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ON SOME PROPERTIES OF FUNCTIONS OF THE GRAND HARDY CLASSES

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Abstract. *Grand Hardy class $H^+_{p)}$, $p > 1$, is defined and some properties of functions belonging to this class are studied in this work. Namely, the analogs of the Riesz and Smirnov theorems as well as the Cauchy's formula for representation of function are proved. Necessary and sufficient condition for the validity of Riesz theorem in grand Hardy spaces $H^+_{p)}$, $p > 1$, is found. Subspace $GH^+_{p)}$ of the grand Hardy space $H^+_{p)}$ generated by this condition is defined.*

Keywords: grand Lebesgue space, grand Hardy class, Riesz theorem, Cauchy's formula

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1. Introduction

In the context of applications to various branches of mathematics, for example, such as theory of partial differential equations, theory of approximations, harmonic analysis, etc., there arose great interest in nonclassical function spaces. As examples of such spaces, we can mention Lebesgue space with variable summability index, Morrey space, grand Lebesgue space, etc. A lot of articles, reviews and monographs have been dedicated to these spaces ([1], [9], [10], [13], [14], [18]–[20], [22]-[24], [28]). Along with this, of course, one has to study approximation matters in suchlike spaces. Approximation matters have been (and are being) relatively well studied in generalized Lebesgue spaces by [3], [20],

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[24], [25], etc. The situation is different with the case of Morrey-type and grand Lebesgue spaces, and only recently the approximation matters began to be studied in these spaces. In this direction, various issues were studied in [2], [4]–[8], [11], [15]–[17], [26], [27].

It is well known that the solution of classical Dirichlet problem for the Laplace equation on the unit disk $\omega = \{z : |z| < 1\}$ is represented by the following Poisson integral:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(t)P(r,t-\theta)dt,$$

where $P(r,t-\theta)$ is a Poisson kernel for unit circle:

$$P(r,t-\theta) = Re\frac{e^{i\theta}+re^{it}}{e^{i\theta}-re^{it}} = \frac{1-r^2}{1-2r\cos(t-\theta)+r^2}, \quad 0 \le r < 1, \quad \theta \in [-\pi,\pi].$$

The desire to weaken the restrictions on the boundary values of the solution leads to the study of the classes of harmonic functions represented by the Poisson-Stieltjes integral. A class h_p of functions $u(r,\theta)$ harmonic in the unit disk ω with

$$\sup_{0<r<1}\int_0^{2\pi} |u(r,\theta)|^p \, d\theta < +\infty$$

is such a class. Namely, the harmonic function in ω is represented in the form of Poisson-Stieltjes integral only in the case where $u(r,\theta) \in h_1$, and in the form of Poisson-Lebesgue integral with $f(t) \in L^p(0,2\pi)$, $p > 1$, only in the case where $u(r,\theta) \in h_p$. Representation of harmonic function through Poisson-Stieltjes integral provides the existence of its limiting values almost everywhere on the unit circle in all non-tangential directions: $f(t) := f(e^{it}) = \lim_{r\to 1} u(r,t)$. Similar classes are defined for the functions analytic in the unit disk, and similar results are obtained for these functions, too. These classes are R. Nevanlinna classes N and Hardy classes H_p, $p > 0$ (for more details see [12], [21]).

In this work, we define the grand Hardy space $H^+_{p)}$, $p > 1$. We study some of the properties of functions belonging to the grand Hardy spaces $H^+_{p)}$. We find a necessary and sufficient condition which provides the validity of the analog of classical Riesz theorem in grand Hardy space $H^+_{p)}$. We prove the analog of Smirnov theorem, and we obtain Cauchy's formula in the space $H^+_{p)}$.

2. Some Auxiliary Concepts and Facts

In this section, we give definitions for grand Lebesgue spaces and classical Hardy classes. We also state some of their properties and auxiliary facts to be used later. By $L^{p)}(0,2\pi)$, $1 < p < +\infty$, we denote a grand Lebesgue space of measurable functions f on $[0,2\pi]$ with the norm

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{2\pi}\int_0^{2\pi} |f(t)|^{p-\varepsilon} \, dt\right)^{\frac{1}{p-\varepsilon}} < +\infty.$$

The following inclusions hold:

$$L^p(0,2\pi) \subset L^{p)}(0,2\pi) \subset L^{p-\varepsilon}(0,2\pi), \quad 1 < p < +\infty.$$

Obviously, the space of infinitely differentiable functions $C^\infty[0,2\pi]$ is embedded in $L^{p)}(0,2\pi)$. The space $L^{p)}(0,2\pi)$ with the norm $\|f\|_{p)}$ is a non-separable Banach space. The space $C_0^\infty[0,2\pi]$ of infinitely differentiable finite functions on $[0,2\pi]$ is not dense in $L^{p)}(0,2\pi)$. The validity of this assertion follows from the statement below:

Statement [15]. *The subspace $\overline{C_0^\infty[0,2\pi]}$ consists of the functions $f \in L^{p)}(0,2\pi)$ which satisfy the condition*

$$\lim_{\varepsilon\to+0} \varepsilon \int_0^{2\pi} |f(t)|^{p-\varepsilon}\,dt = 0, \tag{1}$$

where $\overline{C_0^\infty[0,2\pi]}$ is a closure of $C_0^\infty[0,2\pi]$ in $L^{p)}(0,2\pi)$.

Extending every function $f \in L^{p)}(0,2\pi)$ to the whole axis R and assuming $f(t)=0$, $t \in R\backslash[0,2\pi]$, consider the set $\tilde{G}^{p)}(0,2\pi)$ of functions $f \in L^{p)}(0,2\pi)$ which satisfy the condition

$$\|f(\cdot+\delta) - f(\cdot)\|_{p)} \to 0, \quad \delta \to 0.$$

It is clear that $\tilde{G}^{p)}(0,2\pi)$ is a linear manifold in $L^{p)}(0,2\pi)$. Let $G^{p)}(0,2\pi)$ be its closure in $L^{p)}(0,2\pi)$. The set $C_0^\infty[0,2\pi]$ is dense in $G^{p)}(0,2\pi)$ (see, [26], [27]). Therefore, according to Statement 1, the space $G^{p)}(0,2\pi)$ consists of functions satisfying (1).

Let's recall some facts from the theory of Hardy spaces. Let γ be a unit circle $\gamma = \{z \in C : |z| = 1\}$. The Hardy space H_p^+, $p > 0$, is defined as a space of functions f analytic in $\omega = int\gamma$ which satisfy the condition

$$\sup_{0<r<1} \int_0^{2\pi} \left|f(re^{it})\right|^p dt < +\infty.$$

Clearly $H_p^+ \subset H_q^+$ for every p and q: $0 < q < p$. Every function $f \in H_p^+$ has some limiting values almost everywhere on γ in non-tangential directions. Denote this limit function by f^+. From Fatou's lemma it follows that $f^+ \in L^p(0,2\pi)$. For every function $f \in H_1^+$, the following Poisson formula holds:

$$f(re^{it}) = \int_0^{2\pi} f^+(e^{is}) P(r, s-t) ds. \tag{2}$$

To obtain our main results, we will often use the following classical facts.

Theorem 1. *(Riesz theorem). If $f \in H_p^+$, $p > 0$, then*

$$\lim_{r\to1} \int_0^{2\pi} \left|f(re^{it})\right|^p dt = \int_0^{2\pi} \left|f^+(e^{it})\right|^p dt,$$

$$\lim_{r\to1} \int_0^{2\pi} \left|f(re^{it}) - f^+(e^{it})\right|^p dt = 0.$$

Smirnov theorem below establishes a connection between these classes.

Theorem 2. *(Smirnov theorem). If $f \in H_p^+$, $p > 0$, then*
1) if $\left|f^+(e^{it})\right| \leq M$ almost everywhere on γ, then $|f(z)| \leq M$, $z \in \omega$;
2) if $f^+ \in L^q$, $p < q$, then $f \in H_q^+$.

Cauchy's formula is true for the functions belonging to these classes:

Theorem 3. *(Riesz theorem).*
1) If $f \in H_p^+$, $1 < p < +\infty$, then Cauchy's formula holds:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f^+(\xi)}{\xi - z} d\xi, \quad z \in \omega. \tag{3}$$

2) If $f^+ \in L^p(0, 2\pi)$, $1 < p < +\infty$, then the function f defined by (3) belongs to the class H_p^+.

We will also need the following theorem.

Theorem 4. *(Uniqueness theorem). If the function $f \in H_p^+, p > 0$, is such that f^+ is equal to zero on a set of positive measure on γ, then f is identically equal to zero in ω.*

More details on these facts can be found in [12], [21].

3. Grand-Hardy Space

Consider the problem of representing harmonic function in ω in terms of Poisson formula for $f \in L^{p)}(0, 2\pi)$. Denote by $h_{p)}$, $p > 1$, a class of harmonic functions $u(r, \theta)$ in ω which satisfy the condition

$$\|u\|_{h_{p)}} = \sup_{0<r<1} \|u_r(\cdot)\|_{p)} < +\infty,$$

where $u_r(t) = u(re^{it})$. For $\forall \varepsilon \in (0, p-1)$, the embeddings $h_p \subset h_{p)} \subset h_{p-\varepsilon}$, $p > 1$, are true.

The following analog of Riesz theorem is true:

Theorem 5. *For the harmonic function $u(r, \theta)$ in ω o be represented in terms of Poisson integral with $f \in L^{p)}(0, 2\pi)$, $p > 1$, it is necessary and sufficient that $u \in h_{p)}$. If so,*

$$\|f(\cdot)\|_{p)} = \lim_{r \to 1} \|u_r(\cdot)\|_{p)}, \quad u \in h_{p)}. \tag{4}$$

Proof. Necessity. Let the representation (2) hold for the harmonic function $u(r, \theta)$ in ω with $f \in L^{p)}(0, 2\pi)$. It is clear that $\forall \varepsilon \in (0, p-1)$ we have $f \in L^{p-\varepsilon}(0, 2\pi)$ and, therefore,

$$\int_0^{2\pi} |u_r(t)|^{p-\varepsilon} dt = \int_0^{2\pi} \left| \frac{1}{2\pi} \int_0^{2\pi} f(s) P(r, s-t) ds \right|^{p-\varepsilon} dt \leq$$

$$\leq \left(\frac{1}{2\pi} \right)^{p-\varepsilon} \int_0^{2\pi} \left(\int_0^{2\pi} |f(s)| \, P(r, s-t)^{\frac{1}{p-\varepsilon}} P(r, s-t)^{\frac{p-\varepsilon-1}{p-\varepsilon}} ds \right)^{p-\varepsilon} dt \leq$$

$$\leq \left(\frac{1}{2\pi}\right)^{p-\varepsilon} \int_0^{2\pi} \left(\int_0^{2\pi} |f(s)|^{p-\varepsilon} P(r,s-t)ds \left(\int_0^{2\pi} P(r,s-t)ds \right)^{p-\varepsilon-1} \right) dt =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} |f(s)|^{p-\varepsilon} P(r,s-t)dsdt = \int_0^{2\pi} |f(s)|^{p-\varepsilon} ds.$$

Hence it follows

$$\|u_r(\cdot)\|_{p)} \leq \|f(\cdot)\|_{p)}, \quad 0 < r < 1.$$

Passing to the limit as $r \to 1$, we obtain

$$\overline{\lim_{r\to 1}} \|u_r(\cdot)\|_{p)} \leq \|f(\cdot)\|_{p)}. \tag{5}$$

Sufficiency. Let $u \in h_{p)}$. For $\forall\varepsilon \in (0, p-1)$ we have $u \in h_{p-\varepsilon}$. Then it is clear that the function $u(r,\theta)$ has a representation (2), where $f(t) = u^+(t) \in L^{p-\varepsilon}(0, 2\pi)$ are the limiting values of $u(r,\theta)$ as $r \to 1$ in non-tangential directions. Fatou's lemma implies the validity of the following relation:

$$\int_0^{2\pi} \left|u^+(t)\right|^{p-\varepsilon} dt \leq \underline{\lim}_{r\to 1} \int_0^{2\pi} |u_r(t)|^{p-\varepsilon} dt.$$

We have

$$\left(\frac{\varepsilon}{2\pi} \int_0^{2\pi} \left|u^+(t)\right|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} \leq \underline{lim}_{r\to 1} \left(\frac{\varepsilon}{2\pi} \int_0^{2\pi} |u_r(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} \leq \underline{\lim}_{r\to 1} \|u_r(\cdot)\|_{p)}.$$

Then we obtain

$$\left\|u^+(\cdot)\right\|_{p)} \leq \underline{\lim}_{r\to 1} \|u_r(\cdot)\|_{p)}. \tag{6}$$

(5) and (6) imply (4). ◄

Define the grand Hardy space $H^+_{p)}$, $p > 1$, of functions f analytic in ω which satisfy the condition

$$\|f\|_{H^+_{p)}} = \sup_{0<r<1} \|f_r(\cdot)\|_{p)} < +\infty.$$

The following lemma is true.

Lemma. *The following continuous embeddings hold true:*

$$H^+_p \subseteq H^+_{p)} \subseteq H^+_{p-\varepsilon}, \quad p > 1, \quad 0 < \varepsilon < p-1.$$

Proof. Consider $\forall\varepsilon \in (0, p-1)$. Applying Hölder's inequality with the index $\frac{p}{p-\varepsilon}$, we obtain

$$\left(\int_0^{2\pi} |f_r(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} \leq \left(\int_0^{2\pi} |f_r(t)|^p dt \right)^{\frac{1}{p}} (2\pi)^{\frac{\varepsilon}{p(p-\varepsilon)}}.$$

Consequently,

$$\left(\frac{\varepsilon}{2\pi} \int_0^{2\pi} |f_r(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} \leq \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi} \int_0^{2\pi} |f_r(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} \leq$$

$$\leq (2\pi)^{-\frac{1}{p}}(p-1)\left(\int_0^{2\pi}|f_r(t)|^p\,dt\right)^{\frac{1}{p}},$$

i.e.

$$\left(\frac{\varepsilon}{2\pi}\right)^{\frac{1}{p-\varepsilon}}\|f_r(\cdot)\|_{p-\varepsilon}\leq\|f_r(\cdot)\|_{p)}\leq(2\pi)^{-\frac{1}{p}}(p-1)\,\|f_r(\cdot)\|_p\,.$$

We have

$$\left(\frac{\varepsilon}{2\pi}\right)^{\frac{1}{p-\varepsilon}}\|f\|_{H^+_{p-\varepsilon}}\leq\|f\|_{H^+_{p)}}\leq(2\pi)^{-\frac{1}{p}}\,(p-1)\,\|f\|_{H^+_p}\,.$$

◄

The following theorem, as an analog of the first part of Riesz's Theorem 1, shows that the norm in the large Hardy class can be defined by the norms of the limit function.

Theorem 6. *Every function $f\in H^+_{p)}$, $p>1$, has boundary values $f^+(\cdot)$ almost everywhere on γ in non-tangential directions, $f^+\in L^{p)}(0,2\pi)$ and the relation*

$$\left\|f^+(\cdot)\right\|_{p)}=\lim_{r\to1}\|f_r(\cdot)\|_{p)} \tag{7}$$

holds.

Proof. Consider $\forall\varepsilon\in(0,p-1)$. By Lemma, $f\in H^+_{p-\varepsilon}$ for $\varepsilon\in(0,p-1)$. Therefore, by Theorem 1, the function f has boundary values $f^+(\cdot)$ almost everywhere on γ in non-tangential directions, $f^+\in L^{p-\varepsilon}(0,2\pi)$. By Riesz Theorem 1, we have

$$\lim_{r\to1}\int_0^{2\pi}|f_r(t)|^{p-\varepsilon}\,dt=\int_0^{2\pi}\left|f^+(t)\right|^{p-\varepsilon}dt$$

and

$$\lim_{r\to1}\int_0^{2\pi}\left|f_r(t)-f^+(t)\right|^{p-\varepsilon}dt=0. \tag{8}$$

Consequently,

$$\left(\frac{\varepsilon}{2\pi}\int_0^{2\pi}\left|f^+(t)\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}=\lim_{r\to1}\left(\frac{\varepsilon}{2\pi}\int_0^{2\pi}|f_r(t)|^{p-\varepsilon}\,dt\right)^{\frac{1}{p-\varepsilon}}\leq$$

$$\leq\lim_{r\to1}\sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{2\pi}\int_0^{2\pi}|f_r(t)|^{p-\varepsilon}\,dt\right)^{\frac{1}{p-\varepsilon}}=\lim_{r\to1}\|f_r(\cdot)\|_{p)}\,.$$

Hence it follows

$$\left\|f^+(\cdot)\right\|_{p)}\leq\lim_{r\to1}\|f_r(\cdot)\|_{p)}\,. \tag{9}$$

Let's prove the converse of this inequality. From the inequality

$$\int_0^{2\pi}|f_r(t)|^{p-\varepsilon}\,dt\leq\int_0^{2\pi}\left|f^+(t)\right|^{p-\varepsilon}dt,\forall r\in(0,1),$$

it follows that

$$\|f_r(\cdot)\|_{p)}\leq\left\|f^+(\cdot)\right\|_{p)}\,.$$

Hence, passing to the limit as $r \to 1$, we obtain

$$\lim_{r\to 1} \|f_r(\cdot)\|_{p)} \leq \left\|f^+(\cdot)\right\|_{p)} . \tag{10}$$

(9) and (10) imply (7). ◄

The second part of Riesz theorem is true with additional condition.

Theorem 7. *Let $f \in H^+_{p)}$, $p > 1$. Then the relation*

$$\lim_{r\to 1} \left\|f_r(\cdot) - f^+(\cdot)\right\|_{p)} = 0 \tag{11}$$

holds only when

$$\lim_{\varepsilon\to +0} \varepsilon \int_0^{2\pi} \left|f^+(t)\right|^{p-\varepsilon} dt = 0. \tag{12}$$

Proof. Let (11) be satisfied. Consider arbitrary numbers $\eta > 0$ and $0 < \varepsilon < p - 1$. Then there exists $r_0 \in (0, 1)$ such that for $\forall r$: $r_0 < r < 1$

$$\left\|f_r(\cdot) - f^+(\cdot)\right\|_{p)} < \frac{\eta^{\frac{1}{p-\varepsilon}}}{4\pi}. \tag{13}$$

Fix r and denote $M(r) = \max\limits_{[0,2\pi]} |f_r(t)|$. Let $0 < \varepsilon_0 < p - 1$ be such that for every ε: $0 < \varepsilon < \varepsilon_0$ we have

$$\varepsilon(2M(r))^{p-\varepsilon} < \eta. \tag{14}$$

Using Minkowski's inequality, (13) and (14), we obtain

$$\left(\frac{\varepsilon}{2\pi}\int_0^{2\pi} \left|f^+(t)\right|^{p-\varepsilon} dt\right)^{\frac{1}{p-\varepsilon}} \leq \left(\frac{\varepsilon}{2\pi}\int_0^{2\pi} |f_r(t)|^{p-\varepsilon} dt\right)^{\frac{1}{p-\varepsilon}} +$$

$$+ \left(\frac{\varepsilon}{2\pi}\int_0^{2\pi} \left|f^+(t) - f_r(t)\right|^{p-\varepsilon} dt\right)^{\frac{1}{p-\varepsilon}} \leq$$

$$\leq M(r)\varepsilon^{\frac{1}{p-\varepsilon}} + (2\pi)^{-\frac{1}{p-\varepsilon}} \left(\frac{\varepsilon}{2\pi}\int_0^{2\pi} \left|f^+(t) - f_r(t)\right|^{p-\varepsilon} dt\right)^{\frac{1}{p-\varepsilon}} \leq$$

$$\leq M(r)\varepsilon^{\frac{1}{p-\varepsilon}} + (2\pi)^{-\frac{1}{p-\varepsilon}} \left\|f^+(\cdot) - f_r(\cdot)\right\|_{p)} < \frac{\eta^{\frac{1}{p-\varepsilon}}}{2} + \frac{\eta^{\frac{1}{p-\varepsilon}}}{2} = \eta^{\frac{1}{p-\varepsilon}}.$$

Consequently, for every ε: $0 < \varepsilon < \varepsilon_0$ we have

$$\varepsilon \int_0^{2\pi} \left|f^+(t)\right|^{p-\varepsilon} dt < \eta,$$

i.e. (12) holds.

On the contrary, let's assume (12) is true. Consider an arbitrary number $\eta > 0$. Then it follows directly from (12) that

$$\lim_{\varepsilon\to+0}\left(\frac{\varepsilon}{2\pi}\int_0^{2\pi}\left|f^+(t)\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}=0.$$

Therefore, there exists ε_0: $0<\varepsilon_0<p-1$, such that for every $0<\varepsilon<\varepsilon_0$ we have

$$\left(\frac{\varepsilon}{2\pi}\int_0^{2\pi}\left|f^+(t)\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}<\frac{\eta}{4}. \tag{15}$$

From (8) it follows

$$\lim_{r\to1}\left(\int_0^{2\pi}\left|f_r(t)-f^+(t)\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}=0.$$

Consequently, there exists $r_0\in(0;1)$ such that for $\forall r$: $r_0<r<1$ we have

$$\left(\int_0^{2\pi}\left|f_r(t)-f^+(t)\right|^{p-\varepsilon_0}dt\right)^{\frac{1}{p-\varepsilon_0}}<\frac{\eta}{2(p-1)}. \tag{16}$$

Thus,

$$\left\|f_r(\cdot)-f^+(\cdot)\right\|_{p)}=\sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{2\pi}\int_0^{2\pi}\left|f_r(t)-f^+(t)\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}\le$$

$$\le\sup_{0<\varepsilon\le\varepsilon_0}\left(\frac{\varepsilon}{2\pi}\int_0^{2\pi}\left|f_r(t)-f^+(t)\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}+$$

$$+\sup_{\varepsilon_0<\varepsilon<p-1}\left(\frac{\varepsilon}{2\pi}\int_0^{2\pi}\left|f_r(t)\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}. \tag{17}$$

Denote

$$I_1(r,\varepsilon_0)=\sup_{0<\varepsilon\le\varepsilon_0}\left(\frac{\varepsilon}{2\pi}\int_0^{2\pi}\left|f_r(t)-f^+(t)\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}},$$

$$I_2(r,\varepsilon_0)=\sup_{\varepsilon_0<\varepsilon<p-1}\left(\frac{\varepsilon}{2\pi}\int_0^{2\pi}\left|f_r(t)-f^+(t)\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}.$$

Let's first estimate $I_1(r,\varepsilon_0)$. Using Minkowski's inequality and taking into account the relation

$$\int_0^{2\pi}\left|f_r(t)\right|^{p-\varepsilon}dt\le\int_0^{2\pi}\left|f^+(t)\right|^{p-\varepsilon}dt,$$

we obtain

$$\left(\int_0^{2\pi}\left|f_r(t)-f^+(t)\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}\le\left(\int_0^{2\pi}\left|f_r(t)\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}+$$

$$+\left(\int_0^{2\pi}\left|f^+(t)\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}\leq 2\left(\int_0^{2\pi}\left|f^+(t)\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}.$$

Then, by (15), we have

$$I_1(r,\varepsilon_0)\leq 2\sup_{0<\varepsilon\leq\varepsilon_0}\left(\frac{\varepsilon}{2\pi}\int_0^{2\pi}\left|f^+(t)\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}\leq\frac{\eta}{2}. \tag{18}$$

Now let's estimate $I_2(r,\varepsilon_0)$. Applying Hölder's inequality with $\frac{p-\varepsilon_0}{p-\varepsilon}$ and using (16), for every ε: $\varepsilon>\varepsilon_0$ we obtain

$$\left(\frac{\varepsilon}{2\pi}\int_0^{2\pi}\left|f_r(t)-f^+(t)\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}\leq$$

$$\leq(p-1)(2\pi)^{-\frac{1}{p-\varepsilon_0}}\left(\int_0^{2\pi}\left|f_r(t)-f^+(t)\right|^{p-\varepsilon_0}dt\right)^{\frac{1}{p-\varepsilon_0}}<\frac{\eta}{2}.$$

Consequently,

$$I_2(r,\varepsilon_0)=\sup_{\varepsilon_0<\varepsilon<p-1}\left(\frac{\varepsilon}{2\pi}\int_0^{2\pi}\left|f_r(t)-f^+(t)\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}\leq\frac{\eta}{2}. \tag{19}$$

Then, from (17), (18) and (19) it follows for $\forall r>r_0$

$$\left\|f_r(\cdot)-f^+(\cdot)\right\|_{p)}\leq I_1(r,\varepsilon_0)+I_2(r,\varepsilon_0)\leq\frac{\eta}{2}+\frac{\eta}{2}=\eta,$$

i.e. (11) holds. ◀

Similarly we can prove the following theorem:

Theorem 8. *Let $u\in h_{p)}$, $p>1$. Then the relation*

$$\lim_{r\to 1}\left\|u_r(\cdot)-u^+(\cdot)\right\|_{p)}=0$$

holds only when

$$\lim_{\varepsilon\to+0}\varepsilon\int_0^{2\pi}\left|u^+(t)\right|^{p-\varepsilon}dt=0.$$

The following analogue of Smirnov's theorem also holds.

Theorem 9. *Let $f\in H^+_{p)}$, $p>1$, then*

1) if $\left|f^+(e^{it})\right|\leq M$ almost everywhere on γ, then $|f(z)|\leq M$, $z\in\omega$;

2) if $f^+\in L^{q)}$, $p<q$, then $f\in H^+_{q)}$.

Proof. The proof of the theorem follows directly from Theorem 2 and Lemma. ◀

In theorem below, Cauchy's formula for the functions from grand Hardy class is obtained.

Theorem 10.
1) If $f \in H^+_{p)}$, $1 < p < +\infty$, then the following Cauchy formula holds:

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f^+(\xi)}{\xi - z} d\xi, \quad z \in \omega. \tag{20}$$

2) If $f^+ \in L^{p)}(0, 2\pi)$, $1 < p < +\infty$, then the function f, defined by (20)*, belongs to the class $H^+_{p)}$.*

Proof. Let $f \in H^+_{p)}$, $p > 1$. Then, by Theorem 6, we have $f^+ \in L^{p)}$. From Lemma it follows $f^+ \in L^{p-\varepsilon}$. Using Theorem 3, we obtain Cauchy's formula (20).

On the contrary, let $f^+ \in L^{p)}(0, 2\pi)$. Then $f^+ \in L^{p-\varepsilon}$, and, by Theorem 3, we have Cauchy formula (20). Then the representation

$$f(re^{it}) = \int_0^{2\pi} f^+(e^{is}) P(r, s-t) ds$$

holds. Applying Hölder's inequality and using the relation $\int_0^{2\pi} P(r, s-t) ds = 2\pi$, we obtain

$$\left|f(re^{it})\right| \le \int_0^{2\pi} \left|f^+(e^{is})\right|^{p-\varepsilon} P^{\frac{1}{p-\varepsilon}}(r, s-t) P^{1-\frac{1}{p-\varepsilon}}(r, s-t) ds \le$$

$$\le \left(\int_0^{2\pi} \left|f^+(e^{is})\right|^{p-\varepsilon} P(r, s-t) ds \right)^{\frac{1}{p-\varepsilon}} \left(\int_0^{2\pi} P(r, s-t) ds \right)^{1-\frac{1}{p-\varepsilon}} =$$

$$= (2\pi)^{1-\frac{1}{p-\varepsilon}} \left(\int_0^{2\pi} \left|f^+(e^{is})\right|^{p-\varepsilon} P(r, s-t) ds \right)^{\frac{1}{p-\varepsilon}}.$$

Consequently,

$$\left(\int_0^{2\pi} \left|f(re^{it})\right|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} \le$$

$$\le (2\pi)^{1-\frac{1}{p-\varepsilon}} \left(\int_0^{2\pi} \left|f^+(e^{it})\right|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} \le 2\pi \left(\int_0^{2\pi} \left|f^+(e^{it})\right|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}}.$$

It directly follows $\|f_r(\cdot)\|_{p)} \le 2\pi \|f^+(\cdot)\|_{p)}$, and therefore

$$\sup_{0<r<1} \|f_r(\cdot)\|_{p)} \le 2\pi \left\|f^+(\cdot)\right\|_{p)}.$$

◀

Denote by $L_+^{p)}$ the subspace of $L^{p)}$, generated by the restrictions of functions from $H_{p)}^+$, and let $J : H_{p)}^+ \to L_+^{p)}$ be the corresponding restriction operator:

$$Jf(\xi) = f^+(\xi), \quad \xi \in \gamma.$$

From the uniqueness Theorem 4 and Theorem 10 it follows that the operator J is an isomorphism. Let $G_+^{p)} = G^{p)} \bigcap L_+^{p)}$. It is clear that $G_+^{p)}$ is a subspace of the space $L_+^{p)}$. Let $GH_{p)}^+ = J^{-1}(G_+^{p)})$.

The following theorem is a characterization of the space $GH_{p)}^+$.

Theorem 11. *The space $f \in GH_{p)}^+$, $p > 1$, consists of functions $f \in H_{p)}^+$ satisfying the equality* (11).

Proof. We have $f^+ \in G_+^{p)}$. As $G_{p)}^+ \subset \overline{C_0^\infty[0, 2\pi]}$, by Statement we have $\lim\limits_{\varepsilon \to +0} \varepsilon \int_0^{2\pi} |f^+(t)|^{p-\varepsilon}\, dt = 0$. Therefore from Theorem 7 we obtain (11).

Conversely, if (11) holds for $f \in H_{p)}^+$, then, by Theorem 7, the inclusion the inclusion $f^+ \in G_{p)}$ holds. Therefore, the inclusion $f \in GH_{p)}^+$ is valid. ◀

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