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SOLVABILITY OF BOUNDARY-VALUE PROBLEM WITH BOUNDED OPERATOR IN BOUNDARY CONDITIONS FOR FOURTH-ORDER ELLIPTIC OPERATOR-DIFFERENTIAL EQUATION

E.S. RZAYEV

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Abstract. *In the paper, we derive sufficient conditions for well-posed and unique solvability of a boundary-value problem with a bounded operator in the boundary conditions for a class of fourth-order elliptic operator-differential equations. At the same time, we make estimates for the norms of operators of intermediate derivatives closely related to the solvability conditions, which are expressed using the properties of the operator coefficients of the boundary-value problem.*

Keywords: boundary-value problem, operator-differential equation, operator boundary condition, regular solution, intermediate derivative operators

Mathematics Subject Classification (2020): 34G10, 35J40, 46E40, 47A50, 47D03

1. Introduction and Problem Statement

Let H be a separable Hilbert space with inner (scalar) product (x, y), $x, y \in H$, and A be a positive-definite self-adjoint operator in H ($A = A^* \geq cE$, $c > 0$, E is the identity operator). As is known, the domain of the operator A^γ ($\gamma > 0$) becomes the Hilbert space H_γ with respect to the inner product $(x,y)_\gamma = (A^\gamma x, A^\gamma y)$, $x, y \in D(A^\gamma)$.

Designate by $L_2(\mathbb{R}_+; H)$ the Hilbert space of all vector functions defined on $\mathbb{R}_+$ with values in H and with the norm

$$\|f\|_{L_2(\mathbb{R}_+;H)} = \left(\int_0^{+\infty} \|f(t)\|_H^2 dt\right)^{1/2}.$$

Further, let $L(X, Y)$ designates the set of linear bounded operators acting from a Hilbert space X into another Hilbert space Y; $\sigma(\cdot)$ is the spectrum of the operator $(\cdot)$; henceforth, everywhere derivatives are understood in the sense of the theory of distributions in a Hilbert space [17].

Elvin S. Rzayev
Institute of Mathematics and Mechanics, Baku, Azerbaijan
E-mail: Elvin.Rzayev88@gmail.com

Let us now introduce the following sets:

$$W_2^4(\mathbb{R}_+;H) = \left\{ u(t) : \ u^{(4)}(t) \in L_2(\mathbb{R}_+;H), \ A^4 u(t) \in L_2(\mathbb{R}_+;H) \right\},$$

$$W_{2,T}^4(\mathbb{R}_+;H) =$$

$$= \left\{ u(t) : u(t) \in W_2^4(\mathbb{R}_+;H), u(0) = 0, \ u'(0) = Tu''(0), \ T \in L\left(H_{3/2}; H_{5/2}\right) \right\}.$$

Each of these sets, equipped with the norm

$$\|u\|_{W_2^4(\mathbb{R}_+;H)} = \left(\left\| u^{(4)} \right\|^2_{L_2(\mathbb{R}_+;H)} + \left\| A^4 u \right\|^2_{L_2(\mathbb{R}_+;H)} \right)^{1/2},$$

becomes a Hilbert space [17, Ch.1].

Let us consider the following boundary-value problem in the space H:

$$u^{(4)}(t) + A^4 u(t) + \sum_{j=1}^{4} A_j u^{(4-j)}(t) = f(t), \quad t \in \mathbb{R}_+, \tag{1}$$

$$u(0) = 0, \quad u'(0) = Tu''(0) \tag{2}$$

where $A = A^* \geq cE$, $c > 0$, $T \in L\left(H_{3/2}; H_{5/2}\right)$, A_j, $j = 1,2,3,4$, are linear, generally speaking, unbounded operators, $f(t) \in L_2(\mathbb{R}_+;H)$, $u(t) \in W_2^4(\mathbb{R}_+;H)$.

Definition. *If the vector-function $u(t) \in W_2^4(\mathbb{R}_+;H)$ satisfies Eq. (1) almost everywhere in $\mathbb{R}_+$, and the boundary conditions (2) hold in the following sense:*

$$\lim_{t\to 0} \|u(t)\|_{H_{7/2}} = 0, \quad \lim_{t\to 0} \|u'(t) - Tu''(t)\|_{H_{5/2}} = 0,$$

then $u(t)$ will be called a regular solution of the boundary-value problem (1), (2).

Note that various solvability problems for second-order elliptic operator-differential equations with operator boundary conditions are studied in detail in the works [8], [9], [11], [13], [16], [21]-[24], [27], [28] as well as in references cited there. Similar questions for higher-order operator-differential equations in the case when the coefficients in the boundary conditions are only complex numbers are considered in a broad aspect in the works [2]-[6], [12], [14], [15], [18]-[20], [25], [26] as well as in references cited there. However, all these studies are far from complete. There are relatively few works dedicated to the solvability of fourth-order elliptic operator-differential equations with operator boundary conditions (see, for example, [1], [7], [10]).

In this paper, we derive sufficient conditions for the existence and uniqueness of a regular solution to the boundary-value problem (1), (2), which are expressed by means of the properties of its operator coefficients. Along the way, we construct an explicit representation of the regular solution of the boundary-value problem (1), (2) for $A_j = 0$, $j = 1,2,3,4$.

It should be stressed that boundary-value problems of the form (1), (2) have a number of applications, in particular, in the theory of elasticity.

2. Solvability of Boundary-Value Problem (1), (2) for $A_j = 0$, $j = 1, 2, 3, 4$

First, we denote by P_0 the operator acting from the space $W^4_{2,T}(\mathbb{R}_+; H)$ into the space $L_2(\mathbb{R}_+; H)$ according to the rule

$$P_0 u(t) = u^{(4)}(t) + A^4 u(t), \quad u(t) \in W^4_{2,T}(\mathbb{R}_+; H).$$

Lemma 1. *The operator $P_0 : W^4_{2,T}(\mathbb{R}_+; H) \to L_2(\mathbb{R}_+; H)$ is bounded.*

Proof. The boundedness of the operator $P_0 : W^4_{2,T}(\mathbb{R}_+; H) \to L_2(\mathbb{R}_+; H)$ follows directly from the following inequality using the Cauchy-Schwarz inequality:

$$\|P_0 u\|^2_{L_2(\mathbb{R}_+;H)} = \left\|u^{(4)} + A^4 u\right\|^2_{L_2(\mathbb{R}_+;H)} =$$
$$= \left\|u^{(4)}\right\|^2_{L_2(\mathbb{R}_+;H)} + 2Re\Big(u^{(4)}, A^4 u\Big)_{L_2(\mathbb{R}_+;H)} + \|A^4 u\|^2_{L_2(\mathbb{R}_+;H)} \le$$
$$\le 2\left(\left\|u^{(4)}\right\|^2_{L_2(\mathbb{R}_+;H)} + \|A^4 u\|^2_{L_2(\mathbb{R}_+;H)}\right) = 2\|u\|^2_{W^4_2(\mathbb{R}_+;H)}.$$

◄

The following lemma holds.

Lemma 2. *Let $C = A^{5/2} T A^{-3/2}$ and the point $-\frac{1}{\sqrt{2}} \notin \sigma(C)$. Then the equation $P_0 u(t) = 0$ has a zero (trivial) solution from the space $W^4_{2,T}(\mathbb{R}_+; H)$.*

Proof. Obviously, the general solution of the equation $P_0 u(t) = 0$ from the space $W^4_{2,T}(\mathbb{R}_+; H)$ is represented as follows:

$$u_0(t) = e^{\omega_1 t A}\varphi_0 + e^{\omega_2 t A}\varphi_1,$$

where e^{-tA} is a strongly continuous semigroup of bounded operators generated by the operator $-A$,

$$\varphi_0, \varphi_1 \in H_{7/2},$$
$$\omega_1 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \quad \omega_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i.$$

From the boundary conditions (2), we have:

$$\begin{cases} \varphi_0 + \varphi_1 = 0, \\ \omega_1 A\varphi_0 + \omega_2 A\varphi_1 = TA^2\left(\omega_1^2\varphi_0 + \omega_2^2\varphi_1\right). \end{cases} \tag{3}$$

As a result, from system (3), we obtain:

$$\varphi_1 = -\varphi_0, \tag{4}$$

$$\left(E + \sqrt{2}C\right) A^{7/2}\varphi_0 = 0. \tag{5}$$

Since, according to the condition of the lemma, the point $-\frac{1}{\sqrt{2}} \notin \sigma(C)$, then from Eq. (5) we have that $\varphi_0 = 0$. Then from (4) we obtain that $\varphi_1 = 0$. Thus, $u_0(t) = 0$. ◄

The following theorem holds.

Theorem 1. *Let* $C = A^{5/2}TA^{-3/2}$ *and the point* $-\frac{1}{\sqrt{2}} \notin \sigma(C)$*. Then the equation* $P_0 u(t) = f(t)$ *for any* $f(t) \in L_2(\mathbb{R}_+; H)$ *has a unique regular solution represented in the form*

$$u(t) = \int_0^{+\infty} G(t,s) f(s) ds + \frac{1}{4(\omega_1 - \omega_2)} e^{\omega_1 t A} A^{-\frac{7}{2}} \left(E + \sqrt{2}C\right)^{-1} A^{\frac{5}{2}} \times$$

$$\times \left[-\frac{\omega_4}{\omega_3} \int_0^{+\infty} \left(e^{-\omega_4 A s} - e^{-\omega_3 A s}\right) \left(A^{-2} f(s)\right) ds + \right.$$

$$+T \int_0^{+\infty} \left(\omega_2 e^{-\omega_4 A s} + \omega_1 e^{-\omega_3 A s}\right) \left(A^{-1} f(s)\right) ds -$$

$$\left. -\omega_2{}^2 T A^2 \int_0^{+\infty} \left(\omega_4 e^{-\omega_4 A s} + \omega_3 e^{-\omega_3 A s}\right) \left(A^{-3} f(s)\right) ds \right] -$$

$$-\frac{1}{4} e^{\omega_2 t A} \left[\frac{1}{\omega_1 - \omega_2} A^{-\frac{7}{2}} \left(E + \sqrt{2}C\right)^{-1} A^{\frac{5}{2}} \times \right.$$

$$\times \left(-\frac{\omega_4}{\omega_3} \int_0^{+\infty} \left(e^{-\omega_4 A s} - e^{-\omega_3 A s}\right) \left(A^{-2} f(s)\right) ds + \right.$$

$$+T \int_0^{+\infty} \left(\omega_2 e^{-\omega_4 A s} + \omega_1 e^{-\omega_3 A s}\right) \left(A^{-1} f(s)\right) ds -$$

$$\left. -\omega_2{}^2 T A^2 \int_0^{+\infty} \left(\omega_4 e^{-\omega_4 A s} + \omega_3 e^{-\omega_3 A s}\right) \left(A^{-3} f(s)\right) ds \right) +$$

$$\left. + \int_0^{+\infty} \left(\omega_4 e^{-\omega_4 A s} + \omega_3 e^{-\omega_3 A s}\right) \left(A^{-3} f(s)\right) ds \right],$$

where

$$G(t,s) = \frac{1}{4} \begin{cases} -\left(\dfrac{e^{\omega_2 A(t-s)}}{\omega_1} + \dfrac{e^{\omega_1 A(t-s)}}{\omega_2} \right) A^{-3}, & \text{if } t-s > 0, \\ \left(\dfrac{e^{\omega_4 A(t-s)}}{\omega_3} + \dfrac{e^{\omega_3 A(t-s)}}{\omega_4} \right) A^{-3}, & \text{if } t-s < 0, \end{cases}$$

$$\omega_1 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i, \quad \omega_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i, \quad \omega_3 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i, \quad \omega_4 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i.$$

Proof. For the convenience of further writings, we consider the operator pencil

$$P_0(\lambda; A) = \lambda^4 E + A^4.$$

Then the boundary-value problem (1), (2) for $A_j = 0$, $j = 1, 2, 3, 4$, can be rewritten as an operator equation

$$P_0\left(\frac{d}{dt}; A\right) u(t) = f(t), \tag{6}$$

where $f(t) \in L_2(\mathbb{R}_+; H)$, $u(t) \in W_{2,T}^4(\mathbb{R}_+; H)$.

By Lemma 2, the homogeneous equation

$$P_0\left(\frac{d}{dt}; A\right) u(t) = 0$$

has only a trivial solution from the space $W_{2,T}^4(\mathbb{R}_+; H)$.

Let us show that Eq. (6) has a solution from the space $W_{2,T}^4(\mathbb{R}_+; H)$ for any $f(t) \in L_2(\mathbb{R}_+; H)$.

First, we construct a particular solution of Eq. (1) for $A_j = 0$, $j = 1, 2, 3, 4$.

We extend the function $f(t)$ with a zero for $t < 0$ and consider Eq. (6) on the entire axis. Since the operator A does not depend on t, this extension to the negative half-line is well defined. Applying the direct and inverse Fourier transforms, we obtain

$$u_0(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} P_0^{-1}(i\xi; A) \left(\int_0^{+\infty} f(s) e^{-i\xi s} ds \right) e^{i\xi t} d\xi, \quad t \in \mathbb{R}.$$

It is clear that this vector function satisfies Eq. (6) almost everywhere in $\mathbb{R}$.

Let us show that $u_0(t) \in W_2^4(\mathbb{R}; H)$.

Indeed, by the Plancherel theorem we have:

$$\begin{aligned} \|u_0\|^2_{W_2^4(\mathbb{R};H)} &= \left\| u_0^{(4)} \right\|^2_{L_2(\mathbb{R};H)} + \left\| A^4 u_0 \right\|^2_{L_2(\mathbb{R};H)} = \\ &= \left\| \xi^4 \hat{u}_0(\xi) \right\|^2_{L_2(\mathbb{R};H)} + \left\| A^4 \hat{u}_0(\xi) \right\|^2_{L_2(\mathbb{R};H)} \le \\ &\le \sup_{\xi \in \mathbb{R}} \left\| \xi^4 P_0^{-1}(i\xi; A) \right\|^2_{H \to H} \left\| \hat{f}(\xi) \right\|^2_{L_2(\mathbb{R};H)} + \\ &+ \sup_{\xi \in \mathbb{R}} \left\| A^4 P_0^{-1}(i\xi; A) \right\|^2_{H \to H} \left\| \hat{f}(\xi) \right\|^2_{L_2(\mathbb{R};H)}, \end{aligned} \tag{7}$$

where $\hat{u}_0(\xi)$ and $\hat{f}(\xi)$ are Fourier transforms of functions $u_0(t)$ and $f(t)$, respectively. Since, from the spectral theory of self-adjoint operators, for any $\xi \in \mathbb{R}$ it follows that

$$\left\| \xi^4 P_0^{-1}(i\xi; A) \right\| = \sup_{\mu \in \sigma(A)} \left(\xi^4 \left(\xi^4 + \mu^4 \right) \right)^{-1} \le 1,$$

$$\left\| A^4 P_0^{-1}(i\xi; A) \right\| = \sup_{\mu \in \sigma(A)} \left(\mu^4 \left(\xi^4 + \mu^4 \right) \right)^{-1} \le 1,$$

then from (7) we obtain:

$$\|u_0\|_{W_2^4(\mathbb{R};H)} \leq const\|f\|_{L_2(\mathbb{R};H)}.$$

Therefore, $u_0(t) \in W_2^4(\mathbb{R};H)$.

It's obvious that

$$u_0(t) = \int_0^{+\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\xi^4 E + A^4\right)^{-1} e^{i\xi(t-s)} d\xi \right) f(s)\, ds, \quad t \in \mathbb{R}.$$

We calculate the integral

$$G(t,s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\xi^4 E + A^4\right)^{-1} e^{i\xi(t-s)} d\xi, \qquad t \in \mathbb{R}. \tag{8}$$

Let $\mu \in \sigma(A)$. Then

$$G(t,s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i\xi(t-s)}}{\xi^4 + \mu^4} d\xi. \tag{9}$$

First, to calculate the integral (9), we choose the integration contours on the upper $t-s>0$ and lower $t-s<0$ half-planes, and then apply the residue theorem. As one can see, the poles $z_1 = \omega_1\mu$ and $z_2 = \omega_3\mu$ are in the region that is limited by the integration contour lying in the upper half-plane $t-s>0$, while the poles $z_3 = \omega_2\mu$ and $z_4 = \omega_4\mu$ are in the region that is limited by the integration contour lying in the lower half-plane $t-s<0$ (see Fig. 1 and Fig. 2).

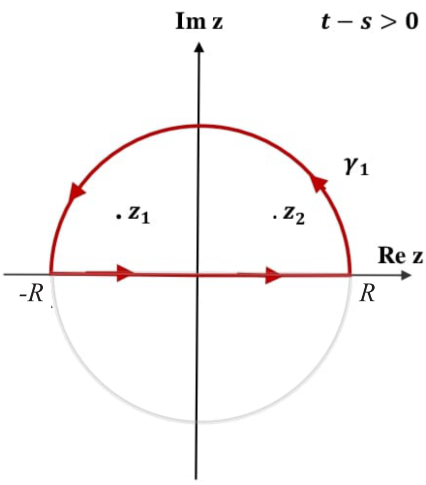

Fig. 1.
Contours on the upper half-plane

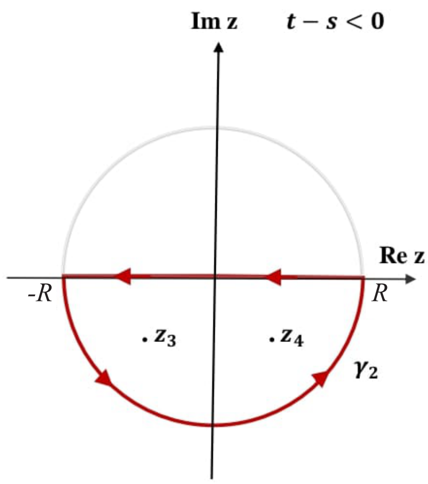

Fig. 2.
Contours on the lower half-plane

It is clear that on the upper half-plane $t-s>0$, integral (9) along the contour γ_1 is equal to the sum of two integrals:

$$\frac{1}{2\pi}\int\limits_{\gamma_1}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}d\xi=\frac{1}{2\pi}\left[\int_{-R}^{R}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}d\xi+\int\limits_{\text{semi-circle }\gamma_1}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}d\xi\right].$$

Note that along the semi-circle γ_1, $\forall\xi:|\xi|=R$, and since

$$\left|\xi^4+\mu^4\right|\geq R^4-\mu^4,\quad \left|\xi^4+\mu^4\right|^{-1}\leq\left(R^4-\mu^4\right)^{-1},$$

then

$$\left|\int\limits_{\text{semi-circle }\gamma_1}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}d\xi\right|\leq\int\limits_{\text{semi-circle }\gamma_1}\left|\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}\right|d\xi\leq\int\limits_{\text{semi-circle }\gamma_1}\left(R^4-\mu^4\right)^{-1}d\xi=$$

$$=\left(R^4-\mu^4\right)^{-1}\int\limits_{\text{semi-circle }\gamma_1}d\xi=\frac{\pi R}{R^4-\mu^4}\to 0\text{ when }R\to\infty.$$

Thus,

$$\lim_{R\to\infty}\int\limits_{\text{semi-circle }\gamma_1}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}d\xi=0.$$

Therefore, when $R\to\infty$

$$\frac{1}{2\pi}\int\limits_{\gamma_1}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}d\xi=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}d\xi\ .$$

Since

$$\underset{\xi=\omega_1\mu}{\text{res}}\ \frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}=\frac{e^{i\omega_1\mu(t-s)}}{4\omega_1^3\mu^3}=\frac{e^{\omega_2\mu(t-s)}}{4\omega_3\mu^3},$$

$$\underset{\xi=\omega_3\mu}{\text{res}}\ \frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}=\frac{e^{i\omega_3\mu(t-s)}}{4\omega_3^3\mu^3}=\frac{e^{\omega_1\mu(t-s)}}{4\omega_1\mu^3},$$

then

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}d\xi=-\frac{1}{4}\left(\frac{e^{\omega_2\mu(t-s)}}{\omega_1\mu^3}+\frac{e^{\omega_1\mu(t-s)}}{\omega_2\mu^3}\right).$$

Similarly, on the lower half-plane $t-s<0$, the integral (9) along the contour γ_2 is equal to the sum of two integrals:

$$\frac{1}{2\pi}\int\limits_{\gamma_2}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}d\xi=\frac{1}{2\pi}\left[\int_{R}^{-R}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}d\xi+\int\limits_{\text{semi-circle }\gamma_2}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}d\xi\right]=$$

$$= \frac{1}{2\pi}\left[-\int_{-R}^{R}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}d\xi \quad + \int\limits_{\text{semi-circle } \gamma_2}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}d\xi\right].$$

Carrying out the corresponding reasoning, we get:

$$\lim_{R\to\infty}\int\limits_{\text{semi-circle } \gamma_2}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}d\xi = 0.$$

Therefore, when $R\to\infty$

$$\frac{1}{2\pi}\int\limits_{\gamma_2}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}d\xi = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}d\xi.$$

And since

$$\operatorname*{res}_{\xi=\omega_2\mu}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4} = \frac{e^{i\omega_2\mu(t-s)}}{4\omega_2^3\mu^3} = \frac{e^{\omega_4\mu(t-s)}}{4\omega_4\mu^3},$$

$$\operatorname*{res}_{\xi=\omega_4\mu}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4} = \frac{e^{i\omega_4\mu(t-s)}}{4\omega_4^3\mu^3} = \frac{e^{\omega_3\mu(t-s)}}{4\omega_2\mu^3},$$

then

$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\xi(t-s)}}{\xi^4+\mu^4}d\xi=\frac{1}{4}\left(\frac{e^{\omega_4\mu(t-s)}}{\omega_3\mu^3}+\frac{e^{\omega_3\mu(t-s)}}{\omega_4\mu^3}\right).$$

Thus, using the spectral expansion of the operator A, for (8) we obtain the representation

$$G(t,s) = \frac{1}{4}\begin{cases} -\left(\dfrac{e^{\omega_2 A(t-s)}}{\omega_1}+\dfrac{e^{\omega_1 A(t-s)}}{\omega_2}\right)A^{-3}, & \text{if } t-s>0,\\ \left(\dfrac{e^{\omega_4 A(t-s)}}{\omega_3}+\dfrac{e^{\omega_3 A(t-s)}}{\omega_4}\right)A^{-3}, & \text{if } \quad t-s<0. \end{cases} \tag{10}$$

Therefore,

$$u_0(t) = \int_0^{+\infty} G(t,s)f(s)\,ds,$$

where $G(t,s)$ is defined in (10).

Obviously, $u_0(t)\in W_2^4(\mathbb{R}_+;H)$. Then according to the Theorem on traces [17, Ch.1]

$$u_0^{(4-j)}(0)\in H_{7/2-j}, \qquad j=0,\ 1,\ 2,\ 3.$$

We are looking for the general solution to equation (6) in the form

$$u(t) = u_0(t) + e^{\omega_1 tA}\psi_0 + e^{\omega_2 tA}\psi_1\ ,$$

where $\psi_0,\psi_1\in H_{7/2}$. From the boundary conditions (2), we have

$$\begin{cases} u_0(0)+\psi_0+\psi_1 = 0,\\ u_0'(0)+\omega_1 A\psi_0+\omega_2 A\psi_1 = T\left(u''_0(0)+\omega_1^2A^2\psi_0+\omega_2^2A^2\psi_1\right). \end{cases} \tag{11}$$

System (11) can be rewritten in the following form:

$$\begin{cases} \psi_0+\psi_1=-u_0\left(0\right), \\ \omega_1 A\psi_0+\omega_2 A\psi_1-{\omega_1}^2 TA^2\psi_0-{\omega_2}^2 TA^2\psi_1=-u'_0\left(0\right)+Tu''_0\left(0\right). \end{cases} \tag{12}$$

Then from system (12), taking into account that the point $-\frac{1}{\sqrt{2}}\notin\sigma\left(C\right)$, we find

$$\psi_0=\frac{1}{4\left(\omega_1-\omega_2\right)}A^{-\frac{7}{2}}\left(E+\sqrt{2}C\right)^{-1}A^{\frac{5}{2}}\times$$
$$\times\left[-\frac{\omega_4}{\omega_3}\int_0^{+\infty}\left(e^{-\omega_4 As}-e^{-\omega_3 As}\right)\left(A^{-2}f\left(s\right)\right)ds+\right.$$
$$+T\int_0^{+\infty}\left(\omega_2 e^{-\omega_4 As}+\omega_1 e^{-\omega_3 As}\right)\left(A^{-1}f\left(s\right)\right)ds-$$
$$\left.-{\omega_2}^2 TA^2\int_0^{+\infty}\left(\omega_4 e^{-\omega_4 As}+\omega_3 e^{-\omega_3 As}\right)\left(A^{-3}f\left(s\right)\right)ds\right],$$
$$\psi_1=-\left(\psi_0+\frac{1}{4}\int_0^{+\infty}\left(\omega_4 e^{-\omega_4 As}+\omega_3 e^{-\omega_3 As}\right)\left(A^{-3}f\left(s\right)\right)ds\right).$$

As a result, we obtain the desired representation of the solution of equation (6)

$$u\left(t\right)=\int_0^{+\infty}G\left(t,s\right)f\left(s\right)ds+\frac{1}{4\left(\omega_1-\omega_2\right)}e^{\omega_1 tA}A^{-\frac{7}{2}}\left(E+\sqrt{2}C\right)^{-1}A^{\frac{5}{2}}\times$$
$$\times\left[-\frac{\omega_4}{\omega_3}\int_0^{+\infty}\left(e^{-\omega_4 As}-e^{-\omega_3 As}\right)\left(A^{-2}f\left(s\right)\right)ds+\right.$$
$$+T\int_0^{+\infty}\left(\omega_2 e^{-\omega_4 As}+\omega_1 e^{-\omega_3 As}\right)\left(A^{-1}f\left(s\right)\right)ds-$$
$$\left.-{\omega_2}^2 TA^2\int_0^{+\infty}\left(\omega_4 e^{-\omega_4 As}+\omega_3 e^{-\omega_3 As}\right)\left(A^{-3}f\left(s\right)\right)ds\right]-$$
$$-\frac{1}{4}e^{\omega_2 tA}\left[\frac{1}{\omega_1-\omega_2}A^{-\frac{7}{2}}\left(E+\sqrt{2}C\right)^{-1}A^{\frac{5}{2}}\times\right.$$
$$\times\left(-\frac{\omega_4}{\omega_3}\int_0^{+\infty}\left(e^{-\omega_4 As}-e^{-\omega_3 As}\right)\left(A^{-2}f\left(s\right)\right)ds+\right.$$
$$+T\int_0^{+\infty}\left(\omega_2 e^{-\omega_4 As}+\omega_1 e^{-\omega_3 As}\right)\left(A^{-1}f\left(s\right)\right)ds-$$
$$\left.-{\omega_2}^2 TA^2\int_0^{+\infty}\left(\omega_4 e^{-\omega_4 As}+\omega_3 e^{-\omega_3 As}\right)\left(A^{-3}f\left(s\right)\right)ds\right)+$$
$$\left.+\int_0^{+\infty}\left(\omega_4 e^{-\omega_4 As}+\omega_3 e^{-\omega_3 As}\right)\left(A^{-3}f\left(s\right)\right)ds\right],$$

where

$$G(t,s) = \frac{1}{4}\begin{cases} -\left(\dfrac{e^{\omega_2 A(t-s)}}{\omega_1} + \dfrac{e^{\omega_1 A(t-s)}}{\omega_2}\right) A^{-3}, & \text{if} \quad t-s>0, \\ \left(\dfrac{e^{\omega_4 A(t-s)}}{\omega_3} + \dfrac{e^{\omega_3 A(t-s)}}{\omega_4}\right) A^{-3}, & \text{if} \quad t-s<0, \end{cases}$$

and we have the inequality

$$\|u\|_{W_2^4(\mathbb{R}_+;H)} \le const \|f\|_{L_2(\mathbb{R}_+;H)}.$$

◄

Taking into account Theorem 1 in combination with Lemmas 1 and 2, we can formulate the following theorem.

Theorem 2. *Let $C = A^{5/2}TA^{-3/2}$ and the point $-\frac{1}{\sqrt{2}} \notin \sigma(C)$. Then the operator P_0 implements an isomorphism between the spaces $W_{2,T}^4(\mathbb{R}_+;H)$ and $L_2(\mathbb{R}_+;H)$.*

From Theorem 2, we have

Corollary 1. *The norm $\|P_0 u\|_{L_2(\mathbb{R}_+;H)}$ is equivalent to the norm $\|u\|_{W_2^4(\mathbb{R}_+;H)}$ in the space $W_{2,T}^4(\mathbb{R}_+;H)$.*

Since the operators of intermediate derivatives

$$A^j \frac{d^{4-j}}{dt^{4-j}} \; : \; W_{2,T}^4(\mathbb{R}_+;H) \to L_2(\mathbb{R}_+;H), \quad j = 1,2,3,4,$$

are continuous [17], their norms can be estimated in terms of the norm $\|P_0 u\|_{L_2(\mathbb{R}_+;H)}$, based on the Corollary 1.

Theorem 3. *Let $C = A^{5/2}TA^{-3/2}$ and $Re\ C \ge 0$. Then for any $u(t) \in W_{2,T}^4(\mathbb{R}_+;H)$ the following inequalities hold:*

$$\left\| A^j u^{(4-j)} \right\|_{L_2(\mathbb{R}_+;H)} \le c_j \|P_0 u\|_{L_2(\mathbb{R}_+;H)}, \quad j = 1,\ 2,\ 3,\ 4, \tag{13}$$

where

$$c_1 = 1, \quad c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{\sqrt{2}}, \quad c_4 = 1.$$

Proof. First, multiply both sides of the equation

$$u^{(4)}(t) + A^4 u(t) = f(t)$$

scalarly by $A^4 u(t)$ in the space $L_2(\mathbb{R}_+;H)$ and integrate by parts taking into account that $u(t) \in W_{2,T}^4(\mathbb{R}_+;H)$. Then, taking into account the condition $Re\ C \ge 0$, we have:

$$Re\ \left(P_0 u,\ A^4 u\right)_{L_2(\mathbb{R}_+;H)} = Re\ \left(u^{(4)} + A^4 u, \quad A^4 u\right)_{L_2(\mathbb{R}_+;H)} =$$

$$= Re\left(CA^{3/2}u''(0),\ A^{3/2}u''(0)\right) + \left\|A^2u''\right\|^2_{L_2(\mathbb{R}_+;H)} + \left\|A^4u\right\|^2_{L_2(\mathbb{R}_+;H)} \geq$$

$$\geq \left\|A^2u''\right\|^2_{L_2(\mathbb{R}_+;H)} + \left\|A^4u\right\|^2_{L_2(\mathbb{R}_+;H)}. \qquad (14)$$

Applying the classical Cauchy-Schwartz and Young inequalities sequentially to the left side of (14), we obtain:

$$\left\|A^2u''\right\|^2_{L_2(\mathbb{R}_+;H)} + \left\|A^4u\right\|^2_{L_2(\mathbb{R}_+;H)} \leq \|P_0u\|_{L_2(\mathbb{R}_+;H)} \left\|A^4u\right\|_{L_2(\mathbb{R}_+;H)} \leq$$

$$\leq \frac{\varepsilon}{2}\|P_0u\|^2_{L_2(\mathbb{R}_+;H)} + \frac{1}{2\varepsilon}\left\|A^4u\right\|^2_{L_2(\mathbb{R}_+;H)}, \quad \varepsilon > 0. \qquad (15)$$

Choosing $\varepsilon = \frac{1}{2}$ in (15), we have:

$$\left\|A^2u''\right\|^2_{L_2(\mathbb{R}_+;H)} \leq \frac{1}{4}\|P_0u\|^2_{L_2(\mathbb{R}_+;H)},$$

therefore,

$$\left\|A^2u''\right\|_{L_2(\mathbb{R}_+;H)} \leq \frac{1}{2}\|P_0u\|_{L_2(\mathbb{R}_+;H)}. \qquad (16)$$

On the other hand, from (15) we obtain

$$\left\|A^4u\right\|^2_{L_2(\mathbb{R}_+;H)} \leq \|P_0u\|_{L_2(\mathbb{R}_+;H)} \left\|A^4u\right\|_{L_2(\mathbb{R}_+;H)},$$

whence

$$\left\|A^4u\right\|_{L_2(\mathbb{R}_+;H)} \leq \|P_0u\|_{L_2(\mathbb{R}_+;H)}. \qquad (17)$$

Let us pass to the estimation of the norm $\left\|A^3u'\right\|_{L_2(\mathbb{R}_+;H)}$. Taking into account $u(t) \in W^4_{2,T}(\mathbb{R}_+;H)$, we integrate by parts and apply the Cauchy-Schwartz inequality:

$$\left\|A^3u'\right\|^2_{L_2(\mathbb{R}_+;H)} = \left(A^3u', A^3u'\right)_{L_2(\mathbb{R}_+;H)} = -\left(A^4u, A^2u''\right)_{L_2(\mathbb{R}_+;H)} \leq$$

$$\leq \left\|A^4u\right\|_{L_2(\mathbb{R}_+;H)} \left\|A^2u''\right\|_{L_2(\mathbb{R}_+;H)}. \qquad (18)$$

Taking into account inequalities (16) and (17) in (18), we have

$$\left\|A^3u'\right\|^2_{L_2(\mathbb{R}_+;H)} \leq \frac{1}{2}\|P_0u\|^2_{L_2(\mathbb{R}_+;H)},$$

therefore,

$$\left\|A^3u'\right\|_{L_2(\mathbb{R}_+;H)} \leq \frac{1}{\sqrt{2}}\|P_0u\|_{L_2(\mathbb{R}_+;H)}.$$

Finally, let us estimate the norm $\|Au'''\|_{L_2(\mathbb{R}_+;H)}$. It was shown in [20] that for any $u(t) \in W^4_2(\mathbb{R}_+;H)$ the following inequality holds

$$\|Au'''\|^2_{L_2(\mathbb{R}_+;H)} \leq 2\left\|A^2u''\right\|_{L_2(\mathbb{R}_+;H)} \left\|u^{(4)}\right\|_{L_2(\mathbb{R}_+;H)}. \qquad (19)$$

Again, taking into account the operation of integration by parts for $u(t) \in W_{2,T}^4(\mathbb{R}_+; H)$, it is easy to see that

$$\|P_0 u\|^2_{L_2(\mathbb{R}_+;H)} = \left\|u^{(4)}\right\|^2_{L_2(\mathbb{R}_+;H)} + \left\|A^4 u\right\|^2_{L_2(\mathbb{R}_+;H)} + 2Re\left(u^{(4)},\ \ A^4 u\right)_{L_2(\mathbb{R}_+;H)} =$$

$$= \left\|u^{(4)}\right\|^2_{L_2(\mathbb{R}_+;H)} + \left\|A^4 u\right\|^2_{L_2(\mathbb{R}_+;H)} +$$

$$+2Re\left(CA^{3/2}u''(0),\ A^{3/2}u''(0)\right) + 2\left\|A^2 u''\right\|^2_{L_2(\mathbb{R}_+;H)}.$$

Hence, taking into account the condition $Re\ C \geq 0$, we obtain

$$\|P_0 u\|_{L_2(\mathbb{R}_+;H)} \geq \left\|u^{(4)}\right\|_{L_2(\mathbb{R}_+;H)}. \tag{20}$$

Then, taking into account (16) and (20) in inequality (19), we have

$$\|Au'''\|^2_{L_2(\mathbb{R}_+;H)} \leq \|P_0 u\|^2_{L_2(\mathbb{R}_+;H)},$$

therefore,

$$\|Au'''\|_{L_2(\mathbb{R}_+;H)} \leq \|P_0 u\|_{L_2(\mathbb{R}_+;H)}.$$

◀

3. Solvability of Boundary-Value Problem (1), (2) for $A_j \neq 0$, $j = 1, 2, 3, 4$

Denote by P_1 the operator acting from the space $W_{2,T}^4(\mathbb{R}_+; H)$ into the space $L_2(\mathbb{R}_+; H)$ according to the rule

$$P_1 u(t) = \sum_{j=1}^{4} A_j u^{(4-j)}(t), \quad u(t) \in W_{2,T}^4(\mathbb{R}_+; H).$$

The following lemma holds.

Lemma 3. *Let* $A_j A^{-j} \in L(H,\ H)$, $j = 1, 2, 3, 4$. *Then the operator* $P_1 : W_{2,T}^4(\mathbb{R}_+; H) \to L_2(\mathbb{R}_+; H)$ *is bounded.*

Proof. For any $u(t) \in W_{2,T}^4(\mathbb{R}_+; H)$, there holds

$$\|P_1 u\|_{L_2(\mathbb{R}_+;H)} \leq \sum_{j=1}^{4} \left\|A_j u^{(4-j)}\right\|_{L_2(\mathbb{R}_+;H)} \leq$$

$$\leq \sum_{j=1}^{4} \left\|A_j A^{-j}\right\|_{H\to H} \left\|A^j u^{(4-j)}\right\|_{L_2(\mathbb{R}_+;H)}. \tag{21}$$

Then, taking into account the theorem on intermediate derivatives [17, Ch. 1], from (21) we obtain

$$\|P_1 u\|_{L_2(\mathbb{R}_+;H)} \leq const\ \|u\|_{W_2^4(\mathbb{R}_+;H)}\ .$$

◀

The results obtained allow us to find conditions for the existence and uniqueness of a regular solution to the boundary-value problem (1), (2).

Denote by P the operator acting from the space $W_{2,T}^4(\mathbb{R}_+;H)$ into the space $L_2(\mathbb{R}_+;H)$ according to the rule

$$Pu(t) = u^{(4)}(t) + A^4 u(t) + \sum_{j=1}^{4} A_j u^{(4-j)}(t), \quad u(t) \in W_{2,T}^4(\mathbb{R}_+;H).$$

Taking into account Lemmas 1 and 3, we have:

$$\|Pu\|_{L_2(\mathbb{R}_+;H)} \leq \|P_0 u\|_{L_2(\mathbb{R}_+;H)} + \|P_1 u\|_{L_2(\mathbb{R}_+;H)} \leq const\ \|u\|_{W_2^4(\mathbb{R}_+;H)}\ .$$

Thus, the following theorem holds.

Theorem 4. *Let* $A_j A^{-j} \in L(H,\ H)$, $j = 1,2,3,4$. *Then the operator* P : $W_{2,T}^4(\mathbb{R}_+;H) \to L_2(\mathbb{R}_+;H)$ *is bounded.*

The following main theorem holds.

Theorem 5. *Let* $C = A^{5/2} T A^{-3/2}$, $ReC \geq 0$ *and* $A_j A^{-j} \in L(H,\ H)$, $j = 1,2,3,4$. *Then, if the inequality holds*

$$\sum_{j=1}^{4} c_j \left\|A_j A^{-j}\right\|_{H \to H} < 1,$$

where the numbers c_j, $j = 1,2,3,4$, *are defined in Theorem 3, then the boundary-value problem (1), (2) for any* $f(t) \in L_2(\mathbb{R}_+;H)$ *has a unique regular solution.*

Proof. We write the boundary-value problem (1), (2) in the form of an operator equation

$$Pu(t) = P_0 u(t) + P_1 u(t) = f(t),$$

where $f(t) \in L_2(\mathbb{R}_+;H)$, $u(t) \in W_{2,T}^4(\mathbb{R}_+;H)$.

It should be stressed that the conditions $C = A^{5/2} T A^{-3/2}$, $Re\ C \geq 0$ ensure the existence of a bounded inverse operator P_0^{-1} acting from the space $L_2(\mathbb{R}_+;H)$ into the space $W_{2,T}^4(\mathbb{R}_+;H)$. After replacing $u(t) = P_0^{-1} v(t)$, where $v(t) \in L_2(\mathbb{R}_+;H)$, we obtain the following equation in the space $L_2(\mathbb{R}_+;H)$:

$$v(t) + P_1 P_0^{-1} v(t) = f(t).$$

In this case, for any $v(t) \in L_2(\mathbb{R}_+;H)$, taking into account inequalities (13), we have:

$$\left\|P_1 P_0^{-1} v\right\|_{L_2(\mathbb{R}_+;H)} = \|P_1 u\|_{L_2(\mathbb{R}_+;H)} \leq$$

$$\leq \sum_{j=1}^{4} \left\| A_j A^{-j} \right\|_{H\to H} \left\| A^j u^{(4-j)} \right\|_{L_2(\mathbb{R}_+;H)} \leq$$

$$\leq \sum_{j=1}^{4} c_j \left\| A_j A^{-j} \right\|_{H\to H} \|P_0 u\|_{L_2(\mathbb{R}_+;H)} = \sum_{j=1}^{4} c_j \left\| A_j A^{-j} \right\|_{H\to H} \|v\|_{L_2(\mathbb{R}_+;H)}.$$

Since by the condition of the theorem

$$\sum_{j=1}^{4} c_j \left\| A_j A^{-j} \right\|_{H\to H} < 1,$$

then the operator $E + P_1 P_0^{-1}$ has an inverse in the space $L_2(\mathbb{R}_+;H)$. Then

$$u(t) = P_0^{-1} \left(E + P_1 P_0^{-1} \right)^{-1} f(t),$$

at that

$$\|u\|_{W_2^4(\mathbb{R}_+;H)} \leq$$

$$\leq \left\| P_0^{-1} \right\|_{L_2(\mathbb{R}_+;H)\to W_2^4(\mathbb{R}_+;H)} \left\| \left(E + P_1 P_0^{-1} \right)^{-1} \right\|_{L_2(\mathbb{R}_+;H)\to L_2(\mathbb{R}_+;H)} \|f\|_{L_2(\mathbb{R}_+;H)} \leq$$

$$\leq const \ \|f\|_{L_2(\mathbb{R}_+;H)}.$$

◀

From Theorem 5, we have

Corollary 2. *Under the conditions of Theorem 5, the operator P implements an isomorphism between the spaces $W_{2,T}^4(\mathbb{R}_+;H)$ and $L_2(\mathbb{R}_+;H)$.*

Remark. Note that in Theorem 5 the condition $Re\ C \geq 0$ allows us to omit the condition $-\frac{1}{\sqrt{2}} \notin \sigma(C)$, where $C = A^{5/2} T A^{-3/2}$.

The case when $Re\ C$ is not a non-negative operator requires a separate study.

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