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ELASTIC-PLASTIC PROBLEM FOR A PERFORATED PLATE UNDER TRANSVERSAL SHEAR

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Abstract. *The problem of solving the problem of the transverse shear of a plate along the edges of the holes and weakened by a two-periodic system of rectilinear through cracks with plastic end zones, collinear to the abscissa and ordinate axes of unequal length, is considered. General representations of solutions are constructed that describe a class of problems with a doubly periodic distribution of moments outside circular holes and straight-line cracks with end zones of plastic deformations. Satisfying the boundary conditions, the solution of the problem of the plate shear theory is reduced to two infinite systems of algebraic equations and two singular integral equations. Then each singular integral equation is reduced to a finite system of linear algebraic equations.*

Keywords: perforated plate, straight cracks with end zones, transverse shear, plastic deformation zones

Mathematics Subject Classification (2020): 74S70, 65E05

1. Introduction

Consider an isotropic elastic plate weakened by a doubly periodic system of rectilinear through cracks collinear to the abscissa and ordinate axes. The plate material is assumed to be elastic-ideally plastic, satisfying the Tresca-Saint-Venant plasticity condition. The action of the transverse shear will stimulate the occurrence of plastic deformation zones at the crack tips. Let us consider the problem of the initial development of plastic deformations on the continuation of cracks. The study of the stress-strain state of a plate with a doubly periodic system of circular holes with rectilinear through cracks shows that the first zones of plastic deformation will appear on the continuation of crack

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lines. In accordance with the Leonov-Panasyuk-Dagdale model [9], the plastic zone will be a narrow layer on the continuation of cracks. It has been experimentally shown that the plastic deformation zones will be segments located on the continuation of the crack. In plates, plastic deformation zones can physically be realized in the form of a slip plane.

The crack edges outside the end zones are free from external loads. It is assumed that in the process of deformation of an isotropic plate, the opposite sides of the cracks do not come into contact with each other. It is required to determine the stress-strain state of an isotropic plate by the boundary conditions, which express the absence of displacements along the contour of circular holes and external loads on the edges of a doubly periodic system of rectilinear through cracks, taking into account plastic deformations in the continuation of cracks.

2. Formulation of the Problem

A plane problem of the theory of elasticity for an isotropic plate is considered, a doubly periodic lattice with circular holes having a radius λ ($\lambda < 1$) and centers at the points is weakened

$$P_{mn} = m\omega_1 + n\omega_2 \quad (m, n = 0, \pm 1, \pm 2, ...),$$

$$\omega_1 = 2, \quad \omega_2 = \omega_1 \cdot he^{i\alpha}, \quad h > 0, \quad Im\omega_2 > 0.$$

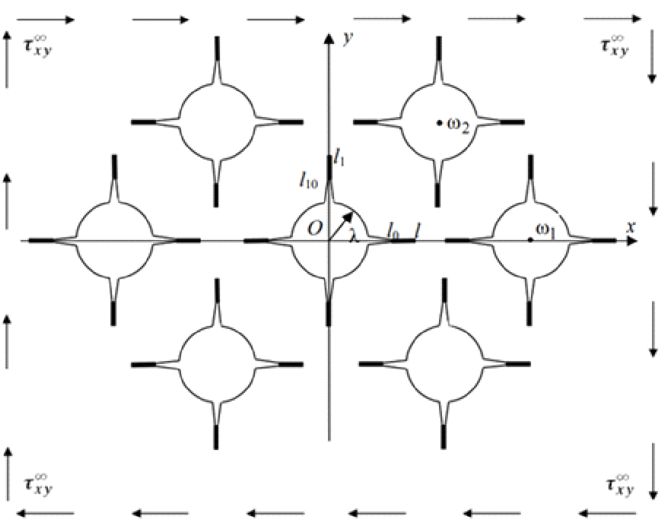

Fig. 1. Calculation scheme of the problem of crack development in a perforated body under transverse shear

The action of the transverse load will stimulate the occurrence of plastic deformation zones at the crack tips (Fig. 1). The contours of the circular holes and the edges of the cracks are free from loads. We will consider the material of the perforated plate to be ideal-elasto-plastic Tresca-Saint-Venant, according to which the maximum shear stress at each point of the body does not exceed the shear yield strength τ_s ($\tau_{xy} = \tau_s$, $\sigma_x = 0$, $\sigma_y = 0$ where τ_s is the transverse shear yield strength). Consider the problem of initial development plastic deformations during transverse shear of the plate by forces τ_{xy}^{∞}.

As the intensity of the external load increases in the body around the holes, zones of increased stresses are formed, the location of which has a doubly periodic character. The first plasticity bands, according to the Leonov–Panasyuk–Dugdale scheme [9], will develop along the lines along the continuation of the cracks. Due to the symmetry of the boundary conditions and the geometry of the region D occupied by the plate material, stresses are doubly periodic functions with basic periods ω_1 and ω_2.

The contours of the circular holes and the edges of the cuts are free from external forces [4], [5], [9]

$\sigma_r - i\tau_{r\theta} = 0$ on the contours of the holes

$\sigma_y - i\tau_{xy} = 0$ on the banks of cracks.

3. Method for Solving the Problem

Stresses and displacements in the plane theory of elasticity can be represented [10] in terms of two analytical functions of the complex variable $z = x + iy$ $\Phi(z)$ and $\Psi(z)$ using the Kolosov-Muskhelishvili formulas:

$$\sigma_y + \sigma_x = \sigma_r + \sigma_\theta = 2\left[\Phi(z) + \overline{\Phi(z)}\right],$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = e^{-2i\theta}(\sigma_\theta - \sigma_r + 2i\tau_{r\theta}) = 2\left[\bar{z}\Phi'(z) + \Psi(z)\right],$$

$$2\mu(u + iv) = k\varphi(z) - z\overline{\Phi(z)} - \overline{\psi(z)},$$

$$\varphi'(z) = \Phi(z), \psi'(z) = \Psi(z),$$

where μ is the shear modulus of the material; v is Poisson's ratio; $k = 3 - 4v$ for plane deformation; $k = (3 - v)/(1 + v)$ for a plane stress state; r, θ polar coordinates.

Based on the Kolosov-Muskhelishvili formulas [10] and the boundary conditions on the contours of circular holes and crack faces, as well as the stress distribution on the plasticity bands, the problem is reduced to determining two functions $\Phi(z)$ and $\Psi(z)$ analytic in the region D:

Based on the contours of the circular holes, we have:

$$\Phi(\tau) + \overline{\Phi(\tau)} - \left[\bar{\tau}\Phi'(\tau) + \Psi(\tau)\right]e^{2i\theta} = 0. \tag{1}$$

The boundary conditions on the faces of cracks and plastic deformation zones have the form:

$$\Phi(t) + \overline{\Phi(t)} + t\overline{\Phi'(t)} + \overline{\Psi(t)} = a \quad \text{on} \quad L_1, \tag{2}$$

$$\Phi(t_1) + \overline{\Phi(t_1)} + t_1\overline{\Phi'(t_1)} + \overline{\Psi(t_1)} = a_* \quad \text{on} \quad L_2, \tag{3}$$

where $\tau = \lambda e^{i\theta} + m\omega_1 + n\omega_2$, $m, n = 0, \pm1, \pm2, ...$, t and t_1 are affixes of points of crack edges directed collinear to the abscissa and ordinate axes, respectively; $a = i\tau_s$ and $a_* = i\tau_s$ plasticity bands directed collinear to the abscissa and ordinate axes, respectively; L_1 and L_2 are a set of crack edges and plastic deformation zones, collinear, respectively, to the abscissa and ordinate axes.

4. Solution of the Boundary Value Problem

The solution of the boundary value problem (1) – (3) will be sought in the form [7]

$$\varPhi(z)=\varPhi_0(z)+\varPhi_1(z)+\varPhi_2(z),$$
$$\varPsi(z)=\varPsi_0(z)+\varPsi_1(z)+\varPsi_2(z), \tag{4}$$
$$\varPhi_0(z)=\phi_0', \quad \varPsi_0(z)=\chi_0'',$$

$$\phi_0(z)=\phi_{01}(z)+\phi_{10}(z), \quad \chi_0(z)=\chi_{01}(z)+\chi_{10}(z),$$
$$\phi_{01}(z)=A_1z+A_2z^3+A_0\xi(z)-\alpha_2\lambda^2\zeta(z), \tag{5}$$
$$\chi_{01}(z)=B_0+B_1z^2+B_2z^4-A_0\xi_*(z)-\beta_2\lambda^2\nu(z)+\alpha_2\lambda^2\zeta_*(z),$$
$$\nu(z)=\int\zeta(z)\,dz, \quad \xi(z)=\int\nu(z)\,dz, \quad \zeta_*(z)=-\int Q(z)\,dz,$$
$$\xi_*(z)=-\int\nu_*(z)\,dz, \quad \nu_*(z)=-\int\zeta_*(z)\,dz,$$

$$\phi_{10}(z)=i\tau_{xy}^{\infty}+i\sum_{k=0}^{\infty}\alpha_{2k+2}\frac{\lambda^{2k+2}\rho^{(2k)}(z)}{(2k+1)!}, \tag{6}$$
$$\chi_{10}(z)=i\tau_{xy}^{\infty}+i\sum_{k=0}^{\infty}\beta_{2k+2}\frac{\lambda^{2k+2}\gamma^{(2k)}(z)}{(2k+1)!}-i\sum_{k=0}^{\infty}\alpha_{2k+2}\frac{\lambda^{2k+2}S^{(2k)}(z)}{(2k+1)!},$$

$$\varPhi_1(z)=\frac{1}{2\omega}\int_{L_1}g(t)\,\zeta(t-z)\,dt,$$
$$\varPsi_1(z)=-\frac{\pi z}{2\omega^2}\int_{L_1}g(t)\left[\zeta(t-z)+S(t-z)-t\gamma(t-z)\right]dt, \tag{7}$$
$$\varPhi_2(z)=\frac{i}{2\omega}\int_{L_2}g_1(t_1)\,\zeta(it_1-z)\,dt_1+A',$$
$$\varPsi_2(z)=-\frac{i}{2\omega}\int_{L_2}\left\{\overline{g_1(t_1)}\zeta(it_1-z)-\left[S(it_1-z)+it_1\gamma(it_1-z)\right]g_1(t_1)\right\},$$

where

$$\rho(z)=\frac{1}{z}+\left[\frac{1}{z-P_{mn}}+\frac{1}{P_{mn}}+\frac{z}{P_{mn}^2}\right], \quad \gamma(z)=\frac{1}{z^2}+\sum_{m,n}{}'\left[\frac{1}{(z-P_{mn})^2}-\frac{1}{P_{mn}^2}\right],$$

$$\frac{\gamma^{(2k)}(z)}{(2k+1)!}=\frac{1}{z^{2k+2}}+\sum_{m,n}{}'\frac{1}{(z-P_{mn})^{2k+2}}-\frac{1}{P_{mn}^2} \quad (k=0,1,2,...),$$

$$S(z)=\sum_{m,n}{}'\left[\frac{P_{mn}}{(z-P_{mn})^2}-\frac{2z}{P_{mn}}-\frac{1}{P_{mn}}\right].$$

The prime at the sum means that the index $m = 0$ is excluded during summation; the integrals in (7) are taken along the lines $L_1 = \{[-\ell, -\lambda] + [\lambda, \ell]$ and $L_2 = [-l_1,\ -\lambda] + [\lambda,\ l_1]\}$; $g(x)$, $g_1(y)$ are the desired functions that characterize the shift of the crack faces with end zones, respectively, A and B are constants.

Let us now present the dependences that the coefficients of expressions (6), (7) must satisfy. From the symmetry conditions with respect to the coordinate axes, we find that [11]

$$Im\alpha_{2k} = 0,\quad Im\beta_{2k} = 0,\quad k = 1, 2, \dots .$$

The conditions for the constancy of the main vector of all forces acting on the arc connecting two congruent points in D, the properties of functions $\rho\,(z)\,,\ \gamma\,(z)\,,\ S\,(z)$ at congruent points leads to the relations [1]

$$A - \overline{A} - \overline{B} = -\frac{1}{\omega_1}\left[\left(\overline{\delta_1} + \delta_1\right) b - \alpha_2\lambda^2\left(\delta_1 - \overline{\gamma_1}\right) + \beta_2\lambda^{2\delta_1}\right], \tag{8}$$

$$b = \frac{1}{2\pi i}\int_L tg\,(t)\,dt.$$

From system (8) the constants A and B are determined, and the quantities A and B are real. Each of the constants A and B is further conveniently represented as the sum of two constants $A = A_* + A_{**}$, $B = B_* + B_{**}$, of which A_{**} and B_{**} depend on the coefficients α_2, β_2 i.e. they are found from system (8) under the assumption that $b = 0$.

The unknown functions $g\,(x)$and $g_1\,(y)$, and the constants α_{2k} and β_{2k} must be determined from the boundary conditions (1)–(3). Complex representations (4) define a class of problems with a periodic stress distribution. Due to the fulfillment of the doubly periodicity condition, the system of boundary conditions (3) degenerates into one functional equation, for example, on the contour $L_{0,0}\left(\tau = \lambda e^{i\theta}\right)$, and the system of boundary conditions (2), (3) degenerates into a boundary condition on L_1 and L_2 [3].

Additional conditions should be added to the main representations of the considered problem (4)-(7)

$$\int_{-l}^{-\lambda} g\,(t)\,dt = 0,\quad \int_{\lambda}^{l} g\,(t)\,dt = 0,\quad \int_{-l_1}^{-\lambda} g_1\,(t_1)\,dt_1 = 0,\quad \int_{\lambda}^{l_1} g\,(t_1)\,dt_1 = 0. \tag{9}$$

These conditions ensure the uniqueness of the angles of rotation of the median plane when bypassing the contours of cracks [5].

To formulate equations for the unknown coefficients α_{2k}, β_{2k}, we transform the boundary condition (1) as follows

$$\varPhi_0(\tau) + \overline{\varPhi_0(\tau)} - e^{2i\theta}\left[\bar{\tau}\varPhi_0'(\tau) + \varPsi_0(\tau)\right] = f_1\,(\theta) + if_2\,(\theta) + \varphi_1\,(\theta) + i\varphi_2\,(\theta)\,, \tag{10}$$

$$f_1\,(\theta) + if_2\,(\theta) = -\varPhi_1(\tau) - \overline{\varPhi_1(\tau)} + e^{2i\theta}\left[\bar{\tau}\varPhi_1'(\tau) + \varPsi_1(\tau)\right]\ ,$$

$$\varphi_1\,(\theta) + i\varphi_2\,(\theta) = -\varPhi_2(\tau) - \overline{\varPhi_2(\tau)} + e^{2i\theta}\left[\bar{\tau}\varPhi_2'(\tau) + \varPsi_2(\tau)\right].$$

Regarding the functions $f_1(\theta)+if_2(\theta)$ and $\varphi_1(\theta)+i\varphi_2(\theta)$, we will assume that they are expanded on the contour $|\tau|=\lambda$ into Fourier series. Based on antisymmetry, these series have the following form

$$f_1(\theta)+if_2(\theta)=\sum_{k=-\infty}^{\infty}A_{2k}e^{2ik\theta},\quad ReA_{2k}=0, \tag{11}$$

$$A_{2k}=\frac{1}{2\pi}\int_0^{2\pi}\left[f_1(\theta)+if_2(\theta)\right]e^{-2ik\theta}d\theta\quad (k=0,\pm1,\pm2,...),$$

$$\varphi_1(\theta)+i\varphi_2(\theta)=\sum_{k=-\infty}^{\infty}B_{2k}e^{2ik\theta},\quad ReB_{2k}=0, \tag{12}$$

$$B_{2k}=\frac{1}{2\pi}\int_0^{2\pi}\left[\varphi_1(\theta)+i\varphi_2(\theta)\right]e^{-2ik\theta}d\theta\quad (k=0,\pm1,\pm2,...).$$

Substituting expressions (11) into (12) and changing the order of integration, after calculating the integrals using the theory of residues, we find the values of A_{2k}, B_{2k}:

$$A_{2k}=-\frac{1}{2\omega}\int_{L_1}g(t)f_{2k}(t)dt,\quad f_0(t)=\gamma(t),\quad f_2(t)=-\frac{\lambda^2}{2}\gamma^{(2)}(t),$$

$$f_{2k}(t)=-\frac{\lambda^{2k}(2k-1)}{(2k)!}\gamma^{(2k)}(t)+\frac{\lambda^{2k-2}}{(2k-3)!}\gamma^{(2k-2)}(t)\quad (k=2,3,\dots),$$

$$f_{-2k}(t)=\frac{\lambda^{2k}}{(2k)!}\gamma^{(2k)}(t)\quad (k=1,2,\dots),\quad \gamma(t)=ctg\frac{\pi}{\omega}t,$$

$$B_{2k}=-\frac{i}{2\omega}\int_{L_2}g_1(t_1)\varphi_{2k}(it_1)dt_1,\quad \varphi_0(it_1)=\delta(it_1)-\overline{\delta(it_1)},$$

$$\varphi_2(it_1)=-\frac{\lambda^2}{2}\delta^{(2)}(it_1)+2\left[\delta(it_1)-it\delta'(it_1)\right],$$

$$\varphi_{2k}(it_1)=\frac{(1-2k)\lambda^{2k}}{(2k)!}\delta^{(2k)}(it_1)+$$

$$+\frac{2\lambda^{2k-2}}{(2k-2)!}\left[k\delta^{(2k-2)}(it_1)-it_1\delta^{(2k-1)}(it_1)\right]\quad (k=2,3,\dots),$$

$$\varphi_{-2k}(it_1)=-\frac{\lambda^{2k}}{(2k)!}\delta^{(2k)}(it_1)\quad (k=1,2,\dots),\quad \delta(it_1)=\mathrm{ctg}\,\frac{\pi}{\omega}(it_1).$$

Now, to solve the boundary value problem (10), we apply the method of power series. By substituting into the left side of the boundary condition (10) instead of $\Phi_0(\tau)$, $\overline{\Phi_0(\tau)}$, $\Phi_0'(\tau)$ and $\Psi_0(\tau)$ their expansions in Laurent series in the vicinity of the zero point, and into the right side of (10) instead of $f_1(\theta)+if_2(\theta)$ and $\varphi_1(\theta)+i\varphi_2(\theta)$ by the Fourier series and equating the coefficients at the same powers of $\exp(i\theta)$ in both parts boundary condition (11), we obtain two infinite systems of algebraic equations with respect to the coefficients α_{2k}, β_{2k}. After some transformations, we obtain an infinite system of linear algebraic equations with respect to α_{2k} [8], [9]:

$$i\alpha_{2j+2} = \sum_{j=0}^{\infty} iA_{j,k}\alpha_{2k+2} + b_j \quad (j = 0,\ 1,\ 2,\ ...)\,, \tag{13}$$

$$b_0 = M_2' - \sum_{k=0}^{\infty} \frac{g_{k+2}\lambda^{2k+4}}{2^{2k+4}} A^*_{-2k+2},$$

$$b_j = A^*_{2j+2} - \frac{(2j+1)\,A_0' g_{j+1}\lambda^{2k+2}}{K2^{2j+2}} - \sum_{k=0}^{\infty} \frac{(2j+2k+3)!g_{j+k+2}\lambda^{2j+2k+4}}{(2j)\,!\,(2k+3)!2^{2j+2k+4}} A^*_{-2k+2},$$

$$M_2' = M_2 + i\tau_{xy}^{\infty}, \quad A_0' = M_0 - 2i\tau_{xy}^{\infty}, \quad M_{2k} = A_{2k} + B_{2k} \quad (k = 0, \pm 1, \pm 2, ...)\,,$$

$$K = 1 - \frac{\pi^2}{12}\lambda^2, \quad g_j = 2\sum_{m=1}^{\infty} \frac{1}{m^{2j}}, \quad A^*_{2k+2} = M_{2k+2}, \quad A^*_{-2k-2} = M_{-2k-2},$$

$$A_{j,k} = (2j+1)\,\gamma_{j,k}\lambda^{2j+2k+2}, \quad \gamma_{0,0} = \frac{3}{8} g_2\lambda^2 + \sum_{i=1}^{\infty} \frac{(2i+1)\,g_{i+1}^2\lambda^{4i+2}}{2^{4i+4}},$$

$$\gamma_{j,k} = -\frac{(2j+2k+2)!g_{j+k+1}}{(2j+1)!\,(2k+1)!2^{2j+2k+2}} + \frac{(2j+2k+4)!g_{j+k+2}}{(2j+2)!\,(2k+2)!2^{2j+2k+4}} +$$

$$+ \sum_{i=0}^{\infty} \frac{(2j+2k+1)!\,(2k+2i+1)!g_{j+k+1}g_{k+i+1}\lambda^{4i+2}}{(2j+1)!\,(2k+1)!\,(2i+1)!\,(2i)!2^{2j+2k+4i+4}} + b_{j,k},$$

$$b_{0,k} = 0, \quad b_{j,0} = 0, \quad b_{j,k} = \frac{g_{j+1}g_{k+1}\lambda^2}{2^{2j+2k+4}} \left(1 + \frac{K_1\lambda^2}{K}\right),$$

$$K_1 = \pi^2/12 \quad (j = 1,\ 2,\ ...) \quad (k = 1, 2, ...)\,.$$

To determine the coefficients β_{2k}, the following equations are obtained

$$\begin{gathered} i\beta_{2k} = \tfrac{1}{K}\left\{-A_0' + 2\textstyle\sum_{k=0}^{\infty} \tfrac{g_{k+1}\lambda^{2k+2}}{2^{2k+2}} i\alpha_{2k+2}\right\}, \\ i\beta_{2j+4} = (2j+3)\,i\alpha_{2j+2} + \sum_{k=0}^{\infty} \tfrac{(2j+2k+3)!\lambda^{2j+2k+4}g_{j+k+2}}{(2j+2)!(2k+1)!2^{2j+2k+4}} i\alpha_{2k+2} - A^*_{-2j-2}. \end{gathered} \tag{14}$$

Thus, solving the boundary value problem (1), the definition of the desired coefficients α_{2k} and β_{2k} is reduced to infinite algebraic equations, on the right side of which there are quantities that depend in the form of integrals on the desired functions $g\,(x)$ and $g_1\,(y)$. To determine the desired functions $g\,(x)$, $g_1\,(y)$, there are boundary conditions (2), (3) on the crack edges.

Requiring that functions (4)-(7) satisfy the boundary condition on the crack edges L_1, we obtain, after some transformations, a singular integral equation with respect to $g(x)$ [6]:

$$\begin{gathered}\frac{1}{\omega}\int_{L_1} g(t)\,\xi(t-x)\,dt-\frac{1}{2\pi i}\int_{L_1}\overline{g(t)}\xi(t-x)\,dt-\\ -\frac{1}{2\pi i}\int_{L_1}\overline{g(t)}\left[\xi(t-x)+S(t-x)-t\gamma(t-x)\right]dt-A+\overline{A}+\overline{B}+H(x)=ia,\end{gathered} \tag{15}$$

where

$$H(x)=\Phi_s(x)+\overline{\Phi_s(x)}+x\overline{\Phi'_s(x)}+\overline{\Psi_s(x)},\Phi_s(x)=\Phi_0(x)+\Phi_2(x),\Psi_s(x)=\Psi_0(x)+\Psi_2(x).$$

Similarly, satisfying the boundary condition on the line L_2, after some transformations, we obtain one more singular integral equation with respect to the desired function $g_1(y)$:

$$\begin{gathered}-\frac{1}{\omega}\int_{L_2}\left\{g_1(t_1)\left[i\xi(it-iy)-i\overline{\xi(it-iy)}\right]+\right.\\ \left.+\overline{g_1(t_1)}\left[iS(it-iy)-(it-iy)\overline{\gamma(it-iy)}-i\overline{\xi(it-iy)}\right]\right\}dt_1+\\ +N(y)-A+\overline{A}+\overline{B}+H(x)=ia_*,\end{gathered} \tag{16}$$

where

$$N(y)=\overline{\Phi_*(iy)}+iy\overline{\Phi'_*(iy)}+\overline{\Psi_8(iy)},\quad \Phi_*(z)=\Phi_0(z)+\Phi_1(z),\quad \Psi_*(z)=\Psi_0(z)+\Psi_1(z).$$

The obtained singular integral equations of the first kind (15) and (16) together with algebraic systems (13)–(14) are the main resolving equations of the problem under consideration. These equations allow us to determine the functions $g(x)$ and $g_1(y)$ and coefficients α_{2k} and β_{2k}. Knowing the functions $g(x)$, $g_1(y)$, $\phi_{10}(z)$ and $\chi_{10}(z)$, one can find the stress-strain state of the perforated plates.

In brittle fracture mechanics [2], the stress intensity factor in the vicinity of the crack tip is of particular interest. In the case under consideration, the crack at one end $x=\lambda$ comes to the surface of a circular hole, free from external forces. In this case, the stresses at the tip $x=\lambda$ are limited and have a well-known singularity at the other end $x=l$. In particular, for the stress intensity factor iK_{II} at the crack tip at the ends of $x=\pm l$, we will have the formula

$$iK_{II}=\lim x\to l\left[\sqrt{2\pi\,|x-l|}g(x)\right].$$

The function $g(x)$ is bounded in a neighborhood $x=\pm\lambda$ and has a singularity of order 1/2 in a neighborhood of $x=\pm l$.

The development of a crack is determined by some additional condition specified at the tip of the crack. For a linearly elastic body, such a condition is the local Griffith–Irwin failure criterion $K_{II}=K_{IIC}$ (K_{IIC} is a constant characterizing the resistance of a material to crack propagation in it). This condition makes it possible to determine the value of the limiting (critical) value of external forces.

The value l, which characterizes the length of the plasticity bands, enters the solution of equation (15), (16) as an unknown parameter, then the solution of the singular integral equation should be sought in the class of everywhere bounded functions (stresses). The condition of limited stresses at the ends of $x = \pm l$ serves to determine the parameter l, knowing which one can find the length of the plastic zones.

Using the expansions of the functions $\xi(z)$, $\gamma(z)$, $S(z)$ in the main parallelogram of periods, and also taking into account $g(x) = -g(-x)$ and applying the change of variables, equation (15) is reduced to the form

$$\gamma(z) = \frac{1}{z^2} + \sum_{j=1}^{\infty} g_{j+1} \frac{(2j+1)\, z^{2j}}{2^{2j+2}}, \quad \zeta(z) = \frac{1}{z} - \sum_{j=1}^{\infty} g_{j+1} \frac{z^{2j+1}}{2^{2j+2}},$$

$$S(z) = \sum_{j=1}^{\infty} \rho_{j+1} \frac{(2j+1)\, z^{2j+1}}{2^{2j+2}},$$

$$x = \pm\lambda, \quad g_k = \sum_{m,n} \frac{1}{T^{2k}}, \quad \rho_k = \sum_{m,n} \frac{T}{T^{2k+1}}, \quad T = \frac{1}{2} P_{m,n}$$

$$(m, n = 0, \pm 1, \pm 2, \ldots, \quad k = 2, 3, \ldots),$$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{p(\tau)}{\tau - \eta} d\tau + \frac{1}{\pi} \int_{-1}^{1} p(\tau)\, B(\eta, \tau)\, d\tau - A + \overline{A} + \overline{B} + H(\eta) = ia, \tag{17}$$

$$P(\tau) = g(t), \quad x = \xi_0 l, \quad \xi_0^2 = u_0, \quad t = \xi l, \quad \xi^2 = u, P(\tau) = g(t), \quad x = \xi_0 l,$$

$$\xi_0^2 = u_0, \quad t = \xi l, \quad \xi^2 = u, B(\eta, \tau) = \frac{1 - \lambda_1^2}{2} \sum_{j}^{\infty} \left(K_j^* - K_j\right) \left(\frac{l}{2}\right)^{2j+2} u_0^j A_j,$$

$$A_j = \left\{ \left[\begin{array}{l} (2j+1) + \frac{(2j+1)(2j)(2j-1)}{1 \cdot 2 \cdot 3} \left(\frac{u}{u_0}\right) + \ldots \\ + \frac{(2j+1)(2j)(2j-1)\ldots[(2j+1)-(2j+1-1)]}{1 \cdot 2 \cdot \ldots\, (2j+1)} \left(\frac{u}{u_0}\right)^j \end{array} \right] \right\}$$

$$u = \frac{1 - \lambda_1^2}{2} (\tau + 1) + \lambda_1^2, \quad u_0 = \frac{1 - \lambda_1^2}{2} (\eta + 1) + \lambda_1^2, \quad \lambda_1 = \frac{\lambda}{l},$$

$$K_1 = \omega_1 \delta_1, \quad K_0^* = -\frac{\omega_1}{2} (\gamma_1 + \delta_1), \quad K_j^* (j+1)(\rho_{j+1} - g_{j+1}), \quad K_j = g_{j+1} \quad (j = 1, 2, \ldots).$$

5. Numerical Solution Technique and Analysis

The solution of equation (17) is required to be in the class of functions (stresses) bounded at the ends of the segment $[-1, 1]$. The ordinary singular integral equation is regularized according to Carleman–Vekua by reducing it to the Fredholm equation. However, when solving problems of interest for applications, it seems to be more expedient to use one of the methods of direct solution of singular equations [6]:

$$p(\eta) = \frac{p_0(\eta)}{\sqrt{1 - \eta^2}}. \tag{18}$$

Here $p_0(\tau)$ is Hlder continuous on $[-1,1]$, and the function $p_0(\tau)$ is replaced by the Lagrange interpolation polynomial constructed from the Chebyshev nodes

$$L_n[P_o,\eta]=\frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}P_k^0\frac{\cos n\theta\cdot\sin\theta_k}{\cos\theta-\cos\theta_k},\quad \eta=cos\theta,$$

$$P_k^0=P_0(\eta_{:}),\quad \eta_{<}=\cos\theta_m,\quad \theta_m=\frac{2m-1}{2n}\pi\quad (m=1,2,3,\ldots,n).$$

Using (18), relations [6]

$$\frac{1}{\pi}\int\frac{\cos n\tau d\tau}{\cos\tau-\cos\theta}=\frac{\sin n\theta}{\sin\theta}\quad (0\le\theta\le\pi),$$

$$\int_{-1}^{1}\frac{F(x)dx}{\sqrt{1-x^2}}=\frac{\pi}{n}\sum_{\nu=1}^{n}F(\cos\theta_\nu)\quad (m=1,2,\ldots),$$

as well as expressions (20), we obtain the quadrature formulas

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{P(\tau)\,d\tau}{\tau-\eta}=\frac{1}{n\sin\theta}\sum_{\nu=1}^{n}P_\nu^0\sum_{m=0}^{n-1}\cos m\theta_\nu\sin m_\theta, \tag{19}$$

$$\frac{1}{2\pi}\int_{-1}^{1}P(\tau)\,B(\eta,\ \tau)\,d\tau=\frac{1}{2n}\sum_{\nu=1}^{n}P_\nu^0B(\eta_\nu,\tau_\nu),\quad \tau_\nu=\eta_\nu, \tag{20}$$

$$A_{2k}=\frac{1-\lambda_1^2}{2}\frac{1}{2n}\sum_{\nu=1}^{n}P_\nu^0 f_{2k}^*(\tau_\nu),$$

$$f_{2k}^*(\tau)=f_{2k}^*\left(\xi^2\right),\quad \xi f_{2k}\left(\xi^2\right)=lf_{2k}(t).$$

Formulas (19), (20) make it possible to replace the basic equations with an infinite system of linear algebraic equations with respect to the approximate values of P_ν^0 of the desired function at the nodal points, as well as the coefficients α_{2k},β_{2k}.

After some calculations, the singular equation is replaced by the following system:

$$\sum_{\nu=1}^{n}a_{m\nu}P_\nu^o+\frac{1}{2}\left[\alpha_2\lambda^2(\delta_1-\overline{\gamma_1})-\beta_2\lambda^2\overline{\delta_1}\right]-H_{**}(\eta_m)=a\quad (m=1,2,\ldots,n),$$

$$a_{m\nu}=\frac{1}{2n}\left[\frac{1}{\sin\theta_m}ctg\frac{\theta_m+(-1)^{|m-\nu|}}{2}+B(\eta_m,\tau_\nu)\right],\quad \tau_m=\eta_m.$$

It is necessary to add the following algebraic equation to the obtained equations

$$\begin{aligned}&\sum_{\nu=1}^{n}(-1)^{\nu+n}P_\nu^0 tg\tfrac{\theta_\nu}{2}=0,\\&\sum_{\nu=1}^{n}(-1)^{\nu}R_\nu^0 ctg\tfrac{\theta_\nu}{2}=0,\end{aligned} \tag{21}$$

which ensures the finiteness of stresses at the point $x = \pm\lambda$, $x = \pm l$ (more precisely, the equality of the stress intensity factor to zero in the sense of (15)).

Equation (21) with marked systems constitutes a closed system for determining all unknowns of the problem. However, the solution of this closed system for a given load τ_{xy}^{∞} is difficult, due to the non-linearity of algebraic equations with respect to the unknown parameter l. Therefore, it is easier to assume that l is given, and to find the load acting on the plate.

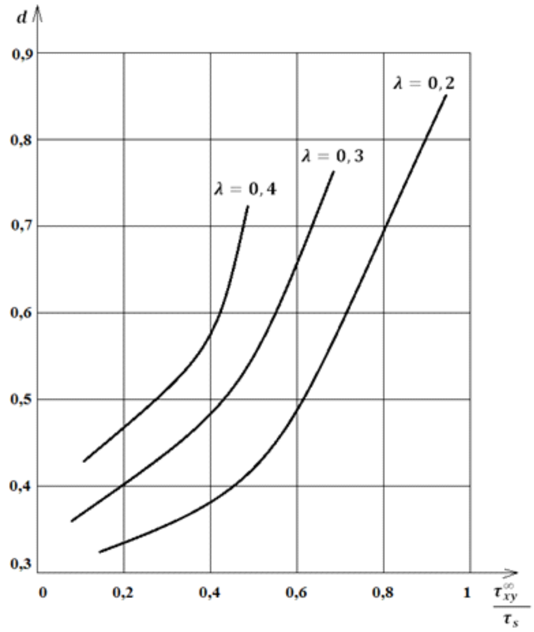

Fig. 2. Dependences of the relative length of the end zone of the crack $d = (\ell - \ell_1)/\lambda$ on the dimensionless value of the intensity of external loading $\tau_{xy}^{\infty}/\tau_*$ for some hole radii: $\lambda = 0,2 \div 0,4$ (curves 1 − 3)

To determine the limiting state of the shear plate, at which crack growth occurs, we use the criterion of brittle fracture of the plate, which is taken as the criterion of critical opening of crack edges at the base of the plastic zone. According to this criterion, crack growth occurs when the opening of crack edges at the tip at the base of the plastic zone reaches the limiting value for the plate material δ_{cr}, i.e. under conditions

$$\int_{-l}^{l_0} g(t)\, dt = -i\delta_{cr}, \qquad \int_{-l_1}^{l_{10}} g_1(t)\, dt = i\delta_{cr}.$$

Let us represent conditions (19) in a discrete form using the above changes of variables and the Gaussian quadrature formula

$$\frac{1 - h_1^2}{2\sqrt{2}M} \sum_{k=1}^{M_1} \frac{p_k^0}{\sqrt{(1 - h_1^2)(1 + \tau_k) + h_1^2}} = \delta_{cr},$$

$$-\frac{1-\lambda_2^2}{2\sqrt{2}M}\sum_{n=1}^{M_2}\frac{p_k^0}{\sqrt{(1-\lambda_2^2)(1+\tau_n)+\lambda_2^2}}=\delta_{cr},$$

where M_1, M_2 is the number of chebyshev nodes belonging to intervals (l_0, l), (l_{10}, l_1).

Using the expansion of the functions $ctg\frac{\pi}{\omega}z$, $sh^{-2}\frac{\pi}{\omega}z$ in the main band of periods, each singular integral equation after some transformations are reduced to a standard form. Using quadrature formulas [7], we reduce each singular integral equation to a finite system of algebraic equations with respect to the values of the required functions $g(x)$ and $g_1(y)$.

Figure 2 shows the dependence of the relative length of the end zone of cracks collinear with the abscissa axis $d = (\ell - \ell_1)/\lambda$ on the dimensionless value of the loading intensity Y for some hole radii: $1 - \lambda = 0,2$; $2 - \lambda = 0,3$; $3 - \lambda = 0,4$.

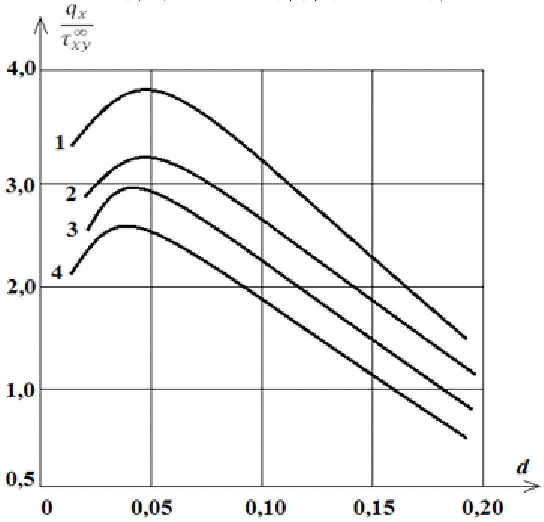

Fig. 3. Dependences of the distribution of shear stresses in the q_x/τ_{xy}^{∞} bonds on the relative size of the end zone of the crack d for some values of the radius of the holes $\lambda = 0,2 \div 0,5$ (curves 1 - 4)

Figure 3 shows the dependence of the forces in the bonds q_x/τ_{xy}^{∞} on the relative size d for some hole radii: $\lambda = 0,2 \div 0,5$ (curves $1-4$).

Calculations show that, under a linear bond deformation law, the stresses in the bonds always have maximum values at the edge of the end zone. The same picture is also observed for the shear values of the edges of the crack end zone. The crack shear at the edge of the end zone has a maximum for linear and nonlinear deformation laws, and with an increase in the relative compliance of the bonds, the crack opening increases.

To determine the limiting equilibrium state of cracks at which they grow, we use the deformation fracture criterion.

Using the obtained solution, the conditions that determine the limiting external load at which a crack develops at point $x = \pm\ell_1$ or $y = \pm h_1$ will be

$$C(\ell_1, q_x(\ell_1))\, q_x(\ell_1) = \delta_{IIc}, \quad C(h_1, q_y(h_1))\, q_y(h_1) = \delta_{IIc}. \tag{22}$$

The solution of the obtained algebraic systems and equation (22) makes it possible to determine the critical value of the external load, the dimensions of the end zones and the forces in the bonds in the state of limit equilibrium, at which cracks grow in the perforated body.

6. Conclusion

The analysis of the limiting equilibrium state of a perforated body under transverse shear, when cracks develop, is reduced to a parametric study of the combined algebraic system and the crack growth criterion (22) for various laws of bond deformation, elastic constants of materials, and geometric characteristics of the perforated body. Directly from the solution of the obtained algebraic systems, the forces in the bonds and the shift of the edges of the end zones before destruction are determined.

References

1. Birger I.A. General algorithms for solving elasticity, plasticity and creeping problems. *In Achievements of Deformed Medium Mechanics*, Nauka, Moscow, 1975, pp. 51-73 (in Russian).
2. Cherepanov G.P. *Mechanics of Brittle Fracture*. Nauka, Moscow, 1974 (in Russian).
3. Goldshtein R.V., Perelmuter M.N. The way to simulate crack growth resistance for composite materials. *Vychisl. Mekh. Sploshnykh Sred*, 2009, **2** (2), pp. 22-39 (in Russian).
4. Ilyushin A.A. *Plasticity*. Logos, Moscow, 2004 (in Russian).
5. Ishlinsky A.Yu., Ivlev D.D. *The Mathematical Theory of Plasticity*. Fizmatlit, Moscow, 2001 (in Russian).
6. Kalandiya A.I. On the approximate solution of one class of singular integral equations. *Dokl. Akad. Nauk SSSR*, 1959, **125** (9), pp. 715-718 (in Russian).
7. Mekhdiev A.K., Mekhtiev R.K. Transverse shear of an isotropic elastic medium in the case when the binder and inclusions are weakened by crack initiation. *Baku Math. J.*, 2022, **1** (2), pp. 125-144.
8. Mehtiyev R.K. *Longitudinal shift of bodies with complex structure, relaxed rectified through crack*. Proc. of the 7th Intern. Conf. Control and Optimization with Industrial Applications (COIA), 1, August 26-28, 2020, Baku, Azerbaijan, pp. 281-283.
9. Mirsalimov V.M. *Multidimentional Elastic-Plastic Problems*. Nauka, Moscow, 1987 (in Russian).
10. Muskhelishvili N.I. *Some Basic Problems of Mathematical Theory Elasticity*. Kluwer, Amsterdam, 1977.
11. Panasyuk V.V. *Mechanics of Quasibrittle Fracture of Materials*. Naukova Dumka, Kiev, 1991 (in Russian).