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TWO-WEIGHTED INEQUALITIES OF MAXIMAL COMMUTATORS IN THE MODIFIED MORREY SPACES, ASSOCIATED WITH THE LAPLACE-BESSEL DIFFERENTIAL OPERATOR

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Abstract. We consider the generalized shift operator, associated with the Laplace-Bessel differential operator $\Delta_B = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum\limits_{i=1}^{k} \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}$. The maximal commutators $M_{b,\gamma}$, associated with the generalized shift operator are investigated. At first, we prove that the maximal commutators is bounded from the modified Morrey space $\widetilde{L}_{p,\lambda,\varphi_1,\gamma}(\mathbb{R}^n_{k,+})$ to $\widetilde{L}_{p,\lambda,\varphi_1,\gamma}(\mathbb{R}^n_{k,+})$ for all $1 < p < \infty$, $b \in BMO_\gamma(\mathbb{R}^n_{k,+})$, $(\varphi_1, \varphi_2) \in \widetilde{A}_{p,\gamma}(\mathbb{R}^n_{k,+})$.

Keywords: generalized shift operator, commutator, maximal function, singular integral operator, modified Morrey space, BMO_γ space

Mathematics Subject Classification (2020): 42B20, 42B25, 42B35

1. Introduction

F. Chiarenza and M. Frasca [5] studied the boundedness of the maximal operator M in Morrey spaces.

Let b be a locally integrable function on $\mathbb{R}^n$ and T be a Calderon-Zygmund operator. The commutator is defined for smooth functions f by $[b,T]f = bT(f) - T(bf)$. Coifman, Rochberg and Weiss [6] stated that $[b,T]$ is a bounded operator on $L_p(\mathbb{R}^n), 1 < p < \infty$, when b is a BMO function. Chanillo [4] proved that commutators with Reisz potentials characterize the function space BMO.

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In the theory of partial differential equations Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ play an important role. They were introduced by C. Morrey in 1938 [21] and defined as follows: *For* $0 \le \lambda \le n$, $1 \le p < \infty$, $f \in L_{p,\lambda}(\mathbb{R}^n)$ *if* $f \in L_p^{loc}(\mathbb{R}^n)$ *and*

$$\|f\|_{L_{p,\lambda}} \equiv \|f\|_{L_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\ r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty.$$

If $\lambda = 0$, then $L_{p,\lambda}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, if $\lambda = n$, then $L_{p,\lambda}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$, if $\lambda < 0$ or $\lambda > n$, then $L_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

These spaces appeared to be quite useful in the study of the local behaviour of the solutions to elliptic partial differential equations, apriori estimates and other topics in the theory of partial differential equations.

Also by $WL_{p,\lambda}(\mathbb{R}^n)$ we denote the weak Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WL_{p,\lambda}} \equiv \|f\|_{WL_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\ r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(\mathbb{R}^n)$ denotes the weak L_p-space.

The maximal operator, potential and related topics associated with the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{k} \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad \gamma_1 > 0, \dots, \gamma_k > 0$$

have been investigated by many researchers, see [1]-[3], [8]-[12], [14], [16], [19], [22], [23].

2. Preliminaries

Let $1 \le k \le n$, $\mathbb{R}^n_{k,+} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n :\ x_1 > 0, \dots, x_k > 0\}$ and $E(x,r) = \{y \in \mathbb{R}^n_{k,+}\ ;\ |x-y| < r\}$, $\gamma = (\gamma_1, \dots, \gamma_k)$, $\gamma_1 > 0, \dots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \ldots + \gamma_k$, $(x')^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$. For a measurable set $E \subset \mathbb{R}^n_{k,+}$ suppose $|E|_\gamma = \int_E (x')^\gamma dx$, then $|E(0,r)|_\gamma = \omega(n,k,\gamma) r^Q$, $Q = n + |\gamma|$, where

$$\omega(n,k,\gamma) = \int\limits_{E(0,1)} (x')^\gamma dx = \frac{\pi^{\frac{n-k}{2}}}{2^k}\ \Gamma^{-1}\left(\frac{Q+2}{2}\right) \prod_{i=1}^{k} \Gamma\left(\frac{\gamma_i+1}{2}\right).$$

$L_{p,\gamma} \equiv L_{p,\gamma}(\mathbb{R}^n_{k,+})$ is the space of all classes of measurable functions f with finite norm

$$\|f\|_{L_{p,\gamma}} = \left(\int\limits_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx\right)^{\frac{1}{p}}, \quad 1 \le p < \infty$$

and $WL_{p,\gamma}(\mathbb{R}^n_{k,+})$, the weak $L_{p,\gamma}$ space defined as the set of all measurable functions f on $\mathbb{R}^n_{k,+}$ with the following finite norm

$$\|f\|_{WL_{p,\gamma}} = \sup_{r>0} r \left|\left\{x \in \mathbb{R}^n_{k,+} \ : \ |f(x)| > r\right\}\right|^{1/p}_{\gamma}, \quad 1 \le p < \infty.$$

For $p = \infty$ the space $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ is defined by means of the usual modification

$$\|f\|_{L_{\infty,\gamma}} = \|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n_{k,+}} |f(x)|.$$

Let $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ be the space of measurable functions on $\mathbb{R}^n_{k,+}$ with finite norm

$$\|f\|_{L_{p,\omega,\gamma}} = \|f\|_{L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})} = \left(\int\limits_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)(x')^\gamma dx\right)^{1/p}, \quad 1 \le p < \infty$$

and for $p = \infty$ the space $L_{\infty,\omega,\gamma}(\mathbb{R}^n_{k,+}) = L_\infty(\mathbb{R}^n_{k,+})$.

Definition 1. *The weight function φ belongs to the class $A_{p,\gamma}(\mathbb{R}^n_{k,+})$ for $1 \le p < \infty$, if*

$$\sup_{x\in\mathbb{R}^n_{k,+},\, r>0} \left(\frac{1}{|E(x,r)|_\gamma} \int\limits_{E(x,r)} \varphi^p(y)(y')^\gamma dy\right)^{\frac{1}{p}} \left(\frac{1}{|E(x,r)|_\gamma} \int\limits_{E(x,r)} \varphi^{-p'}(y)(y')^\gamma dy\right)^{\frac{1}{p'}} < \infty.$$

Definition 2. *The weight function (φ_1, φ_2) belongs to the class $\widetilde{A}_{p,\gamma}(\mathbb{R}^n_{k,+})$ for $1 < p < \infty$, if*

$$\sup_{x\in\mathbb{R}^n_{k,+},\, r>0} \left(\frac{1}{|E(x,r)|_\gamma} \int\limits_{E(x,r)} \varphi_2^p(y)(y')^\gamma dy\right)^{\frac{1}{p}} \left(\frac{1}{|E(x,r)|_\gamma} \int\limits_{E(x,r)} \varphi_1^{-p'}(y)(y')^\gamma dy\right)^{\frac{1}{p'}} < \infty.$$

The generalized shift operator T^y is defined by (see, for example [16], [18])

$$T^y f(x) = C_{\gamma,k} \int\limits_0^\pi \ldots \int\limits_0^\pi f\left((x', y')_\beta, x'' - y''\right) \, d\nu(\beta),$$

where $d\nu(\beta) = \prod\limits_{i=1}^k \sin^{\gamma_i - 1}\beta_i \quad d\beta_1 \ldots d\beta_k$, $x' = (x_1, \ldots, x_k) \in \mathbb{R}^n$, $x'' = (x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-k}$, $(x_i, y_i)_{\beta_i} = (x_i^2 - 2x_i y_i \cos\beta_i + y_i^2)^{\frac{1}{2}}$, $1 \le i \le k$, $(x', y')_\beta = ((x_1, y_1)_{\beta_1}, \ldots, (x_k, y_k)_{\beta_k})$ and

$$C_{\gamma,k} = \pi^{-\frac{k}{2}} \, \Gamma^{-1}\left(\frac{|\gamma|}{2}\right) \prod_{i=1}^k \Gamma\left(\frac{\gamma_i + 1}{2}\right) = \frac{2^{k-1}|\gamma|}{\pi} \left(\frac{|\gamma|}{2} + 1\right) \omega(2, k, \gamma).$$

It is well known that T^y is closely related to the Laplace-Bessel differential operator $\Delta_{B_{k,n}} = \sum\limits_{i=1}^{k} B_i + \sum\limits_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2}$, where $B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$, $i = 1, \ldots, k$. Furthermore, T^y generates the corresponding B -convolution

$$(f \otimes g)(x) = \int\limits_{\mathbb{R}^n_{k,+}} f(y)\; T^y g(x)(y')^\gamma dy.$$

Lemma 1. *Let $1 < p < \infty$, $\varphi \in A_{p,\gamma}(\mathbb{R}^n_{k,+})$, $g \in L_{p,\varphi,\gamma}(\mathbb{R}^n_{k,+})$. For all $x \in \mathbb{R}^n_{k,+}$ the following equality is valid.*

$$\int\limits_{\mathbb{R}^n_{k,+}} T^y |g(x)|^p \varphi(y)(y')^\gamma dy =$$

$$= \int\limits_{\mathbb{R}^n \times (0,\infty)^k} \left| g\left(\sqrt{z_1^2 + \overline{z}_1^2}, \ldots, \sqrt{z_k^2 + \overline{z}_k^2}, z''\right)\right|^p \varphi(z, \overline{z'}) d\nu(z, \overline{z'}).$$

Lemma 2. *Let $g \in L^{loc}_{1,\gamma}(\mathbb{R}^n_{k,+})$. For all $x \in \mathbb{R}^n_{k,+}$ the following equality*

$$\int\limits_{E(0,r)} T^y g(x)(y')^\gamma dy \quad = \quad \int\limits_{E((x,0),r)} g\left(\sqrt{z_1^2 + \overline{z}_1^2}, \ldots, \sqrt{z_k^2 + \overline{z}_k^2}, z''\right) d\nu(z, \overline{z'}),$$

holds, where $E((x,0), r) = \{(z, \overline{z'}) \in \mathbb{R}^n \times (0, \infty)^k \;:\; \left|(x - z, \overline{z'})\right| < r\}$.

The proof of Lemmas 1 and 2 is done straightforwardly via the following substitutions

$$z'' = y'', z_i = y_i \cos\alpha_i, \quad \overline{z_i} = y_i \sin\alpha_i, \quad 0 \le \alpha_i < \pi, \; i = 1, \ldots, k,$$
$$y \in \mathbb{R}^n_{k,+}, \; \overline{z'} = (\overline{z}_1, \ldots, \overline{z}_k), \; (z, \overline{z'}) \in \mathbb{R}^n \times (0, \infty)^k, \; 1 \le k \le n.$$

Definition 3. *Let $1 \le p < \infty$, $0 \le \lambda \le Q$ and $[t]_1 = \min\{1, t\}$. We denote by $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ the modified B-Morrey space and $\widetilde{L}_{p,\lambda,\varphi,\gamma}(\mathbb{R}^n_{k,+})$ the modified weighted B-Morrey space, associated to the Laplace-Bessel differential operator as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_{k,+}$, with finite norm*

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} = \sup_{t>0,\, x \in \mathbb{R}^n_{k,+}} \left([t]_1^{-\lambda} \int_{E_t} T^y |f(x)|^p (y')^\gamma dy\right)^{1/p},$$

$$\|f\|_{\widetilde{L}_{p,\lambda,\varphi,\gamma}} = \sup_{t>0,\, x \in \mathbb{R}^n_{k,+}} \left([t]_1^{-\lambda} \int_{E_t} T^y |f(x)|^p \varphi(y)(y')^\gamma dy\right)^{1/p}.$$

Note that

$$WL_{p,\gamma}(\mathbb{R}^n_{k,+}) = W\widetilde{L}_{p,0,\gamma}(\mathbb{R}^n_{k,+}),$$
$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \subset W\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \;\text{ and }\; \|f\|_{W\widetilde{L}_{p,\lambda,\gamma}} \le \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.$$

The sharp maximal function $M_\gamma^\sharp$ is defined by

$$M_\gamma^\sharp f(x) = \sup_{t>0} |E(0,t)|_\gamma^{-1} \int\limits_{E(0,t)} |T^y f(x) - f_{E(0,t)}(x)|(y')^\gamma dy,$$

where $f_{E(0,t)}(x) = |E(0,t)|_\gamma^{-1} \int\limits_{E(0,t)} T^y f(x)(y')^\gamma dy.$

$B-BMO$ space, $BMO_\gamma(\mathbb{R}^n_{k,+})$, defined as the space of locally integrable functions f with finite norm

$$\|f\|_{BMO_\gamma} = \sup_{t>0,\ x\in\mathbb{R}^n_{k,+}} |E(0,t)|_\gamma^{-1} \int\limits_{E(0,t)} |T^y f(x) - f_{E(0,t)}(x)|(y')^\gamma dy$$

or

$$\|f\|_{BMO_\gamma} = \inf_C \sup_{t>0,\ x\in\mathbb{R}^n_{k,+}} |E(0,t)|_\gamma^{-1} \int\limits_{E(0,t)} |T^y f(x) - C|(y')^\gamma dy.$$

The following theorem holds.

Theorem 1. *[13] 1) Let $f \in L^{loc}_{1,\gamma}(\mathbb{R}^n_{k,+})$. If*

$$\sup_{t>0,\ x\in\mathbb{R}^n_{k,+}} \left(|E(0,t)|_\gamma^{-1} \int\limits_{E(0,t)} |T^y f(x) - f_{E(0,t)}(x)|^p (y')^\gamma dy \right)^{1/p} = \|f\|_{BMO_{p,\gamma}} < \infty,$$

then for any $1 < p < \infty$

$$\|f\|_{BMO_\gamma} \le \|f\|_{BMO_{p,\gamma}} \le A_p \|f\|_{BMO_\gamma},$$

where the constant A_p depends only on p.

2) Let $f \in BMO_\gamma(\mathbb{R}^n_{k,+})$. Then, there is a constant $C > 0$ such that

$$|f_{E(0,r)} - f_{E(0,t)}| \le C\|f\|_{BMO_\gamma} \ln\frac{t}{r}, \qquad 0 < 2r < t,$$

where C is independent of f, x, r and t.

Lemma 3. *Let $1 < p < \infty$, $\varphi \in A_{p,\gamma}(\mathbb{R}^n_{k,+})$, $b \in BMO_\gamma(\mathbb{R}^n_{k,+})$. Then*

$$\|b\|_{BMO_\gamma} \approx \sup_{x\in\mathbb{R}^n_{k,+}, r>0} \frac{\left\|T^{\cdot} b(x) - b_{E(0,r)}\right\|_{L_{p,\varphi,\gamma}(E(0,r))}}{\|\varphi\|_{L_{p,\gamma}(E(0,r))}}.$$

Proof. From Hölder's inequality, we get

$$\|b\|_{BMO_\gamma} \lesssim \sup_{x\in\mathbb{R}^n_{k,+}, r>0} \frac{\left\|T^{\cdot}b(x)-b_{E(0,r)}\right\|_{L_{p,\varphi,\gamma}(E(0,r))}}{\|\varphi\|_{L_{p,\gamma}(E(0,r))}}.$$

Now we obtain that

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{\left\|T^{\cdot}b(x)-b_{E(0,r)}\right\|_{L_{p,\varphi,\gamma}(E(0,r))}}{\|\varphi\|_{L_{p,\gamma}(E(0,r))}} \lesssim \|b\|_{BMO_\gamma}.$$

We can assume without loss of generality that $\|b\|_{BMO_\gamma}=1$; otherwise, we replace b by $b/\|b\|_{BMO_\gamma}$ it follows that

$$\int_{E(0,r)} \left(\frac{|T^{\cdot}b(x)-b_{E(0,r)}|\varphi(y)}{\|b\|_{BMO_\gamma}}\right)^p dy =$$

$$= \int_{E(0,r)} \left(|T^{\cdot}b(x)-b_{B(x,r)}|\varphi(y)\right)^p dy \lesssim 1.$$

◀

Now we define the B-maximal operator by

$$M_\gamma f(x) = \sup_{r>0} |E(0,r)|_\gamma^{-1} \int\limits_{E(0,r)} T^y[|f|](x)(y')^\gamma dy.$$

The following theorem is about the boundedness of the B-maximal operator in $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ spaces which was proved in [7].

Theorem 2. *1. If $f\in L_{1,\omega,\gamma}(\mathbb{R}^n_{k,+})$, $\omega\in A_{1,\gamma}(\mathbb{R}^n_{k,+})$, then $M_\gamma f\in WL_{1,\omega,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|M_\gamma f\|_{WL_{1,\omega,\gamma}} \le C_1\|f\|_{L_{1,\omega,\gamma}},$$

where C_1 depends only on ω,γ and n.

2. If $f\in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$, $\omega\in A_{p,\gamma}(\mathbb{R}^n_{k,+})$, $1<p<\infty$, then $M_\gamma f\in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|M_\gamma f\|_{L_{p,\omega,\gamma}} \le C_2\|f\|_{L_{p,\omega,\gamma}},$$

where C_2 depends only on p,ω,γ and n.

3. Main Results

Also, in the works [17], [20] it was proved:

Proposition. *Let $1 \le p < \infty$, $(\varphi, \varphi_1) \in \widetilde{A}_p(Y)$. Then M_ν is bounded from $f \in L_{p,\varphi_1}(Y)$ to $f \in L_{p,\varphi_2}(Y)$, where (Y, d, ν) homogeneous type space.*

Theorem 3. *Let $1 < p < \infty$, $0 \le \lambda < Q$ and $(\varphi_1, \varphi_2) \in \widetilde{A}_{p,\gamma}(\mathbb{R}^n_{k,+})$. Then M_γ is bounded from $\widetilde{L}_{p,\lambda,\varphi_1,\gamma}(\mathbb{R}^n_{k,+})$ to $\widetilde{L}_{p,\lambda,\varphi_2,\gamma}$.*

Proof. We need to introduce the maximal operator defined on a space of homogeneous type (Y, d, ν). By this we mean a topological space $Y = \mathbb{R}^n \times (0, \infty)^k$ equipped with a continuous pseudometric d and a positive measure ν satisfying

$$\nu(E((x, \overline{x'}), 2r)) \le C_1 \nu(E((x, \overline{x'}), r)) \tag{1}$$

with a constant C_1 independent of $(x, \overline{x'})$ and $r > 0$. Here $E((x, \overline{x'}), r) = \{(y, \overline{y'}) \in Y : d(((x, \overline{x'}), (y, \overline{y'})) < r\}$, $d\nu(y, \overline{y'}) = (\overline{y'})^{\gamma-1} dy\, d\overline{y'}$, $(\overline{y'})^{\gamma-1} = (\overline{y_1})^{\gamma_1 - 1} \cdots (\overline{y_k})^{\gamma_k - 1}$, $d((x, \overline{x'}), (y, \overline{y'})) = |(x, \overline{x'}) - (y, \overline{y'})| \equiv (|x - y|^2 + (\overline{x'} - \overline{y'})^2)^{\frac{1}{2}}$.

Let (Y, d, ν) be a space of homogeneous type. Define

$$M_\nu \overline{f}(x, \overline{x'}) = \sup_{r>0} \nu(E((x, \overline{x'}), r))^{-1} \int_{E((x,\overline{x'}),r)} \left|\overline{f}(y, \overline{y'})\right| d\nu(y),$$

where $\overline{f}(x, \overline{x'}) = f\left(\sqrt{x_1^2 + \overline{x}_1^2}, \ldots, \sqrt{x_k^2 + \overline{x}_k^2}, x''\right)$.

It is well known that the fractional maximal operator M_ν is bounded on $L_{p,\psi_1}(Y, d\nu)$ to $L_{p,\psi_2}(Y, d\nu)$ for $1 < p < \infty$, $(\psi_1, \psi_2) \in \widetilde{A}_p(Y)$ (see [15]). Here we are concerned with the fractional maximal operator defined by $d\nu(y, \overline{y'}) = (\overline{y'})^{\gamma-1} dy\, d\overline{y'}$. It is clear that this measure satisfies the doubling condition (1).

It can be proved that

$$M_\gamma f\left(\sqrt{z_1^2 + \overline{z}_1^2}, \ldots, \sqrt{z_k^2 + \overline{z}_k^2}, z''\right) = M_\nu \overline{f}\left(\sqrt{z_1^2 + \overline{z}_1^2}, \ldots, \sqrt{z_k^2 + \overline{z}_k^2}, z'', 0\right) \tag{2}$$

and

$$M_\gamma f(x) = M_\nu \overline{f}(x, 0). \tag{3}$$

Indeed, Lemma 1 and $\psi_1(y) = \varphi_1(y)(M_\nu \chi_{E((x,0),r)}(y))^\theta$, $\psi_2(y) = \varphi_2(y)(M_\nu \chi_{E((x,0),r)}(y))^\theta$, for any $0 < \theta < 1$, $(\psi_1, \psi_2) \in \widetilde{A}_p(Y)$, we have

$$\int_{\mathbb{R}^n_{k,+}} T^y |f(x)|^p \varphi_1(y)(M_\gamma \chi_{E_r}(y))^\theta (y')^\gamma\, dy =$$

$$= \int_Y \left|\overline{f}\left(\sqrt{y_1^2 + \overline{y}_1^2}, \ldots, \sqrt{y_k^2 + \overline{y}_k^2}, y'', 0\right)\right|^p \varphi_1(y, \overline{y'})(M_\nu \chi_{E((x,0),r)}(y, \overline{y'}))^\theta\, d\nu(y, \overline{y'})$$

and

$$|E_r|_\gamma = \nu E\left(\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\right),r\right)$$

imply (2). Furthermore, taking $\overline{z}_k = 0$ in (2) we get (3).

Using Lemma 1 and equality (2) we have

$$\int_{E_r} T^y\left(M_\gamma f(x)\right)^p \varphi_2(y)(y')^\gamma\, dy \le$$

$$\le \int_{\mathbb{R}^n_{k,+}} T^y\left(M_\gamma f(x)\right)^p \varphi_2(y)(M_\gamma \chi_{E_r}(y))^\theta (y')^\gamma\, dy =$$

$$= \int_{\mathbb{R}^n\times(0,\infty)^k}\left(M_\gamma f\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2},z''\right)\right)^p \times$$

$$\times \varphi_2(z,\overline{z'})(M_\gamma\chi_{E((x,0),r)}(z,\overline{z'}))^\theta\, d\nu(z,\overline{z'}) =$$

$$= \int_Y\left(M_\nu \overline{f}\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\right)\right)^p \times$$

$$\times \varphi_2(z,\overline{z'})(M_\nu\chi_{E((x,0),r)}(z,\overline{z'}))^\theta\, d\nu(z,\overline{z'}).$$

By the Proposition we have

$$\left(\int_{E_r} T^y\left(M_\gamma f(x)\right)^p \varphi_2(y)(y')^\gamma dy\right)^{\frac{1}{p}} \le$$

$$\le \Bigg(\int_Y\left(M_\nu \overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\dots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right)^p \times$$

$$\times \varphi_2(y,\overline{y'})(M_\nu\chi_{E((x,0),r)}(y,\overline{y'}))^\theta\, d\nu(y,\overline{y'})\Bigg)^{\frac{1}{p}} =$$

$$= \left(\int_Y\left(M_\nu \overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\dots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right)^p \psi_2(y,\overline{y'})\, d\nu(y,\overline{y'})\right)^{\frac{1}{p}} \le$$

$$\le C_2\left(\int_Y\left|\overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\dots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right|^p \psi_1(y,\overline{y'})\, d\nu(y,\overline{y'})\right)^{\frac{1}{p}} =$$

$$= C_2\Bigg(\int_Y\left|\overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\dots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right|^p \times$$

$$\times \varphi_1(y,\overline{y'})(M_\nu\chi_{E((x,0),r)}(y,\overline{y'}))^\theta\, d\nu(y,\overline{y'})\Bigg)^{\frac{1}{p}} =$$

$$= C_2\Bigg(\int_Y\left|f\left(\sqrt{y_1^2+\overline{y}_1^2},\dots,\sqrt{y_k^2+\overline{y}_k^2},y''\right)\right|^p \times$$

$$\times \varphi_1(y,\overline{y'})(M_\nu\chi_{E((x,0),r)}(y,\overline{y'}))^\theta\, d\nu(y,\overline{y'})\Bigg)^{\frac{1}{p}=}$$

$$= C_2 \left(\int_{\mathbb{R}^n_{k,+}} T^y[|f|\ ]^p(x)\varphi_1(y)(M_\gamma\chi_{E_r}(y))^\theta (y')^\gamma dy \right)^{\frac{1}{p}} \le$$

$$\le C_2 \left(\int_{E_r} T^y[|f|\ ]^p(x)\varphi_1(y)(y')^\gamma dy + \right.$$

$$\left. + \sum_{j=1}^{\infty} \int_{E_{2^{j+1}r}\setminus E_{2^j r}} T^y[|f|\ ]^p(x)\varphi_1(y)(M_\gamma\chi_{E_r}(y))^\theta (y')^\gamma dy \right)^{\frac{1}{p}} \le$$

$$\le C_2 \left(\int_{E_r} T^y[|f|\ ]^p(x)\varphi_1(y)(y')^\gamma dy + \right.$$

$$\left. + \sum_{j=1}^{\infty} \int_{E_{2^{j+1}r}\setminus E_{2^j r}} T^y[|f|\ ]^p(x)\varphi_1(y)\frac{r^{Q\theta}}{(|y|+r)^{Q\theta}}(y')^\gamma dy \right)^{\frac{1}{p}} \le$$

$$\le C_3\ \|f\|_{\widetilde{L}_{p,\omega_1,\varphi_1,\gamma}} \left([r]_1^\lambda + \sum_{j=1}^{\infty} \frac{1}{(2^j+1)^{Q\theta}}[2^{j+1}r]_1^\lambda \right)^{\frac{1}{p}} \le C_3\ [r]_1^{\frac{\lambda}{p}} \|f\|_{\widetilde{L}_{p,\omega_1,\varphi_1,\gamma}}.$$

Then

$$\|M_\gamma f\|_{\widetilde{L}_{p,\lambda,\varphi_2,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\ t>0} t^{-\frac{\lambda}{p}} \|T^{\cdot}(M_\gamma f(x))\|_{L_{p,\varphi_2,\gamma}(E_t)} \le C_4 \|f\|_{\widetilde{L}_{p,\lambda,\varphi_1,\gamma}}.$$

◄

If $\lambda = 0$, then we get the following result from Theorem 3:

Corollary. *Let $1 < p < \infty$ and $(\varphi_1, \varphi_2) \in \widetilde{A}_{p,\gamma}(\mathbb{R}^n_{k,+})$, then the operator M_γ is bounded from $L_{p,\varphi_1,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\varphi_2,\gamma}(\mathbb{R}^n_{k,+})$.*

The commutator generated by the B-maximal operator M_γ for a given measurable function b is formally defined by

$$[M_\gamma, b]f = M_\gamma(bf) - bM_\gamma(f)$$

and for a given measurable function b, the B-maximal commutator is defined by

$$M_{b,\gamma}(f)(x) := \sup_{r>0} |E(0,r)|_\gamma^{-1} \int\limits_{E(0,r)} T^y |(b(x)-b(y))f(x)|(y')^\gamma dy$$

for all $x \in \mathbb{R}^n_{k,+}$.

Lemma 4. *[13] Let $1 < s < \infty$ and $b \in BMO(\mathbb{R}^n_{k,+})$. Then there exists $C > 0$ such that for all $x \in \mathbb{R}^n_{k,+}$*

$$M_\gamma(M_{b,\gamma}f)(x) \le C\|b\|_{BMO_\gamma} \left((M_\gamma(M_\gamma f)^s)^{\frac{1}{s}}(x) + M_\gamma\left(M_\gamma|f|^s\right)^{\frac{1}{s}}(x) \right)$$

holds.

Theorem 4. *Let $b \in BMO_\gamma(\mathbb{R}^n_{k,+})$, $1 < p < \infty$ and $(\varphi_1, \varphi_2) \in \widetilde{A}_{p,\gamma}(\mathbb{R}^n_{k,+})$, $\varphi_1 \in A_{p,\gamma}(\mathbb{R}^n_{k,+})$. Then $M_{b,\gamma}$ is bounded from $L_{p,\varphi_1,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\varphi_2,\gamma}$.*

Proof. By using Lemma 4 and Corollary, we have $M_{b,\gamma}$ is bounded from $L_{p,\varphi_1,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\varphi_2,\gamma}$. ◀

Operators $M_{b,\gamma}$ and $[M_\gamma, b]$ are essentially different from each other. For example, $M_{b,\gamma}$ is a positive and sublinear operator, but $[M_\gamma, b]$ is neither positive nor sublinear. However, if b satisfies some additional conditions, then the operator $M_{b,\gamma}$ is controlled by $[M_\gamma, b]$.

Theorem 5. *Let* $1 < p < \infty$, $0 \le \lambda < Q$, $b \in BMO_\gamma(\mathbb{R}^n_{k,+})$ *and* $(\varphi_1, \varphi_2) \in \widetilde{A}_{p,\gamma}(\mathbb{R}^n_{k,+})$, $\varphi_1 \in A_{p,\gamma}(\mathbb{R}^n_{k,+})$. *Then the commutator* $M_{b,\gamma}$ *is bounded from* $\widetilde{L}_{p,\lambda,\varphi_1,\gamma}(\mathbb{R}^n_{k,+})$ *to* $\widetilde{L}_{p,\lambda,\varphi_2,\gamma}$.

Proof. *Sufficiency:* Let $1 < p < \infty$, $0 \le \lambda < Q$, $f \in \widetilde{L}_{p,\lambda,\varphi_1,\gamma}(\mathbb{R}^n_{k,+})$. We have

$$\int\limits_{E(0,t)} T^y \left[M_{b,\gamma} f\right]^p(x) \varphi_2(y)(y')^\gamma dy \le$$

$$\le \int\limits_{\mathbb{R}^n_{k,+}} T^y \left[M_{b,\gamma} f\right]^p(x) \varphi_2(y) (M_\gamma \chi_{E(0,t)}(y))^\delta (y')^\gamma dy, \;\; x \in \mathbb{R}^n_{k,+}.$$

Taking into account the properties of $\widetilde{A}_{p,\gamma}(\mathbb{R}^n_{k,+})$ we can easily see that $\varphi_1(y)(M_\nu \chi_{E((x,0),r)}(y))^\theta$, $\varphi_2(y)(M_\nu \chi_{E((x,0),r)}(y))^\theta$, for any $0 < \theta < 1$. Then by using Theorem 4 we obtain

$$\int\limits_{E(0,t)} T^y \left[M_{b,\gamma} f\right]^p(x) \varphi_2(y)(y')^\gamma dy \le$$

$$\le C \;\|b\|^p_{BMO_\gamma} \int\limits_{\mathbb{R}^n_{k,+}} T^y[|f|\;]^p(x) \varphi_1(y) (M_\gamma \chi_{E(0,r)}(y))^\delta (y')^\gamma dy \le$$

$$\le C \;\|b\|^p_{BMO_\gamma} \int\limits_{E(0,r)} T^y[|f|\;]^p(x) \varphi_1(y)(y')^\gamma dy +$$

$$+C \,\|b\|^p_{BMO_\gamma} \sum_{j=1}^{\infty} \int\limits_{E(0,2^{j+1}r)\setminus E(0,2^j r)} T^y[|f|\;]^p(x) \varphi_1(y) (M_\gamma \chi_{E(0,r)}(y))^\theta (y')^\gamma dy \le$$

$$\le C \;\|b\|^p_{BMO_\gamma} \int\limits_{E(0,r)} T^y[|f|\;]^p(x) \varphi_1(y)(y')^\gamma dy +$$

$$+C \,\|b\|^p_{BMO_\gamma} \sum_{j=1}^{\infty} \int\limits_{E(0,2^{j+1}r)\setminus E(0,2^j r)} T^y[|f|\;]^p(x) \varphi_1(y) \frac{r^{Q\theta}}{(|y|+r)^{Q\theta}} (y')^\gamma dy \le$$

$$\le C \;\|b\|^p_{BMO_\gamma} \|f\|^p_{\widetilde{L}_{p,\lambda,\varphi_1,\gamma}} \left([r]_1^\lambda + \sum_{j=1}^{\infty} \frac{1}{(2^j+1)^{Q\theta}} [2^{j+1} r]_1^\lambda \right) \le$$

$$\le C \; [r]_1^\lambda \, \|b\|^p_{BMO_\gamma} \, \|f\|^p_{\widetilde{L}_{p,\lambda,\varphi_1,\gamma}}.$$

◀

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