Baku Mathematical Journal
2023, Vol. 2, No 1, PP. 69-76
https://doi.org/10.32010/j.bmj.2023.06

# ON A FINDING THE GUARANTEED WITH RESPECT TO THE COST FUNCTION SOLUTION IN THE INTEGER PROGRAMMING PROBLEM

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Received: 29.12.2022 / Revised: 07.02.2023 / Accepted: 27.02.2023

**Abstract.** *In the paper the definitions are given of the guaranteed with respect to the cost function solution and guaranteed suboptimal solution in the integer programming problem. A method for finding a solution is developed that guarantees that the value of the functional is not less than a given number. This method is based on the variation of the coefficients of the functional in some intervals. The essence of this method is that the total income generated by raising and lowering market prices is greater. The calculation experiments based on the proposed method are given.*

**Keywords**: integer programming problem, guaranteed with respect to the cost function solution, guaranteed suboptimal solution, dichotomies principle, calculation experiment

**Mathematics Subject Classification (2020):** 90C10, 91B32

## 1. Introduction

Consider the following known integer programming problem

$$\sum_{j=1}^{n} c_j x_j \to \max, \tag{1}$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \left(i = \overline{1, m}\right), \tag{2}$$

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$$0 \leq x_j \leq d_j \quad \left(j = \overline{1,n}\right) \quad \text{are integers .} \tag{3}$$

Here $c_j > 0$, $a_{ij} \geq 0$, $b_i > 0$, $d_j > 0$ $(i = \overline{1,m};\ j = \overline{1,n})$ are given integers. First we give an economical interpretation for problem (1)-(3). Assume any enterprise should produce $n$ type of product expressed by numbers. Let for the producing the one unit of the $j$-th $\left(j = \overline{1,n}\right)$ product $a_{ij}$ $(i = \overline{1,m};\ j = \overline{1,n})$ volume from the given resources $b_i$ $\left(i = \overline{1,m}\right)$ is used. From the selling of one unit of this product the benefit $c_j$ $\left(j = \overline{1,n}\right)$ is taken. Also we assume that each product of the $j$-th type will be produced no more than $d_j$ $\left(j = \overline{1,n}\right)$ number. Then we naturally come to the problem: how many from each product should be produced under the condition that the used volume of the resources would not exceed the given ones and the benefit be maximal.

Thus we arrive to the mathematical model (1)-(3) of the above described problem assuming that the numbers $x_j$ $(0 \leq x_j \leq d_j)$ $\left(j = \overline{1,n}\right)$ take integer values. Here $x_j$ $\left(j = \overline{1,n}\right)$ represent the appropriate number of products to be produced.

In the present paper we assume that some optimal or suboptimal solution of problem (1)-(3) is already found by some method and the corresponding value of functional (1) is calculated. It is required to find a solution that guarantees that this value (the volume of the benefit) will be greater than the given number. Finding such a solution due to the change of resources $b_i$ $\left(i = \overline{1,m}\right)$ from the right hand side of system (2) was performed in works [7]–[9], etc.

Note that in monograph [9] the methods have been developed for the finding of the guaranteed solutions using the variation of the right hand sides for the different classes of the integer programming problems.

In this work we find the guaranteed solution not using the right hand side of system (2), but the variation in some intervals the coefficients of function (1). The similar problem for the Bull programming problem with single or more restrictions were solved in [9], [10].

It should be noted that since problem (1)-(3) is from NP–integer class there are not methods working in real time for its solution in multidimensional case [1], [2], [4], etc.

In [3], [5] mentioned in the literature, the problems of investment and stability of the solution that come to the model (1)-(3) were considered. In [6], [13], problems of linear programming with data intervals were investigated. However, the issue that we are considering is completely different from these cases.

## 2. Problem Statement

Let the optimal $X^* = (x_1^*, x_2^*, \ldots, x_n^*)$ and suboptimal $X^s = (x_1^s, x_2^s, \ldots, x_n^s)$ solution of problem (1)-(3) is found and corresponding values

$$f^* = \sum_{j=1}^{n} c_j x_j^* \quad \text{or} \quad f^s = \sum_{j=1}^{n} c_j x_j^s$$

are determined. Let it is required to find solutions that guarantees to increase the values $f^*$ and $f^s$ by at least $\triangle^*$ and $\triangle^s$, correspondingly. Here one can take

$$\triangle^* = \left[f^* \frac{p}{100}\right] \quad \text{or} \quad \triangle^s = \left[f^s \frac{p}{100}\right]$$

and $p$ is the increasing percentage of the optimal or suboptimal solution. Here we not changing the numbers $a_{ij}$ and $b_i$ $\left(i=\overline{1,m};\ \ j=\overline{1,n}\right)$ should admit such a minimal the variation of the coefficients $c_j$ $\left(j=\overline{1,n}\right)$ in the intervals $[\alpha_j,\beta_j]$ $\left(j=\overline{1,n}\right)$ to provide that the value of function (1) would not less than the numbers $f^*+\triangle^*$ or $f^s+\triangle^s$ correspondingly. Here naturally the relations $\alpha_j\le 0$ and $\beta_j\ge 0$ $\left(j=\overline{1,n}\right)$ must be hold. Because in the final guaranteed solution the coefficients of the function (1) should be varied. Note that the problem of increasing the benefit by regulation of the market costs $c_j$ $\left(j=\overline{1,n}\right)$ not changing the financial resources $b_i$ $\left(i=\overline{1,m}\right)$ is an actual problem.

Thus we arrive at the following mathematical model of the above described problem

$$\sum_{j=1}^{n}(c_j+\delta_j)x_j \ \to \max, \tag{4}$$

$$\sum_{j=1}^{n}a_{ij}x_j \ \ \le b_i \ \left(i=\overline{1,m}\right)\ , \tag{5}$$

$$0\le x_j\le d_j \quad \left(j=\ \overline{1,n}\ \right) \text{ are integers,} \tag{6}$$

$$\alpha_j\le\delta_j\le\beta_j \quad \left(j=\ \overline{1,n}\ \right) \text{ are integers.} \tag{7}$$

Here $x_j$, $\delta_j$ $\left(j=\overline{1,n}\right)$ are nonnegative integer variables; $c_j>0$, $a_{ij}\ge 0$, $d_j>0$, $b_i>0$, $\alpha_j\le 0$, $\beta_j\ge 0$ $(i=\overline{1,m};\ j=\overline{1,n})$ are given integers. Note that if for any number $j_*$ $(j_*\in\ \{1,\ 2,\ldots,\ n\}\ )$ we have $\delta_{j_*}\in[\alpha_{j_*},0)$ then the value $c_{j_*}$ must decrease by $\delta_{j_*}$ and if $\delta_{j_*}\in(0,\beta_{j_*}]$ the the value $c_{j_*}$ must increase by $\delta_{j_*}$. If the case $\delta_{j_*}=0$ takes place then the value of $c_{j_*}$ will not be changed.

As we see, the presence of products $\delta_j{\cdot}x_j$ in the function (4) turns this problem into a non-linear integer programming problem. That is why the solution of problem (4)-(7) becomes complicated.

Considering all above it is expedient to rewrite the mathematical model of the considered problem in the form

$$\delta_j \ \ \to min, \quad \left(j=\overline{1,n}\right), \tag{8}$$

$$\sum_{j=1}^{n}(c_j+\delta_j)x_j \ \ \ge f^*+\ \triangle^*, \tag{9}$$

$$\sum_{j=1}^{n}a_{\mathrm{ij}}x_j \ \ \le b_i,\ \left(i=\overline{1,m}\right), \tag{10}$$

$$0\le x_j\le d_j,\ \left(j=\overline{1,n}\right) \text{ are integers,} \tag{11}$$

$$\alpha_j\le\delta_j\le\beta_j, \quad \left(j=\ \overline{1,n}\ \right) \text{ are integers.} \tag{12}$$

# 3. Mathematical Justification of the Method

First we give some definitions based on works [1], [4], [13].

**Definition 1.** *The vector* $X = (x_1, x_2, \ldots, x_n)$ *is called an admissible solution to problem (8)-(12) if the conditions (9)-(12) will be satisfied for any value of the parameters* $\delta_j$ $(j = \overline{1,n})$ *from the interval* $[\alpha_j, \beta_j]$ $\left(j = \overline{1,n}\right)$.

**Definition 2.** *The* $n$ *dimensional admissible solution* $X = (x_1, x_2, \ldots, x_n)$ *is called guaranteed with respect to functional solution for problem (8)-(12) if the parameters* $\delta_j$ $(j = \overline{1,n})$ *take minimal value.*

Note that since (8)-(12) is muticriteria ad nonlinear integer programming problem it belongs to the NP-integer class. Therefore when the number of the variables is large finding of the optimal solution to this problem requires non real machine time. But in practice we meet necesaty to find the solution of such problems and also suboptimal solution of mathematical problem (8)-(12). For this purpose we developed a numerical method to find the approximate solution of problem (8)-(12). Found by this way solution we call guaranteed with respect to the functional suboptimal solution.

Here we purpose an algorithm for solution of the problem obtained from (8)-(12) by replacing $f^*$ and $\triangle^*$ by $f^s$ and $\triangle^s$ respectively. Our goal is to find such minimal values of the parameters $\delta_j,\ (j = \overline{1,n})$ within condition (12) that provide fulfilment of conditions (9)-(11). The calculation procedure below was carried out to reach this goal.

First $c_j$ $(j = \overline{1,n})$ we remember the numbers $c_j^{'} := c_j$ and $\delta_j := \alpha_j$ $(j = \overline{1,n})$. Then setting $c_j := c_j^{'} + \delta_j$ $(j = \overline{1,n})$ we find by certain method some suboptimal solution $X^{s_0} = (\ x_1^{s_0}, x_2^{s_0}, \ldots, x_n^{s_0})$ of problem (1)-(3) (one can use for instant the methods given in [2], [6], [12]) and calculate

$$f^{s_0} = \sum_{j=1}^{n} c_j\ x_j^{s_0}.$$

It is clear that since the numbers $\delta_j$ $(j = \overline{1,n})$ are negative we have $f^{s_0} \leq\ f^s + \triangle^s$.

To formulate the next intermediate problem (1)-(3) we set $\alpha_j^1 :=\ \ \alpha_j,\ \beta_j^1 := \beta_j$ $\left(j = \overline{1,n}\right)$ and using dichotomies principle $\delta_j^1 := \left[\frac{\alpha_j^1 + \beta_j^1}{2}\right]$ $\left(j = \overline{1,n}\right)$ determine the coefficients $c_j := c_j^{'} + \delta_j^1$ $\left(j = \overline{1,n}\right)$. Then we find the solution $X^{s_1} = (x_1^{s_1}, x_2^{s_1}, \ldots, x_n^{s_1})$ of the intermediate problem and corresponding value of function (1)

$$f^{s_1} = \sum_{j=1}^{n} c_j\ x_j^{s_1}.$$

It is clear that it must be $f^{s_1} <\ f^s + \triangle^s$ or $f^{s_1} \geq\ f^s + \triangle^s$ .

If $f^{s_1}\ <\ f^s + \triangle^s$ then setting $\alpha_j^2 := \delta_j^1,\ \beta_j^2 := \beta_j^1$ $\left(j = \overline{1,n}\right)$ by the dichotomies principle we take $\delta_j^2 := \left[\frac{\alpha_j^2 + \beta_j^2}{2}\right]$ $\left(j = \overline{1,n}\right)$. Then accepting $c_j := c_j^{'} + \delta_j^2$ $\left(j = \overline{1,n}\right)$ we obtain suboptimal solution $X^{s_2} = (x_1^{s_2}, x_2^{s_2}, \ldots, x_n^{s_2})$ of problem (1)-(3) and determine the corresponding value

$$f^{s_2} = \sum_{j=1}^{n} c_j \ x_j^{s_2} \tag{8}$$

of functional (1).

If we have $f^{s_1} \geq \ f^s + \triangle^s$ then the solution $X^{s_1} = (x_1^{s_1}, x_2^{s_1}, \ldots, x_n^{s_1})$ may be a suboptimal solution. Therefore in order to remember it we set $X^Z := X^{S_1}$ and $\overline{\delta_j} = \delta_j^1 \left(j = \overline{1,n}\right)$ and taking $\alpha_{\text{j}}^2 := \alpha_j^1, \beta_j^2 := \delta_j^1$ by the dichotomies principle $\delta_j^2 := \left[\frac{\alpha_j^2 + \beta_j^2}{2}\right] \ \left(j = \overline{1,n}\right)$ and define $c_j := c_j^{'} + \delta_j^2 \ \left(j = \overline{1,n}\right)$. Since our goal is to minimize the numbers $\delta_j \ \left(j = \overline{1,n}\right)$, we find the solution $X^{s_2} = (x_1^{s_2}, x_2^{s_2}, \ldots, x_n^{s_2})$ of the constructed intermediate problem (1)-(3) and corresponding value

$$f^{s_2} = \sum_{j=1}^{n} c_j \ x_j^{s_2} \ . \tag{14}$$

Note that since only one of the cases $f^{s_1} < \ f^s + \triangle^s$ and $f^{s_1} \geq \ f^s + \triangle^s$ can take place the values $f^{s_2}$ in expressions (13) and (14) will be different. In the case $f^{s_1} < \ f^s + \triangle^s$ one should use the number $f^{s_2}$ from formula (13) otherwise the number $f^{s_2}$ from formula will be used. The process will be finished when for some $k - th$ step the conditions $\left|\beta_j^k - \alpha_j^k\right| \leq 1 \ \left(j = \overline{1,n}\right)$ would be satisfied, or application of the dichotomies principle will not give any new number.

Here we oriif the following

**Theorem.** *In order to* $X^{s_k} = (x_1^{s_k}, x_2^{s_k}, \ldots, x_n^{s_k})$ *found in some* $k - th$ *step would be a guaranteed with respect to the functional solution of problem (1)-(3) it is necessary and sufficient the fulfilment of the relation* $\left|\beta_j^k - \alpha_j^k\right| \leq 1 \quad \left(j = \overline{1,n}\right)$ *for any numbers* $j \ \left(j = \overline{1,n}\right)$.

*Proof. Sufficiency.* Suppose that the relations $\left|\beta_j^k - \alpha_j^k\right| \leq 1 \quad \left(j = \overline{1,n}\right)$ are valid in some $k - th$ step foe the arbitrary numbers $j \ \left(j = \overline{1,n}\right)$. We show that the solution $X^{s_k} = (x_1^{s_k}, x_2^{s_k}, \ldots, x_n^{s_k})$ is a guaranteed with respect to the functional solution of problem (1)–(3).

By $\theta_j^k$ parameters we denote the midpoints of the intervals $\left[\alpha_j^k, \beta_j^k\right] \quad \left(j = \overline{1,n}\right)$

$$\theta_j^k = \left[\frac{\alpha_j^k + \beta_j^k}{2}\right] \ . \tag{15}$$

Here $[z]$ stands for the integer part of the number z. Then $-1 \leq \beta_j^k - \alpha_j^k \leq 1$ should be valid. If to consider the integrity condition of the numbers $\alpha_j^k$ and $\beta_j^k \ \left(j = \overline{1,n}\right)$ then we get the following three variants:

I. $\beta_j^k - \alpha_j^k = -1$;

II. $\beta_j^k - \alpha_j^k = 0$;

III. $\beta_j^k - \alpha_j^k = 1$.

For case I. we have $\beta_j^k = \alpha_j^k - 1$. Since $\theta_j^k \in \left[\alpha_j^k, \beta_j^k\right]$ $\left(j = \overline{1,n}\right)$, considering this in relation (15) we get

$$\theta_j^k = \left[\frac{\beta_j^k + \alpha_j^k}{2}\right] = \left[\frac{\alpha_j^k - 1 \quad + \alpha_j^k}{2}\right] = \left[\alpha_j^k - \frac{1}{2}\right] = \left[\alpha_j^k\right].$$

For the case II. We have $\beta_j^k = \alpha_j^k$ . Considering this in (15) we have $\theta_j^k = \left[\frac{\beta_j^k + \alpha_j^k}{2}\right] = \left[\frac{\alpha_j^k + \alpha_j^k}{2}\right] = \left[\alpha_j^k\right]$.

In the case III. Is valid $\beta_j^k = \alpha_j^k + 1$ . Since $\theta_j^k \in \left[\alpha_j^k, \beta_j^k\right]$ $\left(j = \overline{1,n}\right)$ for this case, considering this in (13) we get $\theta_j^k = \left[\frac{\beta_j^k + \alpha_j^k}{2}\right] = \left[\frac{\alpha_j^k + 1 + \alpha_j^k}{2}\right] = \left[\alpha_j^k + \frac{1}{2}\right] = \left[\alpha_j^k\right]$.

If in the cases I, II, III to replace the numbers $\alpha_j^k$ $(j = \overline{1,n})$ by the numbers $\beta_j^k$ $(j = \overline{1,n})$ and then consider this in (15) we obtain $\theta_j^k = \left[\beta_j^k\right]$.

Thus if the conditions $\left|\beta_j^k - \alpha_j^k\right| \le 1$ are satisfied for the numbers $j$ $\left(j = \overline{1,n}\right)$, then the number $\theta_j^k$ coincide with the number $\alpha_j^k$ (midpoint of the interval $\left[\alpha_j^k, \beta_j^k\right]$) or with the number $\beta_j^k$. So the new solution coincides with the last one. It means that we cannot find new different solution.

*Neccesarity.* By assuming the opposite, the necessity can be proved easily. ◀

The work was reported in the international Conference COIA-2018 [11].

# 4. Results of the Calculational Experiments

To find out the quality of the method we proposed above, some computational experiments were carried out. The coefficients of the solved problems are random integers that satisfy the following conditions:

$$0 < c_j \le 999, \quad 0 < a_j \le 99, \; b = \left[0.4 \sum_{j=1}^{n} a_j d_j\right], \; 0 \le d_j \le 10 \; \left(j = \overline{1,n}\right).$$

The results of the experiments are presented in the table below. In the problems being solved, the suboptimal solution was found by the method given in [2], [9]. Here the number p represents the marked percentage increase in the number $f^{s_0}$. $p = 20\%$ is accepted in the calculation.

The main labels in the table are given in the text above the article. However, the values $\theta\left(c_j\right)$ and $\theta(a_j)$ indicate the average percentage increase in the functional and the average percentage decrease in the coefficients of the constraint condition, respectively. So :

$$\theta(f) = \left(f^{zs} - f^{s_0}\right) / \; f^{s_0} \cdot 100\%, \quad \theta\left(c_j\right) = \sum_{j=1}^{n} \left(c_j^{'} - c_j\right) / \sum_{j=1}^{n} c_j \cdot 100\%.$$

| | $n = 100$ | $n = 200$ | $n = 500$ | $n = 1000$ |
|---|---|---|---|---|
| $f^{s_0} = \sum_{j=1}^{n} c_j x_j^{s_0}$ | 65369 | 163651 | 365798 | 698459 |
| $\Delta^s = [f^{s_0} \frac{p}{100}]$ | 13073 | 32730 | 73159 | 139691 |
| $f^{s_0} + \Delta^s$ | 78442 | 196381 | 438957 | 838150 |
| $f^{zs}$ | 78631 | 197183 | 441445 | 843948 |
| $f^{zs} - f^{s_0}$ | 13262 | 33532 | 75647 | 145489 |
| $\theta(f)$ | 20.29% | 20.49% | 20.68% | 20.83% |
| $\sum_{j=1}^{n} c_j$ | 25142 | 52798 | 145823 | 304218 |
| $\sum_{j=1}^{n} c_j^{'}$ | 30573 | 64317 | 177904 | 372364 |
| $\sum_{j=1}^{n} (c_j^{'} - c_j)$ | 5431 | 11519 | 32081 | 68146 |
| $\theta(c_j)$ | 21.60% | 21.82% | 21.99% | 22.40% |

**Table 1. Result of the experiments**

# 5. Conclusions

We can draw the following conclusions based on the data in the table:

- based on the guaranteed suboptimal solution, if we set the functional increase from the initial value to 20%, the total increase obtained varied between 20.29%-20.83%;
- the average percentage of increase of functional coefficients was 21.60%-22.40%;
- with the increase in the number of unknowns in all solving issues, the growth percentages of the guaranteed price of the functional have increased, and the average growth percentages of the functional coefficients have also increased;
- these results once again show that the method proposed in the article reflects the reality, and significant results can be obtained by applying this method to practical issues.

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