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THE STUDY OF APPROXIMATE SOLUTION OF ONE CLASS OF HYPERSINGULAR INTEGRAL EQUATIONS OF FIRST KIND

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Abstract. *This work presents the method for calculating the approximate solution to the integral equation of the first kind of the Neumann boundary value problems for the Helmholtz equation in two-dimensional space*

Keywords: Helmholtz equation, Neumann boundary value problems, integral equation of first kind

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1. Introduction and Problem Statement

As is known, in many cases it is impossible to find an exact solution of Neumann boundary value problems for the Helmholtz equation in two-dimensional space. Therefore, there is an interest in the study of approximate solution of these boundary value problems with theoretical justification. One of the methods to solve the Neumann boundary value problem for the Helmholtz equation in two-dimensional space is to reduce it to an integral equation of the first kind. Note that the main advantage of applying the method of integral equations to the exterior boundary value problems is that this method allows reducing the problem for an unbounded domain to the one for a bounded domain of lesser dimension.

Let $D \subset R^2$ be a bounded domain with twice continuously differentiable boundary L, and f be a given continuous function on L. Consider the Neumann boundary value problems for the Helmholtz equation:

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Interior Neumann problem. Find a function u, which is twice continuously differentiable on D, continuous on L, has a normal derivative in the sense of uniform convergence, and satisfies the Helmholtz equation $\Delta u + k^2 u = 0$ in D and the boundary condition $\dfrac{\partial u}{\partial \nu} = f$ on L, where Δ is a Laplace operator, k is a wave number with $Imk \geq 0$, and $\nu(x)$ is an outer unit normal at the point $x \in L$.

Exterior Dirichlet problem. Find a function u, which is twice continuously differentiable on $R^2 \backslash \bar{D}$, continuous on L, has a normal derivative in the sense of uniform convergence, satisfies the Helmholtz equation in $R^2 \backslash \bar{D}$, Sommerfeld radiation condition

$$\left(\frac{x}{|x|}, \;\; grad\, u(x) \right) - ik\, u(x) = o\left(\frac{1}{|x|^{1/2}} \right), \quad x \to \infty,$$

uniformly in all directions $x/|x|$ and the boundary condition $u = f$ on L.

It was shown in [1] that the double-layer potential

$$u\,(x) = \int_L \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, \varphi(y)\, dL_y, \quad x \in R^2 \backslash L,$$

with density $\varphi \in \mathrm{N}(L)$ ($\mathrm{N}(L)$ is a space of all continuous functions φ, whose double-layer potential with density φ has continuous normal derivatives on both sides of the curve L) is a solution of the interior and exterior Neumann boundary value problems for the Helmholtz equation if φ is a solution of the hypersingular integral equation of the first kind

$$T\varphi = 2f, \tag{1}$$

where

$$(T\varphi)(x) = 2\frac{\partial}{\partial \nu(x)} \left(\int_L \frac{\partial \Phi(x,\,y)}{\partial \nu(y)}\, \varphi(y) dL_y \right), \quad x \in L,$$

$\Phi(x,\,y)$ is a fundamental solution of the Helmholtz equation, i.e.

$$\Phi(x,\,y) = \begin{cases} \dfrac{1}{2\pi} \ln \frac{1}{|x-y|} & \text{for } \; k = 0, \\ \\ \dfrac{i}{4} H_0^{(1)}(k\,|x-y|) & \text{for } \; k \neq 0, \end{cases}$$

$H_0^{(1)}$ is a zero degree Hankel function of the first kind defined by the formula $H_0^{(1)}(z) = J_0(z) + i\,N_0(z)$,

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{z}{2} \right)^{2m}$$

is a Bessel function of zero degree,

$$N_0(z) = \frac{2}{\pi} \left(\ln \frac{z}{2} + C \right) J_0(z) + \sum_{m=1}^{\infty} \left(\sum_{l=1}^{m} \frac{1}{l} \right) \frac{(-1)^{m+1}}{(m!)^2} \left(\frac{z}{2} \right)^{2m}$$

is a Neumann function of zero degree, and $C = 0.57721...$ is an Euler's constant.

Note that the integral equations of the first kind do not fit into the Riesz-Fredholm theory. Moreover, the operator T is unbounded in the space $\mathrm{N}(L)$. But, it was shown in [1] that if $Imk > 0$, then, for every right-hand side $f \in C(L)$, the hypersingular integral equation (1) is uniquely solvable in the space $\mathrm{N}(L)$, and the solution of the integral equation (1) has the form

$$\varphi = -2\,S\,\left(I - \tilde{K}\right)^{-1}\left(I + \tilde{K}\right)^{-1} f, \tag{2}$$

where

$$(Sf)(x) = 2\int_L \Phi(x,y)\, f(y)\, dL_y, \quad x \in L,$$

$$\left(\tilde{K}f\right)(x) = 2\int_L \frac{\partial \Phi(x,\,y)}{\partial \nu(x)}\, f(y)\, dL_y, \quad x \in L,$$

I is a unit operator in $C(L)$, and $C(L)$ denotes the space of all continuous functions on L with the norm $\|\varphi\|_\infty = \max\limits_{x\in L}|\varphi(x)|$. Note that in spite of invertibility of the operators $I - \tilde{K}$ and $I + \tilde{K}$, the explicit forms of the inverse operators $\left(I - \tilde{K}\right)^{-1}$ and $\left(I + \tilde{K}\right)^{-1}$ are unknown. Nevertheless, in [6], the method has been given to study the approximate solution to the integral equation of the first kind (1) of the Neumann boundary value problems for the Helmholtz equation in three-dimensional space. As is known, the fundamental solution of the Helmholtz equation in three-dimensional space has the form

$$\Phi_k(x,\,y) = \frac{\exp(ik\,|x-y|)}{4\pi\,|x-y|}, \quad x, y \in R^3, \quad x \neq y,$$

which is strictly different from the fundamental solution of the Helmholtz equation in two-dimensional space. Note that in [10], the quadrature formulas have been constructed for simple-layer and double-layer potentials using the asymptotic formula for the zero degree Hankel functions of the first kind, which does not allow to find the convergence rate of these quadrature formulas. But, in [7], a more practical method was used to construct quadrature formulas for simple-layer and double-layer potentials, and in [8], the quadrature formulas for the normal derivative of simple-layer potential have been constructed and the error estimates for the constructed quadrature formulas have been obtained.

Despite the successes in the field of numerical solution of integral equations of the first kind (see [2]–[5], [12]), the approximate solution of Neumann boundary value problems for the Helmholtz equation in two-dimensional space has not yet been studied by the method of integral equations of the first kind. This work is just about it.

2. The Study of Approximate Solution of the Equation (1)

Assume that the curve L is defined by the parametric equation $x(t) = (x_1(t), x_2(t))$, $t \in [a,b]$. Let's divide the interval $[a,b]$ into $n > 2M_0(b-a)/d$ equal parts: $t_p = a +$

$\frac{(b-a)\,p}{n}$, $p=\overline{0,n}$, where

$$M_0=\max_{t\in[a,b]}\sqrt{\left(x_1'\left(t\right)\right)^2+\left(x_2'\left(t\right)\right)^2}<+\infty$$

(see [11]) and d is a standard radius (see [13]). As control points, we consider $x\left(\tau_p\right)$, $p=\overline{1,n}$, where $\tau_p=a+\frac{(b-a)\,(2p-1)}{2n}$. Then the curve L is divided into elementary parts: $L=\bigcup\limits_{p=1}^{n}L_p$, where $L_p=\{\,x\left(t\right):\ \ t_{p-1}\le t\le t_p\}$.

It is known (see [9]) that

(1) $\forall p\in\{1,2,...,n\}$: $r_p(n)\sim R_p(n)$, where

$$r_p\left(n\right)=\min\left\{\,\left|x\left(\tau_p\right)-x\left(t_{p-1}\right)\right|,\ \left|x\left(t_p\right)-x\left(\tau_p\right)\right|\,\right\},$$

$$R_p\left(n\right)=\max\left\{\,\left|x\left(\tau_p\right)-x\left(t_{p-1}\right)\right|\,,\,\left|x\left(t_p\right)-x\left(\tau_p\right)\right|\,\right\},$$

and $a\left(n\right)\sim b\left(n\right)$ means $C_1\le\dfrac{a\left(n\right)}{b\left(n\right)}\le C_2$, where C_1 and C_2 are positive constants independent of n.

(2) $\forall p\in\{1,2,...,n\}:R_p\left(n\right)\le d/2$;

(3) $\forall p,j\in\{1,2,...,n\}:r_j\left(n\right)\sim r_p\left(n\right)$;

(4) $r\left(n\right)\sim R\left(n\right)\sim\frac{1}{n}$, where $R\left(n\right)=\max\limits_{p=\overline{1,\,n}}R_p\left(n\right)$, $r\left(n\right)=\min\limits_{p=\overline{1,\,n}}r_p\left(n\right)$.

Let

$$\Phi_n(x,\,y)=\frac{i}{4}H_{0,n}^{(1)}\left(k\left|x-y\right|\right),\quad x,y\in L,\quad x\ne y,$$

where

$$H_{0,n}^{(1)}\left(z\right)=J_{0,n}\left(z\right)+i\,N_{0,n}\left(z\right),J_{0,n}\left(z\right)=\sum_{m=0}^{n}\frac{\left(-1\right)^m}{\left(m!\right)^2}\left(\frac{z}{2}\right)^{2m}$$

and

$$N_{0,n}\left(z\right)=\frac{2}{\pi}\left(\ln\frac{z}{2}+C\right)\,J_{0,n}\left(z\right)+\sum_{m=1}^{n}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left(-1\right)^{m+1}}{\left(m!\right)^2}\left(\frac{z}{2}\right)^{2m}.$$

It is not difficult to show that

$$\frac{\partial\Phi_n\left(x,y\right)}{\partial\nu\left(x\right)}=\frac{i}{4}\left(\frac{\partial J_{0,n}\left(k\left|x-y\right|\right)}{\partial\nu\left(x\right)}+i\,\frac{\partial N_{0,n}\left(k\left|x-y\right|\right)}{\partial\nu\left(x\right)}\right),$$

where

$$\frac{\partial J_{0,n}\left(k\left|x-y\right|\right)}{\partial\nu\left(x\right)}=\left(x-y,\,\nu\left(x\right)\right)\sum_{m=1}^{n}\frac{\left(-1\right)^m k^{2m}\left|x-y\right|^{2m-2}}{2^{2m-1}\left(m-1\right)\,!\,m\,!}$$

and

$$\frac{\partial N_{0,n}\left(k\left|x-y\right|\right)}{\partial\nu\left(x\right)}=\frac{2}{\pi}\left(\ln\frac{k\,\left|x-y\right|}{2}+C\right)\frac{\partial J_{0,n}\left(k\left|x-y\right|\right)}{\partial\nu\left(x\right)}+$$

$$+\frac{2\,\left(x-y,\,\nu\left(x\right)\right)}{\pi\,\left|x-y\right|^2}J_{0,n}\left(k\left|x-y\right|\right)+$$

$$+\left(x-y,\,\nu\left(x\right)\right)\sum_{m=1}^{n}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left(-1\right)^{m+1}k^{2m}\left|x-y\right|^{2m-2}}{2^{2m-1}\left(m-1\right)!\,m!}.$$

Consider the matrices $S^n=\left(s_{pj}\right)_{p,j=1}^{n}$ and $\tilde{K}^n=\left(\tilde{k}_{pj}\right)_{p,j=1}^{n}$ with the elements

$$s_{pj}=\frac{2\,\left|sgn\left(p-j\right)\right|\,\left(b-a\right)}{n}\mathit{\Phi}_n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)\,\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}$$

and

$$\tilde{k}_{pj}=\frac{2\,\left|sgn\left(p-j\right)\right|\,\left(b-a\right)}{n}\frac{\partial\mathit{\Phi}_n\left(x\left(\tau_p\right),\,x\left(\tau_j\right)\right)}{\partial\nu\left(x\left(\tau_p\right)\right)}\,\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2},$$

respectively. It was proved in [7], [8] that if $f\in C\left(L\right)$, then the expressions

$$\left(S_nf\right)\left(x\left(\tau_p\right)\right)=\sum_{j=1}^{n}s_{pj}f\left(x\left(\tau_j\right)\right)$$

and

$$\left(\tilde{K}_nf\right)\left(x\left(\tau_p\right)\right)=\sum_{j=1}^{n}\tilde{k}_{pj}f\left(x\left(\tau_j\right)\right)$$

are the quadrature formulas for the integrals $\left(Sf\right)\left(x\right)$ and $\left(\tilde{K}f\right)\left(x\right)$ at the control points $x\left(\tau_p\right),\,p=\overline{1,n}$, respectively, with

$$\max_{p=\overline{1,\,n}}\left|\left(Sf\right)\left(x\left(\tau_p\right)\right)-\left(S_nf\right)\left(x\left(\tau_p\right)\right)\right|\le M^1\left(\omega\left(f,\,1/n\right)+\left\|f\right\|_\infty\frac{\ln n}{n}\right)$$

and

$$\max_{p=\overline{1,\,n}}\left|\left(\tilde{K}f\right)\left(x\left(\tau_p\right)\right)-\left(\tilde{K}_nf\right)\left(x\left(\tau_p\right)\right)\right|\le M\left(\omega\left(f,\,1/n\right)+\left\|f\right\|_\infty\frac{\ln n}{n}\right),$$

where $\omega\left(f,\,\delta\right)$ is the modulus of continuity of the function f, i.e.

$$\omega\left(f,\,\delta\right)=\max_{\substack{|x-y|\le\delta\\ x,\,y\in L}}\left|f\left(x\right)-f\left(y\right)\right|,\delta>0.$$

It is known (see [1]) that if $Im\,k>0$, then for every right-hand side $g\in C\left(L\right)$ the integral equations

$$\varphi\pm\tilde{K}\varphi=g \tag{3}$$

are uniquely solvable in the space $C\left(L\right)$. Then, proceeding as in [6], it is not difficult to prove the following lemmas.

[1] Hereinafter denotes a positive constant which can be different in different inequalities.

Lemma 1. *If $Imk > 0$, then there exists an inverse matrix $\left(I^n + \tilde{K}^n\right)^{-1}$ such that*

$$M_1 = \sup_n \left\| \left(I^n + \tilde{K}^n\right)^{-1} \right\| < +\infty$$

and

$$\max_{l=\overline{1,n}} \left| \left(\left(I + \tilde{K}\right)^{-1} g \right) (x(\tau_l)) - \sum_{j=1}^{n} \tilde{k}_{lj}^{+} g(x(\tau_l)) \right| \leq M \left(\omega\,(g, 1/n) + \|g\|_\infty \frac{\ln n}{n} \right),$$

where I^n is a unit operator in C^n, and $\tilde{k}_{lj}^{+}$ is an element of the matrix $\left(I^n + \tilde{K}^n\right)^{-1}$ in the $l-$th row and $j-$th column.

Lemma 2. *If $Imk > 0$, then there exists an inverse matrix $\left(I^n - \tilde{K}^n\right)^{-1}$ such that*

$$M_2 = \sup_n \left\| \left(I^n - \tilde{K}^n\right)^{-1} \right\| < +\infty$$

and

$$\max_{l=\overline{1,n}} \left| \left(\left(I - \tilde{K}\right)^{-1} g \right) (x(\tau_l)) - \sum_{j=1}^{n} \tilde{k}_{lj}^{-} g(x(\tau_l)) \right| \leq M \left(\omega\,(g, 1/n) + \|g\|_\infty \frac{\ln n}{n} \right),$$

where $\tilde{k}_{lj}^{-}$ is an element of the matrix $\left(I^n - \tilde{K}^n\right)^{-1}$ in the $l-$th row and $j-$th column.

Now, using the formula (2), let's consider the approximate solution of the equation (1).

Theorem. *Let $Imk > 0$ and the function $f(x)$ be continuous on L. Then the expression*

$$\varphi_n(x(\tau_l)) = -2 \sum_{j=1}^{n} s_{lj} \left(\sum_{p=1}^{n} \tilde{k}_{jp}^{-} \left(\sum_{m=1}^{n} \tilde{k}_{pm}^{+} f(x(\tau_m)) \right) \right)$$

is an approximate value of the solution $\varphi(x)$ of the equation (1) at the points $x(\tau_l)$, $l = \overline{1,n}$, with

$$\max_{l=\overline{1,n}} |\varphi(x(\tau_l)) - \varphi_n(x(\tau_l))| \leq M \left(\omega(f, 1/n) + \|f\|_\infty \frac{\ln n}{n} \right).$$

Proof. Taking into account the error estimates for the quadrature formulas for the integral $(Sf)(x)$, $x \in L$, at the control points $x(\tau_l)$, $l = \overline{1, n}$, and Lemmas 1 and 2, we obtain

$$|\varphi(x(\tau_l)) - \varphi_n(x(\tau_l))| \le 2 \left| \left(S \left(I - \tilde{K}\right)^{-1} \left(I + \tilde{K}\right)^{-1} f \right)(x(\tau_l)) - \right.$$

$$\left. - \sum_{j=1}^{n} s_{lj} \left(\left(I - \tilde{K}\right)^{-1} \left(I + \tilde{K}\right)^{-1} f \right)(x(\tau_j)) \right| +$$

$$+2 \left| \sum_{j=1}^{n} s_{lj} \left[\left(\left(I - \tilde{K}\right)^{-1} \left(I + \tilde{K}\right)^{-1} f \right)(x(\tau_j)) - \sum_{p=1}^{n} \tilde{k}_{jp}^{-} \left(\left(I + \tilde{K}\right)^{-1} f \right)(x(\tau_p)) \right] \right| +$$

$$+2 \left| \sum_{j=1}^{n} s_{lj} \left(\sum_{p=1}^{n} \tilde{k}_{jp}^{-} \left[\left(\left(I + \tilde{K}\right)^{-1} f \right)(x(\tau_p)) - \sum_{m=1}^{n} \tilde{k}_{pm}^{+} f(x(\tau_m)) \right] \right) \right| \le$$

$$\le M \left[\left\| \left(I - \tilde{K}\right)^{-1} \left(I + \tilde{K}\right)^{-1} f \right\|_{\infty} \frac{\ln n}{n} + \omega \left(\left(I - \tilde{K}\right)^{-1} \left(I + \tilde{K}\right)^{-1} f,\, 1/n \right) \right] +$$

$$+M \left[\left\| \left(I + \tilde{K}\right)^{-1} f \right\|_{\infty} \frac{\ln n}{n} + \omega \left(\left(I + \tilde{K}\right)^{-1} f,\, 1/n \right) \right] \sum_{j=1}^{n} |s_{lj}| +$$

$$+M \left[\| f \|_{\infty} \frac{\ln n}{n} + \omega(f,\, 1/n) \right] \sum_{j=1}^{n} \left(|s_{lj}| \sum_{p=1}^{n} \left| \tilde{k}_{jp}^{-} \right| \right). \tag{4}$$

As the integral $\left(\tilde{K}\varphi\right)(x)$, $x \in L$, is weakly singular (see [1]), it is not difficult to show that

$$\omega\left(\tilde{K}\varphi, \delta\right) \le M \, \|\varphi\|_{\infty} \, \delta \, |\ln \delta|, \delta > 0.$$

Then, due to the unique solvability of the equation (3) in the space $C(L)$, we have

$$\omega\left(\left(I \pm \tilde{K}\right)^{-1} g,\, 1/n \right) = \omega(\varphi_*,\, 1/n) =$$

$$= \omega\left(g - \tilde{K}\varphi_*,\, 1/n\right) \le \omega(g,\, 1/n) + \omega\left(\tilde{K}\varphi_*,\, 1/n\right) \le$$

$$\le \omega(g,\, 1/n) + M \, \|\varphi_*\|_{\infty} \frac{\ln n}{n} = \omega(g,\, 1/n) + M \left\| \left(I + \tilde{K}\right)^{-1} g \right\|_{\infty} \frac{\ln n}{n} \le$$

$$\le \omega(g,\, 1/n) + M \left\| \left(I + \tilde{K}\right)^{-1} \right\| \|g\|_{\infty} \frac{\ln n}{n},$$

where φ_* is a solution of the equation (3). Then

$$\omega\left(\left(I - \tilde{K}\right)^{-1} \left(I + \tilde{K}\right)^{-1} f,\, 1/n \right) \le$$

$$\leq \omega \left(\left(I + \tilde{K} \right)^{-1} f,\, 1/n \right) + M \left\| \left(I + \tilde{K} \right)^{-1} f \right\|_{\infty} \frac{\ln n}{n} \leq$$

$$\leq \omega \ (f,\, 1/n) + M \ \| f \|_{\infty} \frac{\ln n}{n} + M \left\| \left(I + \tilde{K} \right)^{-1} f \right\|_{\infty} \frac{\ln n}{n}.$$

Besides, proceeding as in [7], we can show that the expression $\sum\limits_{j=1}^{n} |s_{l\,j}|$ is a quadrature formula for the integral

$$2 \int_L |\Phi(x, y)| \ dL_y$$

at the points $x(\tau_l)$, $l = \overline{1, N}$, with

$$\max_{l=\overline{1,\,n}} \left| 2 \int_L |\Phi(x(\tau_l), y)| \ dL_y - \sum_{j=1}^{n} |s_{l\,j}| \right| \leq M \frac{\ln n}{n}.$$

Consequently,

$$\max_{l=\overline{1\,,\,n}} \sum_{j=1}^{n} |s_{l\,j}| \leq 2 \max_{x \in L} \int_L |\Phi(x, y)| \ dL_y + M \frac{\ln n}{n} \leq M.$$

Besides, by Lemmas 1 and 2 we obtain

$$\max_{j=\overline{1\,,\,n}} \sum_{p=1}^{n} \left| \tilde{k}^{+}_{j\,p} \right| \leq M_1, \max_{j=\overline{1\,,\,n}} \sum_{p=1}^{n} \left| \tilde{k}^{-}_{j\,p} \right| \leq M_2.$$

As a result, considering the above obtained inequalities in (4), we get the validity of the theorem. ◄

Corollary. *Let $Im\, k > 0$,*

$$\varphi_n(x(\tau_l)) = -2 \sum_{j=1}^{n} s_{l\,j} \left(\sum_{p=1}^{n} \tilde{k}^{-}_{j\,p} \left(\sum_{m=1}^{n} \tilde{k}^{+}_{p\,m} f(x(\tau_m)) \right) \right)$$

and $x_0 \in D$ ($x_0 \in R^2 \backslash \bar{D}$). Then the sequence

$$u_n(x_0) = \frac{b-a}{n} \sum_{l=1}^{n} \frac{\partial \Phi(x_0, x(\tau_l))}{\partial \nu(x(\tau_l))} \varphi_n(x(\tau_l)) \sqrt{(x_1'(\tau_j))^2 + (x_2'(\tau_j))^2}$$

converges to the value $u(x_0)$ of the solution $u(x)$ to the interior (exterior) Neumann boundary value problem for the Helmholtz equation at the point x_0, with

$$|u_n(x_0) - u(x_0)| \leq M \left[\| f \|_{\infty} \frac{\ln n}{n} + \omega \ (f,\, 1/n) \right].$$

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