infty$, $0 < \delta < 1$ and $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$. Then*

$$\|Mf\|_{L_{p,\omega_2^\delta}(B(x,r))} \le C\|\omega_2^\delta\|_{L_p(B(x,r))} \sup_{t>r} \|f\|_{L_{p,\omega_1^\delta}(B(x,t))} \|\omega_2^\delta\|^{-1}_{L_p(B(x,t))} \tag{3}$$

for every $f \in L_{p,\omega_1^\delta}(X)$, where C does not depend on f, x and r.

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{B(x,2kr)}(y), \quad f_2(y) = f(y)\chi_{X\setminus B(x,2kr)}(y), \ r > 0,$$

and have

$$\|Mf\|_{L_{p,\omega_2^\delta}(B(x,r))} \le \|Mf_1\|_{L_{p,\omega_2^\delta}(B(x,r))} + \|Mf_2\|_{L_{p,\omega_2^\delta}(B(x,r))}.$$

By Theorem 2 we obtain

$$\|Mf_1\|_{L_{p,\omega_2^\delta}(B(x,r))} \le \|Mf_1\|_{L_{p,\omega_2^\delta}(X)} \le C\|f_1\|_{L_{p,\omega_1^\delta}(X)} = C\|f\|_{L_{p,\omega_1^\delta}(B(x,2kr))}, \tag{4}$$

where C does not depend on f. From (4) we obtain

$$\|Mf_1\|_{L_{p,\omega_2^\delta}(B(x,r))} \le C\|\omega_2^\delta\|_{L_p(B(x,r))} \sup_{t>r} \|f\|_{L_{p,\omega_1^\delta}(B(x,t))} \|\omega_2^\delta\|^{-1}_{L_p(B(x,t))}, \tag{5}$$

which is easily obtained from the fact that $\|f\|_{L_{p,\omega_1^\delta}(B(x,2kr))}$ is non-decreasing in r, therefore $\|f\|_{L_{p,\omega_1^\delta}(B(x,2kr))}$ on the right-hand side of (4) is dominated by the right-hand side of (5).

For $y \in B(x,r)$ we get

$$\begin{aligned} Mf_2(y) &= \sup_{t>0} \mu\left(B\left(y,t\right)\right)^{-1} \int_{B(y,t)} |f_2(z)| d\mu\left(z\right) \\ &= \sup_{t>0} \mu\left(B\left(y,t\right)\right)^{-1} \int_{\complement B(x,2kr)\cap B(y,t)} |f(z)| d\mu\left(z\right) \\ &\le \sup_{t>r} \mu\left(B\left(y,t\right)\right)^{-1} \int_{B(x,2kt)} |f(z)| d\mu\left(z\right) \\ &\le C \sup_{t>r} \mu\left(B\left(y,2kt\right)\right)^{-1} \int_{B(x,2kt)} |f(z)| d\mu\left(z\right) \\ &= C \sup_{t>2kr} \mu\left(B\left(y,t\right)\right)^{-1} \int_{B(x,t)} |f(z)| d\mu\left(z\right) \end{aligned}$$

from (1) the doubling condition.

By Hölder inequality we obtain

$$
\begin{aligned}
Mf_2(y) &\leq C \sup_{t>2kr} \mu\left(B\left(y,t\right)\right)^{-1} \|f\|_{L_{p,\omega_1^\delta}(B(x,t))} \|\chi_{B(x,t)}\omega_1^{-\delta}\|_{L_{p'}(X)} \\
&\leq C \sup_{t>r} t^{-Q} \|f\|_{L_{p,\omega_1^\delta}(B(x,t))} \|\omega_1^{-\delta}\|_{L_{p'}(B(x,t))} \\
&\leq C \sup_{t>r} \|f\|_{L_{p,\omega_1^\delta}(B(x,t))} \|\omega_2^\delta\|^{-1}_{L_p(B(x,t))}.
\end{aligned}
$$

Then we have

$$
\|Mf_2\|_{L_{p,\omega_2^\delta}(B(x,r))} \leq C\|\omega_2^\delta\|_{L_p(B(x,r))} \sup_{t>r} \|f\|_{L_{p,\omega_1^\delta}(B(x,t))} \|\omega_2^\delta\|^{-1}_{L_p(B(x,t))}. \tag{6}
$$

From (5) and (6) we obtain (3). ◄

Theorem 7. *Let $1 < p < \infty$, $0 < \delta < 1$, $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$ and the function $\varphi_1(x, r)$ and $\varphi_2(x, r)$ satisfy the condition*

$$
\sup_{t>r} \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\|\omega_1^\delta\|_{L_p(B(x,s))}}{\|\omega_2^\delta\|_{L_p(B(x,t))}} \leq C\varphi_2(x,r), \tag{7}
$$

where C does not depend on x and t.

Then the operator M is bounded from the space $\mathcal{M}^{p,\varphi_1}_{\omega_1^\delta}(X)$ to the space $\mathcal{M}^{p,\varphi_2}_{\omega_2^\delta}(X)$.

Proof. Let $f \in \mathcal{M}^{p,\varphi_1}_{\omega_1^\delta}(X)$. By (7), Theorems 3, 6 with $v_2 = \frac{1}{\varphi_2(x,t)}$, $g = \|f\|_{L_{p,\omega_1^\delta}(B(x,t))}$, $u = \|\omega_2^\delta\|^{-1}_{L_p(B(x,t))}$ and $v_1 = \frac{1}{\varphi_1(x,t)\|\omega_1^\delta\|_{L_p(B(x,t))}}$ we get

$$
\begin{aligned}
&\|Mf\|_{\mathcal{M}^{p,\varphi_2}_{\omega_2^\delta}(X)} \\
&\leq C \sup_{x\in X,\, r>0} \frac{\|\omega_2^\delta\|_{L_p(B(x,r))}}{\varphi_2(x,r)\|\omega_2^\delta\|_{L_p(B(x,r))}} \sup_{t>r} \|f\|_{L_{p,\omega_1^\delta}(B(x,t))} \|\omega_2^\delta\|^{-1}_{L_p(B(x,t))} \\
&\leq C \sup_{x\in X,\, r>0} \frac{1}{\varphi_1(x,r)\|\omega_1^\delta\|_{L_p(B(x,r))}} \|f\|_{L_{p,\omega_1^\delta}(B(x,r))} = C\|f\|_{\mathcal{M}^{p,\varphi_1}_{\omega_1^\delta}(X)},
\end{aligned}
$$

which completes the proof. ◄

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