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BOUNDEDNESS OF THE DISCRETE HILBERT TRANSFORM IN DISCRETE HÖLDER SPACES

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Abstract. *The Hilbert transform plays an important role in the theory and practice of signal processing operations in continuous system theory. The Hilbert transform was the motivation for the development of modern harmonic analysis. Its discrete version is also widely used in many areas of science and technology and plays an important role in digital signal processing. The essential motivation behind thinking about discrete transforms is that experimental data are most frequently not taken in a continuous manner but sampled at discrete time values. Since much of the data collected in both the physical sciences and engineering are discrete, the discrete Hilbert transform is a rather useful tool in these areas for the general analysis of this type of data. The Hilbert transform has been well studied on classical function spaces such as Hölder, Lebesgue, Morrey, etc. But its discrete version, which also has numerous applications, has not been fully studied in discrete analogues of these spaces. In this paper, we discuss the discrete Hilbert transform in discrete Hölder spaces and obtain its boundedness in these spaces.*

Keywords: Hilbert transform, singular integral, approximation, Hölder space

Mathematics Subject Classification (2020): 44A15, 40A05, 42B35

1. Introduction

Let the function f be defined on the real axis and $\alpha \in (0, 1]$. If there exists a number $M > 0$ such that for any $x, y \in R$

$$|f(x) - f(y)| \leq M \cdot |x - y|^{\alpha} \tag{1}$$

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and for any $x, y \in R\backslash\{0\}$

$$|f(x) - f(y)| \leq M \cdot \left|\frac{1}{x} - \frac{1}{y}\right|^{\alpha}, \tag{2}$$

then the function f is said to be Hölder continuous with the exponent α on the real axis (see [9], [17]). The class of Hölder continuous functions with the exponent α on the real axis together with norm

$$\|f\|_{\alpha} = \max_{x\in R} |f(x)| + \sup_{x,y\in R, x\neq y} \frac{|f(x) - f(y)|}{|x-y|^{\alpha}} + \sup_{x,y\in R\backslash\{0\}, x\neq y} \frac{|f(x) - f(y)|}{|1/x - 1/y|^{\alpha}}$$

forms a Banach space and is denoted by $H_{\alpha}(R)$.

It follows from (1) and (2) that for any $f \in H_{\alpha}(R)$ there exists $f(\infty) = \lim_{x\to\pm\infty} f(x)$ and for any $x \neq 0$

$$|f(x) - f(\infty)| \leq \frac{\|f\|_{\alpha}}{|x|^{\alpha}}.$$

Denote

$$H_{\alpha}^{0}(R) = \{f \in H_{\alpha}(R) : f(\infty) = 0\} \subset H_{\alpha}(R).$$

The Hilbert transform of a function $f \in H_{\alpha}^{0}(R)$, $\alpha \in (0, 1]$ is defined as the Cauchy principle value integral

$$(Hf)(t) = \frac{1}{\pi}\int_{R} \frac{f(\tau)}{t-\tau} d\tau \equiv \frac{1}{\pi} \lim_{\varepsilon\to 0+} \int_{R\backslash(t-\varepsilon, t+\varepsilon)} \frac{f(\tau)}{t-\tau} d\tau, \quad t \in R.$$

It is well known (see [9], [18]) that the Hilbert transform of the function $f \in H_{\alpha}^{0}(R)$, $\alpha \in (0, 1]$, exists for any $t \in R$. In case $\alpha \in (0, 1)$, the Hilbert transform is a bounded map in the space $H_{\alpha}^{0}(R)$.

For a numerical sequence $b = \{b_n\}_{n\in Z}$, the sequence $H(b) = \{(Hb)_n\}_{n\in Z}$, where

$$(Hb)_n = \sum_{m\in Z, m\neq n} \frac{b_m}{n-m}, n \in Z,$$

is called the discrete Hilbert transform of the sequence $b = \{b_n\}_{n\in Z}$.

The Hilbert transform plays an important role in the theory and practice of signal processing operations in continuous system theory. The Hilbert transform was the motivation for the development of modern harmonic analysis. Its discrete version is also widely used in many areas of science and technology and plays an important role in digital signal processing. The essential motivation behind thinking about discrete transforms is that experimental data are most frequently not taken in a continuous manner but sampled at discrete time values. Since much of the data collected in both the physical sciences and engineering are discrete, the discrete Hilbert transform is a rather useful tool in these areas for the general analysis of this type of data. The Hilbert transform has been well studied on classical function spaces such as Hölder, Lebesgue, Morrey, Campanato, Besov, Lorentz, Sobolev, etc. (see [1], [6]–[10], [12]–[14], [16], [17], [21]–[23]). But its discrete version, which also has numerous applications, has not been fully studied in discrete analogues of these spaces. M. Riesz proved (see [20]) that the discrete Hilbert transform is a bounded map in the space l_p, $1 < p < \infty$, that is if $b \in l_p$, $1 < p < \infty$, then $H(b) \in l_p$ and the inequality

$$\|Hb\|_{l_p} \le c_p \cdot \|b\|_{l_p}$$

holds, where c_p is an absolute constant. Weighted analogues of (1) are investigated in the works [5], [15], [19], [24]. If $b \in l_1$ then the sequence $H(b)$ belongs to the class $\bigcap\limits_{p>1} l_p$, but, generally speaking, it does not belong to the class l_1. In this case, R.Hunt, B.Muckenhoupt and R.Wheeden (see [11]) proved that the distribution function $(Hb)(\lambda) := \sum\limits_{\{n \in Z:\ |(Hb)_n| > \lambda\}} 1$ of the Hilbert transform of the sequence b satisfies the weak condition

$$\forall \lambda > 0 \quad |(Hb)(\lambda)| \le \frac{c_0}{\lambda} \|b\|_{l_1},$$

where c_0 is an absolute constant.

In [2]–[5], [19], [24], they studied the properties of the discrete Hilbert transform in the discrete Lebesgue and Morrey spaces. In this paper we discuss the discrete Hilbert transform on discrete Hölder spaces and obtain its boundedness in these spaces.

2. Discrete Hölder Spaces

Let $b = \{b_n\}_{n \in Z}$ be a numerical sequence and $\alpha > 0$. If there exists a number $M > 0$ such that for any $m, n \in Z \backslash \{0\}$

$$|b_m - b_n| \le M \cdot \left| \frac{1}{m} - \frac{1}{n} \right|^{\alpha}, \tag{3}$$

then the sequence $b = \{b_n\}_{n \in Z}$ is called a sequence satisfying the Hölder condition with the exponent α. The class of sequences satisfying the Hölder condition with the exponent α with norm

$$\|b\|_{\alpha} = \max_{n \in Z} |b_n| + \sup_{m,n \in Z \backslash \{0\}, m \ne n} \frac{|b_m - b_n|}{|1/m - 1/n|^{\alpha}}$$

forms a Banach space and is denoted by $h_{\alpha}(Z)$.

It follows from (3) that for any $b \in h_{\alpha}(Z)$ there exist $b_{\infty} = \lim\limits_{n \to \pm\infty} b_n$ and for any $m \ne 0$

$$|b_m - b_{\infty}| \le \frac{\|b\|_{\alpha}}{|m|^{\alpha}}. \tag{4}$$

Denote

$$h_{\alpha}^0(Z) = \{b \in h_{\alpha}(R) : b(\infty) = 0\} \subset h_{\alpha}(Z).$$

Note that in case $\alpha > 1$ the space $h_{\alpha}(Z)$ does not consist only of stationary sequences. Therefore, in this case, the discrete Hölder transform is also of interest.

3. Boundedness of the Discrete Hilbert Transform in Discrete Hölder Spaces

Theorem. *For any $0 < \alpha < 1$, the discrete Hilbert transform is a bounded map in the space $h^0_\alpha(Z)$, that is, there exists a constant $c_\alpha > 0$ depending only on $\alpha \in (0,1)$ such that for any $b \in h^0_\alpha(Z)$*

$$\|Hb\|_\alpha \le c_\alpha \cdot \|b\|_\alpha. \tag{5}$$

Proof. Let $b \in h^0_\alpha(Z)$. First we proof that for any $n \neq 0$

$$|(Hb)_n| \le \frac{6}{\alpha(1-\alpha)} \cdot \frac{\|b\|_\alpha}{|n|^\alpha}. \tag{6}$$

Assume, without loss of generality, that $n \ge 1$ (the case $n \le -1$ is studied similarly). Let n_0 be the integer part of the number $\dfrac{n}{2}$. Then

$$|(Hb)_n| \le \sum_{k=1}^{\infty} \frac{1}{k}|b_{n+k}-b_{n-k}| = \sum_{k=1}^{n_0} \frac{1}{k}|b_{n+k}-b_{n-k}| + \sum_{k=n_0+1}^{\infty} \frac{1}{k}|b_{n+k}-b_{n-k}| = J_1 + J_2, \tag{7}$$

where in case $n_0 = 0$ we take $J_1 = 0$. In case $n_0 \ge 1$ it follows from $b \in h^0_\alpha(Z)$ that

$$J_1 \le \sum_{k=1}^{n_0} \frac{\|b\|_\alpha}{k} \left| \frac{1}{n+k} - \frac{1}{n-k} \right|^\alpha =$$

$$= \|b\|_\alpha \sum_{k=1}^{n_0} \frac{1}{k} \left| \frac{2k}{n^2-k^2} \right|^\alpha \le 2^\alpha \|b\|_\alpha \sum_{k=1}^{n_0} \frac{k^{\alpha-1}}{|n^2 - n^2/4|^\alpha} =$$

$$= \left(\frac{8}{3n^2}\right)^\alpha \|b\|_\alpha \sum_{k=1}^{n_0} \frac{1}{k^{1-\alpha}} \le \left(\frac{8}{3n^2}\right)^\alpha \|b\|_\alpha \int_0^{n/2} \frac{1}{x^{1-\alpha}} dx \le \frac{4}{3\alpha} \cdot \frac{\|b\|_\alpha}{n^\alpha}.$$

Thus for any $n \ge 1$

$$J_1 \le \frac{4}{3\alpha} \cdot \frac{\|b\|_\alpha}{n^\alpha}. \tag{8}$$

Let us estimate the sum J_2:

$$J_2 \le \sum_{k=n_0+1}^{\infty} \frac{1}{k}|b_{n+k}| + \sum_{k=n_0+1}^{n-1} \frac{1}{k}|b_{n-k}| + \sum_{k=n+1}^{\infty} \frac{1}{k}|b_{n-k}| = J_2^{(1)} + J_2^{(2)} + J_2^{(3)}, \tag{9}$$

where in cases $n = 1, 2$ we take $J_2^{(2)} = 0$. It follows from (4) that for any $k \neq 0$

$$|b_k| \le \frac{\|b\|_\alpha}{|k|^\alpha}.$$

Therefore for any $n \ge 3$

$$J_2^{(2)} \le \sum_{k=n_0+1}^{n-1} \frac{1}{k} \cdot \frac{\|b\|_\alpha}{(n-k)^\alpha} \le$$

$$\leq \frac{2\|b\|_\alpha}{n} \sum_{k=n_0+1}^{n-1} \frac{1}{(n-k)^\alpha} \leq \frac{2\|b\|_\alpha}{n} \cdot \int_0^{n/2} \frac{dx}{x^\alpha} = \frac{2^\alpha \|b\|_\alpha}{(1-\alpha)n^\alpha}.$$

Thus for any $n \geq 1$

$$J_2^{(2)} \leq \frac{2^\alpha \|b\|_\alpha}{(1-\alpha)n^\alpha}. \tag{10}$$

Let us estimate the sums $J_2^{(1)}$ and $J_2^{(3)}$:

$$J_2^{(1)} \leq \sum_{k=n_0+1}^{\infty} \frac{1}{k} \cdot \frac{\|b\|_\alpha}{(n+k)^\alpha} \leq \|b\|_\alpha \cdot \sum_{k=n_0+1}^{\infty} \frac{1}{k^{1+\alpha}} \leq \|b\|_\alpha \cdot \int_{n/2}^{\infty} \frac{dx}{x^{1+\alpha}} = \frac{2^\alpha \|b\|_\alpha}{\alpha n^\alpha},$$

$$J_2^{(3)} \leq \sum_{k=n+1}^{\infty} \frac{1}{k} \cdot \frac{\|b\|_\alpha}{|n-k|^\alpha} = \|b\|_\alpha \cdot \sum_{k=1}^{\infty} \frac{1}{(k+n)k^\alpha} \leq \|b\|_\alpha \cdot \int_0^{\infty} \frac{dx}{(x+n)x^\alpha} \leq$$

$$\leq \|b\|_\alpha \cdot \left[\int_0^n \frac{dx}{n \cdot x^\alpha} + \int_n^\infty \frac{dx}{x^{1+\alpha}} \right] = \left[\frac{1}{\alpha} + \frac{1}{1-\alpha} \right] \cdot \frac{\|b\|_\alpha}{n^\alpha}.$$

It follows from the last inequalities here and from (9), (10) that for any $n \geq 1$

$$J_2 \leq \frac{2^{1+\alpha}}{\alpha(1-\alpha)} \cdot \frac{\|b\|_\alpha}{n^\alpha}. \tag{11}$$

From (7), (8), and (11) we obtain (6). Hence from the inequality

$$|(Hb)_0| = \left| \sum_{m \in Z, m \neq 0} \frac{b_m}{m} \right| \leq 2 \sum_{m=1}^{\infty} \frac{\|b\|_\alpha}{m^{1+\alpha}} \leq 2 \left(1 + \frac{1}{\alpha}\right) \|b\|_\alpha$$

we obtain

$$\max_{n \in Z} |(Hb)_n| \leq \frac{6}{\alpha(1-\alpha)} \cdot \|b\|_\alpha. \tag{12}$$

Now we show that for any $m, n \in Z \backslash \{0\}$

$$|(Hb)_m - (Hb)_n| \leq \left(\frac{42}{\alpha} + \frac{101}{1-\alpha} \right) \|b\|_\alpha \cdot \left| \frac{1}{m} - \frac{1}{n} \right|^\alpha. \tag{13}$$

Consider the following cases:

1) $m \geq 1$, $n \leq -1$ (or $m \leq -1$, $n \geq 1$). In this case it follows from (6) that

$$|(Hb)_m - (Hb)_n| \leq \frac{6\|b\|_\alpha}{\alpha(1-\alpha)} \left[\frac{1}{|m|^\alpha} + \frac{1}{|n|^\alpha} \right] \leq$$

$$\leq \frac{12\|b\|_\alpha}{\alpha(1-\alpha)} \left[\frac{1}{|m|} + \frac{1}{|n|} \right]^\alpha = \frac{12\|b\|_\alpha}{\alpha(1-\alpha)} \left[\frac{1}{m} - \frac{1}{n} \right]^\alpha.$$

This shows that in this case (13) is satisfied.

2) $m \geq 1, n \geq 1$. Without loss of generality, we assume that $m > n$. Denote $k = m-n$. If $k \geq n/3$, then

$$|(Hb)_{n+k} - (Hb)_n| \leq \frac{6\|b\|_\alpha}{\alpha(1-\alpha)} \left[\frac{1}{|n+k|^\alpha} + \frac{1}{|n|^\alpha}\right] \leq \frac{42\|b\|_\alpha}{\alpha(1-\alpha)} \left[\frac{1}{n} - \frac{1}{n+k}\right]^\alpha,$$

that is, in this case (13) is satisfied. Let $k < n/3$. Then

$$(Hb)_n - (Hb)_{n+k} = \sum_{p=1}^{k} \frac{b_{n-p} - b_{n+p}}{p} - \sum_{p=1}^{k} \frac{b_{n+k-p} - b_{n+k+p}}{p} +$$

$$+ \sum_{p=k+1}^{\infty} \left(\frac{b_{n-p}}{p} - \frac{b_{n+p}}{p} + \frac{b_{n+k+p}}{p} - \frac{b_{n+k-p}}{p}\right) =$$

$$= \sum_{p=1}^{k} \frac{b_{n-p} - b_{n+p}}{p} - \sum_{p=1}^{k} \frac{b_{n+k-p} - b_{n+k+p}}{p} - \sum_{p=k+1}^{2k} \frac{b_{n+k-p}}{p} - \sum_{p=k+1}^{2k} \frac{b_{n+p}}{p} +$$

$$+ \sum_{p=k+1}^{\infty} \left(\frac{1}{p} - \frac{1}{p+k}\right) b_{n+k+p} + \sum_{p=k+1}^{\infty} \left(\frac{1}{p} - \frac{1}{p+k}\right) b_{n-p} =$$

$$= \sum_{p=1}^{k} \frac{b_{n-p} - b_{n+p}}{p} + \sum_{p=1}^{k} \frac{b_{n+k+p} - b_{n+k-p}}{p} +$$

$$+ \sum_{l=1}^{k} \sum_{s=1}^{\infty} \left(\frac{1}{sk+l} - \frac{1}{(s+1)k+l}\right) [b_{n+(s+1)k+l} - b_{n+k+l}] +$$

$$+ \sum_{l=1}^{k} \sum_{s=1}^{\infty} \left(\frac{1}{sk+l} - \frac{1}{(s+1)k+l}\right) [b_{n-sk-l} - b_{n-l}] = I_1 + I_2 + I_3 + I_4. \quad (14)$$

Let us estimate the sums I_1, I_2, I_3 and I_4:

$$|I_1| \leq \|b\|_\alpha \cdot \sum_{p=1}^{k} \frac{1}{p} \left|\frac{1}{n-p} - \frac{1}{n+p}\right|^\alpha =$$

$$= \|b\|_\alpha \cdot \sum_{p=1}^{k} \frac{1}{p} \left|\frac{2p}{n^2-p^2}\right|^\alpha \leq \left(\frac{2}{n^2-k^2}\right)^\alpha \|b\|_\alpha \cdot \sum_{p=1}^{k} \frac{1}{p^{1-\alpha}} \leq$$

$$\leq \left(\frac{2}{n^2-k^2}\right)^\alpha \|b\|_\alpha \cdot \int_0^k \frac{dx}{x^{1-\alpha}} = \left(\frac{2}{n^2-k^2}\right)^\alpha \|b\|_\alpha \cdot \frac{k^\alpha}{\alpha} \leq \frac{4^\alpha}{\alpha} \|b\|_\alpha \cdot \left|\frac{1}{n} - \frac{1}{n+k}\right|^\alpha; \quad (15)$$

$$|I_2| \leq \|b\|_\alpha \cdot \sum_{p=1}^{k} \frac{1}{p} \left|\frac{1}{n+k-p} - \frac{1}{n+k+p}\right|^\alpha =$$

$$= \|b\|_\alpha \cdot \sum_{p=1}^{k} \frac{1}{p} \left| \frac{2p}{(n+k)^2 - p^2} \right|^\alpha \leq \frac{4^\alpha}{\alpha} \|b\|_\alpha \cdot \left| \frac{1}{n} - \frac{1}{n+k} \right|^\alpha ; \tag{16}$$

$$|I_3| \leq \|b\|_\alpha \cdot \sum_{l=1}^{k} \sum_{s=1}^{\infty} \frac{k}{(sk+l)(sk+k+l)} \left(\frac{sk}{(n+sk+k+l)(n+k+l)} \right)^\alpha \leq$$

$$\leq k\|b\|_\alpha \cdot \sum_{s=1}^{\infty} \frac{k}{(sk)(sk+l)} \left(\frac{sk}{(n+sk+k)(n+k)} \right)^\alpha =$$

$$= \left(\frac{k}{n+k} \right)^\alpha \|b\|_\alpha \cdot \sum_{s=1}^{\infty} \frac{1}{s^{1-\alpha}(s+1)(n+sk+k)^\alpha} \leq$$

$$\leq 2 \left(\frac{k}{n+k} \right)^\alpha \|b\|_\alpha \cdot \int_1^\infty \frac{dx}{x^{2-\alpha}(n+kx)^\alpha} =$$

$$= 2 \left(\frac{k}{n+k} \right)^\alpha \|b\|_\alpha \cdot \frac{k^{1-\alpha}}{n} \cdot \int_{k/n}^\infty \frac{dt}{t^{2-\alpha}(1+t)^\alpha}$$

$$\leq \frac{2k\|b\|_\alpha}{n(n+k)^\alpha} \cdot \int_{k/n}^\infty \frac{dt}{t^{2-\alpha}} = \frac{2k^\alpha}{(1-\alpha)n^\alpha(n+k)^\alpha} \cdot \|b\|_\alpha = \frac{2}{1-\alpha} \|b\|_\alpha \cdot \left| \frac{1}{n} - \frac{1}{n+k} \right|^\alpha . \tag{17}$$

To estimate the sum I_4, suppose that $n = p_0 k + l_0$, $0 < l_0 \leq k$. Let us write the sum I_4 in the form

$$I_4 = \sum_{l=1}^{k} \sum_{s=1}^{p_0-2} \left(\frac{1}{sk+l} - \frac{1}{(s+1)k+l} \right) [b_{n-sk-l} - b_{n-l}] +$$

$$+ \sum_{l=1}^{k} \left(\frac{1}{(p_0-1)k+l} - \frac{1}{p_0 k+l} \right) [b_{n-(p_0-1)k-l} - b_{n-l}] +$$

$$+ \sum_{l=1}^{k} \left(\frac{1}{p_0 k+l} - \frac{1}{(p_0+1)k+l} \right) [b_{n-p_0 k-l} - b_{n-l}] +$$

$$+ \sum_{l=1}^{k} \left(\frac{1}{(p_0+1)k+l} - \frac{1}{(p_0+2)k+l} \right) [b_{n-(p_0+1)k-l} - b_{n-l}] +$$

$$+ \sum_{l=1}^{k} \sum_{s=p_0-2}^{\infty} \left(\frac{1}{sk+l} - \frac{1}{(s+1)k+l} \right) [b_{n-sk-l} - b_{n-l}] =$$

$$= I_4^{(1)} + I_4^{(2)} + I_4^{(3)} + I_4^{(4)} + I_4^{(5)}.$$

Then it follows from the following estimates:

$$I_4^{(1)} \leq \|b\|_\alpha \sum_{l=1}^{k} \sum_{s=1}^{p_0-2} \frac{k}{(sk+l)(sk+k+l)} \cdot \left(\frac{sk}{(n-sk-l)(n-l)} \right)^\alpha \leq$$

$$\leq k\|b\|_\alpha \sum_{s=1}^{p_0-2} \frac{k}{(sk)(sk+k)} \cdot \left(\frac{sk}{(p_0k-sk-k)(n-k)}\right)^\alpha =$$

$$= \frac{\|b\|_\alpha}{(n-k)^\alpha} \sum_{s=1}^{p_0-2} \frac{1}{s^{1-\alpha}(1+s)(p_0-s-1)^\alpha} \leq$$

$$\leq \frac{2\|b\|_\alpha}{(n-k)^\alpha} \left[\int_1^{p_0-2} \frac{dx}{x^{2-\alpha}(p_0-2-x)^\alpha} + \frac{1}{2(p_0-2)^\alpha}\right] \leq$$

$$\leq \frac{2\|b\|_\alpha}{(n-k)^\alpha} \left[\int_1^{\frac{p_0-2}{2}} \frac{dx}{x^{2-\alpha}\left(\frac{p_0-2}{2}\right)^\alpha} + \int_{\frac{p_0-2}{2}}^{p_0-2} \frac{dx}{\left(\frac{p_0-2}{2}\right)^{2-\alpha}(p_0-2-x)^\alpha} + \frac{1}{2(p_0-2)^\alpha}\right] \leq$$

$$\leq \frac{2\|b\|_\alpha}{(n-k)^\alpha} \left[\left(\frac{2}{p_0-2}\right)^\alpha \cdot \frac{1}{1-\alpha} + \frac{1}{1-\alpha} \cdot \frac{2}{p_0-2} + \frac{1}{2(p_0-2)^\alpha}\right] \leq$$

$$\leq \frac{\|b\|_\alpha}{(n-k)^\alpha} \cdot \frac{9}{(1-\alpha)(p_0-2)^\alpha} \leq \left(\frac{2}{n+k}\right)^\alpha \cdot \frac{9\|b\|_\alpha}{(1-\alpha)(p_0-2)^\alpha} \leq$$

$$\leq \left(\frac{2}{n+k}\right)^\alpha \cdot \frac{9\|b\|_\alpha}{(1-\alpha)} \cdot \left(\frac{4k}{n}\right)^\alpha \leq \frac{72\|b\|_\alpha}{1-\alpha} \left|\frac{1}{n} - \frac{1}{n+k}\right|^\alpha;$$

$$I_4^{(2)} \leq \|b\|_\alpha \sum_{l=1}^{k} \frac{k}{((p_0-1)k+l)(p_0k+l)} \cdot \left(\frac{(p_0-1)k}{(n-(p_0-1)k-l)(n-l)}\right)^\alpha \leq$$

$$\leq \|b\|_\alpha \sum_{l=1}^{k} \frac{k}{(n-2k)(n-k)} \cdot \left(\frac{n}{(l_0+k-l)(n-k)}\right)^\alpha \leq$$

$$\leq \frac{k\|b\|_\alpha}{(n-2k)(n-k)} \cdot \left(\frac{n}{n-k}\right)^\alpha \sum_{s=1}^{k} \frac{1}{s^\alpha} \leq$$

$$\leq \frac{k\|b\|_\alpha}{\frac{n}{3} \cdot \frac{2n}{3}} \cdot \left(\frac{2n}{n+k}\right)^\alpha \cdot \frac{k^{1-\alpha}}{1-\alpha} \leq \frac{9\|b\|_\alpha}{1-\alpha} \left|\frac{1}{n} - \frac{1}{n+k}\right|^\alpha;$$

$$I_4^{(3)} \leq \|b\|_\alpha \sum_{\substack{l=1\\ l\neq l_0}}^{k} \frac{k}{(p_0k+l)(p_0k+k+l)} \cdot \left(\frac{p_0k}{|n-p_0k-l|(n-l)}\right)^\alpha +$$

$$+ \frac{k(|b_0|+|b_{n-l_0}|)}{(p_0k+l_0)(p_0k+k+l_0)} \leq \|b\|_\alpha \sum_{\substack{l=1\\ l\neq l_0}}^{k} \frac{k}{(n-k)n} \cdot \left(\frac{n}{|l-l_0|(n-k)}\right)^\alpha +$$

$$+ \frac{2k\|b\|_\alpha}{n(n+k)} \leq 2\|b\|_\alpha \frac{k}{(n-k)n} \cdot \left(\frac{n}{n-k}\right)^\alpha \sum_{s=1}^{k} \frac{1}{s^\alpha} +$$

$$+ \left(\frac{k}{n(n+k)}\right)^\alpha \cdot 2\|b\|_\alpha \leq \frac{8\|b\|_\alpha}{1-\alpha} \left|\frac{1}{n} - \frac{1}{n+k}\right|^\alpha;$$

$$I_4^{(4)} \le \|b\|_\alpha \sum_{l=1}^{k} \frac{k}{((p_0+1)k+l)((p_0+2)k+l)} \cdot \left(\frac{(p_0+1)k}{|n-(p_0+1)k-l|(n-l)} \right)^\alpha \le$$

$$\le \frac{k\|b\|_\alpha}{(n+k)n} \cdot \left(\frac{n}{n-k} \right)^\alpha \sum_{s=1}^{k} \frac{1}{s^\alpha} \le \frac{2\|b\|_\alpha}{1-\alpha} \left| \frac{1}{n} - \frac{1}{n+k} \right|^\alpha ;$$

$$I_4^{(5)} \le \|b\|_\alpha \sum_{l=1}^{k} \sum_{s=p_{0+2}}^{\infty} \frac{k}{(sk+l)(sk+k+l)} \cdot \left(\frac{sk}{(sk+l-n)(n-l)} \right)^\alpha \le$$

$$\le k\|b\|_\alpha \sum_{s=p_{0+2}}^{\infty} \frac{k}{(sk)^2} \cdot \left(\frac{sk}{(sk-(p_0+1)k)(n-k)} \right)^\alpha =$$

$$= \frac{\|b\|_\alpha}{(n-k)^\alpha} \sum_{s=p_{0+2}}^{\infty} \frac{1}{s^{2-\alpha}(s-(p_0+1))^\alpha} =$$

$$= \frac{\|b\|_\alpha}{(n-k)^\alpha} \sum_{q=1}^{\infty} \frac{1}{q^\alpha (p_0+q+1)^{2-\alpha}} \le \frac{2\|b\|_\alpha}{(n-k)^\alpha} \int_1^\infty \frac{dx}{x^\alpha (x+p_0)^{2-\alpha}} =$$

$$= \frac{2\|b\|_\alpha}{(n-k)^\alpha p_0} \int_{1/p_0}^\infty \frac{dt}{t^\alpha (1+t)^{2-\alpha}} \le \frac{2\|b\|_\alpha}{(n-k)^\alpha p_0} \left[\int_{1/p_0}^1 \frac{dt}{t^\alpha} + \int_1^\infty \frac{dt}{t^2} \right] \le$$

$$\le \frac{2\|b\|_\alpha}{(n-k)^\alpha p_0} \cdot \frac{2-\alpha}{1-\alpha} \le \frac{4\|b\|_\alpha}{1-\alpha} \cdot \left(\frac{2}{n+k} \right)^\alpha \cdot \frac{k}{n} \le \frac{8\|b\|_\alpha}{1-\alpha} \left| \frac{1}{n} - \frac{1}{n+k} \right|^\alpha$$

that

$$|I_4| \le \frac{99\|b\|_\alpha}{1-\alpha} \cdot \left| \frac{1}{n} - \frac{1}{n+k} \right|^\alpha . \tag{18}$$

The estimates (14), (15), (16), (17), (18) show that (13) is satisfied in this case.

3) $m \le -1$, $n \le -1$. This case is studied in a similar way.

Thus we obtain that for any $m, n \in Z \backslash \{0\}$ (13) is satisfied. It follows from (12) and (13) that the inequality (5) holds. This completes the proof of the theorem. ◄

References

1. Agora E., Carro M.J., Soria J. Boundedness of the Hilbert transform on weighted Lorentz spaces. *J. Math. Anal. Appl.*, 2012, **395** (1), pp.218-229.
2. Aliev R.A., Ahmadova A.N. Boundedness of the discrete Hilbert transform on discrete Morrey spaces. *Ufa Math. J.*, 2021, **13** (1), pp. 98-109.
3. Aliev R.A., Amrahova A.F. On the summability of the discrete Hilbert transform. *Ural Math. J.*, 2018, **4** (2), pp. 6-12.
4. Aliev R.A., Amrahova A.F. Properties of the discrete Hilbert transform. *Complex Anal. Oper. Theory*, 2019, **13** (8), pp. 3883-3897.
5. Andersen K.F. Inequalities with weights for discrete Hilbert transforms. *Canad. Math. Bul.*, 1977, **20** (1), pp. 9-16.

6. Belov Y., Mengestie T.Y., Seip K. Unitary discrete Hilbert transforms. *JAMA*, 2010, **112**, pp. 383-393.
7. Belov Y., Mengestie T.Y., Seip K. Discrete Hilbert transforms on sparse sequences. *Proc. Lond. Math. Soc.*, 2011, **103** (1), pp. 73-105.
8. Edmunds D.E., Kokilashvili V.M., Meskhi A. *Bounded and Compact Integral Operators.* Kluwer, Dordrecht, 2002.
9. Gakhov F.D. *Boundary Value Problems.* Pergamon Press, Oxford, 1966.
10. Garnett J. *Bounded Analytic Functions.* Academic Press, New York, 1981.
11. Hunt R., Muckenhoupt B., Wheeden R. Weighted norm inequalities for the conjugate function and Hilbert transform. *Trans. Amer. Math. Sos.*, 1973, **176**, pp. 227-251.
12. Kokilashvili V., Meskhi A., Rafeiro H., Samko S. *Integral Operators in Non-Standard Function Spaces. Vol. 2: Variable Exponent Hölder, Morrey-Campanato and Grand Spaces.* Birkhäuser, Basel, 2016.
13. Koosis P. *Introduction to H_p Spaces.* Cambridge Univ. Press, New York, 2008.
14. Lifanov I.K. *Singular Integral Equations and Discrete Vortices.* VSP, Utrecht, 1996.
15. Liflyand E. *Weighted estimates for the discrete Hilbert transform.* In: Ruzhansky, M., Tikhonov, S. (eds) Methods of Fourier Analysis and Approximation Theory. Applied and Numerical Harmonic Analysis. Birkhäuser, Cham, 2016, pp. 59-69.
16. Mikhlin S.G., Prössdorf S. *Singular Integral Operators.* Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1986.
17. Muskhelishvili N.I. *Singular Integral Equations: Boundary Problems of Function Theory and Their Application to Mathematical Physics*, 2nd edition. Dover Publications, New York, 2008.
18. Pesenson I.Z. Sampling and interpolation for the discrete Hilbert and Kak-Hilbert transforms. *Canad. Math. Bul.*, First View, pp. 1-16. DOI: https://doi.org/10.4153/S0008439522000169
19. Rakotondratsimba Y. Two weight inequality for the discrete Hilbert transform. *Soochow J. Math.*, 1999, **25** (4), pp. 353-373.
20. Riesz M. Sur les fonctions conjuguées. *Math. Z.*, 1928, **27**, pp. 218-244.
21. Stein E.M. *Singular Integrals and Differentiability Properties of Functions.* Princeton Univ. Press, Princeton, New Jersey, 1970.
22. Stein E.M., Weiss G. *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton Univ. Press, Princeton, New Jersey, 1971.
23. Stepanov V.D. On the boundedness of the Hilbert transform from weighted Sobolev space to weighted Lebesgue space. *J. Fourier Anal. Appl.*, 2022, **28** (3), Paper No. 46, pp. 1-17.
24. Stepanov V.D., Tikhonov S.Y. Two weight inequalities for the Hilbert transform of monotone functions. *Dokl. Math.*, 2011, **83** (2), pp. 241-242.