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PERIODIC AND ANTIPERIODIC BOUNDARY VALUE PROBLEMS FOR THE STURM-LIOUVILLE EQUATION WITH THE DISCONTINUOUS COEFFICIENT

K.D. ÇOŞKUN, A.A. NABIEV*, S. SALTAN

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Abstract. *The present paper studies the boundary-value problem in a finite segment generating with the discontinuous Sturm-Liouville equation and periodic(antiperiodic) boundary conditions. We prove completeness of the system of eigenfunctions and generalized eigenfunctions in the space* $L_2(0, \pi; \rho)$*, obtain the asymptotic formulas for the solution and prove the asymptotic formulas for the eigenvalues of the periodic and antiperiodic boundary value problems.*

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1. Introduction

We consider two boundary value problems generated by the Sturm-Liouville equation

$$-y'' + q(x)y = \lambda^2 \rho(x)y, \quad 0 \le x \le \pi, \tag{1}$$

subject to the boundary conditions

$$y(0) = \pm y(\pi), \quad y'(0) = \pm y'(\pi), \tag{2}$$

* Corresponding author.

Keziban D. Çoşkun
Suleyman Demirel University, Isparta, Türkiye
E-mail: kdc26021990@hotmail.com

Anar A. Nabiev
Suleyman Demirel University, Isparta, Türkiye
E-mail: anaradiloglu@sdu.edu.tr

Suna Saltan
Suleyman Demirel University, Isparta, Türkiye
E-mail: sunasaltan@sdu.edu.tr

where $q(x)$ is a complex valued function in $L_2(0,\pi)$, λ is a spectral parameter,

$$\rho(x) = \begin{cases} 1 \ , & 0 \le x \le a, \\ \alpha^2 \ , & a < x < \pi, \end{cases}$$

with $a \in (0,\pi)$, $\alpha > 0$, $\alpha \neq 1$. Note that the boundary conditions (2) is called periodic for the plus sign and antiperiodic for the minus sign.

Boundary value problems for the Sturm-Liouville equation with discontinuous leading coefficients arise in geophysics, electromagnetics, elasticity and other fields of engineering and physics; for example, modelling toroidal vibrations and free vibrations of the earth, reconstructing the discontinuous material properties of a nonabsorbing media, as a rule leads to direct and inverse problems for the Sturm-Liouville equation with discontinuous coefficients (see [4], [9], [19]).

Sturm-Liouville operators with periodic and antiperiodic boundary conditions were investigated by Stankevich [16], Sadovnichii [14], Marchenko and Ostrovskiy [10], [11]. The inverse problems for the Sturm-Liouville operators with nonseparated boundary conditions were investigated by Sadovnichii [13], Plaksina [17], [18], Yurko [20] and other authors. For an instance, in the studies of Gasymov et. al., [5], Guseinov and Nabiev [6]–[8], the spectral properties were investigated, the uniqueness theorems were proved, necessary and sufficient conditions were obtained for the solution of the inverse problem for the Sturm-Liouville operators with nonseparated boundary conditions . In the studies of Sadovnichii et. al., [15] and Akhtyamov [1] the the uniqueness of the inverse problem were investigated for the general operators by different approach. A direct and inverse scattering problem for the one-dimensional perturbed Hill operator was investigated in [2], [3].

In the present paper investigating the boundary-value problem $(1)-(2)$, in Section 2 we prove completeness of the system of eigenfunctions and generalized eigenfunctions in the space $L_2(0,\pi;\rho)$. In Section 3 we obtain the asymptotic formulas for the solution of Eq. (1) and in the Section 4 we prove the asymptotic formulas for the eigenvalues of the periodic and antiperiodic boundary value problems for the Eq. (1).

2. Completeness of the System of Eigenfunctions of the Boundary Value Problem

Consider the boundary value problem $(1)-(2)$. The values of the parameter $\mu = \lambda^2$ for which the boundary value problem has a non-trivial solution, are called eigenvalues and corresponding solutions are called eigenfunctions.

Let $s(x,\lambda)$ and $c(x,\lambda)$ be linearly-independent solutions of Eq. (1) with initial conditions

$$s(0,\lambda) = c'(0,\lambda) = 0, \quad s'(0,\lambda) = c(0,\lambda) = 1.$$

Then any solution $y(x,\lambda)$ of Eq. (1) can be represented as a linear combination of solutions $s(x,\lambda)$ and $c(x,\lambda)$:

$$y(x,\lambda) = Ac(x,\lambda) + Bs(x,\lambda).$$

Boundary conditions (2) give the following relations:

$$A\left(c(\pi,\lambda)\mp 1\right)+Bs(\pi,\lambda)=0,$$

$$Ac'(\pi,\lambda)+B\left(s'(\pi,\lambda)\mp 1\right)=0.$$

Hence eigenvalues of the boundary value problem $(1)-(2)$ coincide with squares of the roots of the characteristic function

$$\Delta_{\pm}\left(\lambda\right)=\begin{vmatrix} c(\pi,\lambda)\mp 1 & s(\pi,\lambda) \\ c'(\pi,\lambda) & s'(\pi,\lambda)\mp 1 \end{vmatrix}.$$

Since the Wronskian $W\left(c,s\right)=c(x,\lambda)s'(x,\lambda)-c'(x,\lambda)s(x,\lambda)=1$ we have

$$\Delta_{\pm}\left(\lambda\right)=2\mp c(\pi,\lambda)\mp s'(\pi,\lambda).$$

Now consider the solutions $w_1^{\pm}\left(x,\lambda\right)$ and $w_2^{\pm}\left(x,\lambda\right)$ defined as

$$w_1^{\pm}\left(x,\lambda\right)=\mp s(\pi,\lambda)c\left(x,\lambda\right)-\left(1\mp c(\pi,\lambda)\right)s(x,\lambda), \tag{3}$$

$$w_2^{\pm}\left(x,\lambda\right)=\left(1\mp s'(\pi,\lambda)\right)c(x,\lambda)\pm c'(\pi,\lambda)s(x,\lambda). \tag{4}$$

It is easy to compute that

$$w_1^{\pm}\left(0,\lambda\right)=\pm w_1^{\pm}\left(\pi,\lambda\right), \quad w_1^{\pm\prime}\left(0,\lambda\right)\mp w_1^{\pm\prime}\left(\pi,\lambda\right)=-\Delta_{\pm}\left(\lambda\right)$$

and

$$w_2^{\pm}\left(0,\lambda\right)\mp w_2^{\pm}\left(\pi,\lambda\right)=\Delta_{\pm}\left(\lambda\right), \quad w_2^{\pm\prime}\left(0,\lambda\right)=\pm w_2^{\pm\prime}\left(\pi,\lambda\right).$$

An eigenvalue $\mu_n^{\pm}$ of the boundary value problem $(1)-(2)$ is called an eigenvalue with multiplicity p if $\mu_n^{\pm}$ is a root of multiplicity p of the function $\Delta_{\pm}\left(\sqrt{\mu}\right)$. It is easy to see that the functions

$$w_{i,k}^{\pm}\left(x\right)=\frac{\left(-1\right)^k}{k!}\frac{\partial^k}{\partial\mu^k}w_i^{\pm}\left(x,\lambda\right)\left(\mu=\lambda^2\right), \quad k=\overline{0,p-1}, \quad i=1,2,...,$$

satisfy the boundary conditions (2) for $\lambda^2=\mu_n$. Clearly $w_{i,0}^{\pm}\left(x\right),...,w_{i,p-1}^{\pm}\left(x\right)$ $(i\geq 1,2)$ form a chain in which the first nonzero element $w_{i,\ell_i}^{\pm}\left(x\right)$ is an eigenfunction, and the elements following this eigenfunction are the corresponding generalized eigenfunctions. Differentiating the equation (1) k times with respect to $\mu=\lambda^2$, we see that the eigenfunction and generalized eigenfunctions of the chain $w_{i,0}^{\pm}\left(x\right),...,w_{i,p-1}^{\pm}\left(x\right)$, $(i\geq 1,2)$ satisfy the equation

$$-w_{i,k}^{\prime\prime\pm}\left(x\right)+q(x)w_{i,k}^{\pm}\left(x\right)=\rho(x)\left(\mu_n w_{i,k}^{\pm}\left(x\right)-w_{i,k-1}^{\pm}\left(x\right)\right)$$

and boundary conditions (2). It is important to note that two chains $w_{1,0}^{\pm}\left(x\right),...,w_{1,p-1}^{\pm}\left(x\right)$ and $w_{2,0}^{\pm}\left(x\right),...,w_{2,p-1}^{\pm}\left(x\right)$ can consist of the same functions. Beside it we mention that together with eigenvalues and generalized eigenvalues each chain may contain only zero functions.

Let us consider the question of completeness in $L_2(0,\pi;\rho)$ of the system of eigenfunction and generalized eigenfunctions of the boundary value problem $(1)-(2)$.

As it is known a system of vectors is complete in a Hilbert space if and only if the only vector orthogonal to all elements of this system is the null element. Let $M^{\pm}$ is the set of all eigenvalues $\mu_n^{\pm}$, i.e the spectrum of the boundary value problem $(1)-(2)$ and denote the multiplicity of $\mu_n^{\pm}$ by $p_n^{\pm}$. As we have mentioned above, the functions

$$\frac{(-1)^k}{k!}\frac{\partial^k}{\partial\mu^k}w_1^{\pm}\left(x,\sqrt{\mu}\right)|_{\mu=\mu_n^{\pm}} \quad \left(0\le k\le p_n^{\pm}-1,\ \mu_n^{\pm}\in M^{\pm},\ i=1,2\right)$$

either are identically zero, or are the eigenfunction and generalized eigenfunctions of the problem $(1)-(2)$. Let us show that if $f(x)\in L_2(0,\pi)$ and

$$\int_0^{\pi}\rho(x)\frac{\partial^k}{\partial\mu^k}w_1^{\pm}\left(\sqrt{\mu},x\right)f(x)dx|_{\mu=\mu_n^{\pm}}=0 \tag{5}$$

for all $\mu_n^{\pm}\in M^{\pm}$, $k=0,1,...,p_n^{\pm}-1$, $i=1,2$, then $f(x)=0$ a.e. It is clear that the characteristic function $\Delta_{\pm}\left(\sqrt{\mu}\right)$ and the function

$$w_i^{\pm}(f,\sqrt{\mu})=\int_0^{x}\rho(x)w_i^{\pm}\left(x,\sqrt{\mu}\right)f(x)dx \quad (i=1,2)$$

are entire functions of μ. From the formula (5) we have that each $p_n^{\pm}$ fold root $\mu_n^{\pm}$ of the function $\Delta_{\pm}\left(\sqrt{\mu}\right)$ is also a root of functions $w_i^{\pm}(f,\sqrt{\mu})$ $(i=1,2)$ with multiplicity at least $p_n^{\pm}$. Therefore, the equality (5) holds if and only if $w_i^{\pm}(f,\lambda)\left[\Delta_{\pm}(\lambda)\right]^{-1}$ $(i=1,2)$ are entire functions of λ. Consequently, for proving the completeness of the system of the system of eigenfunctions and generalized eigenfunctions of the boundary value problem $(1)-(2)$ it is enough to prove that $f(x)=0$ a.e. if and only if $w_i^{\pm}(f,\lambda)\left[\Delta_{\pm}(\lambda)\right]^{-1}$ $(i=1,2)$ are an entire functions of the parameter λ.

Note that for $q(x)\equiv 0$ the characteristic function $\Delta_0^{\pm}(\lambda)$ of the boundary value problem $(1)-(2)$ has the form

$$\Delta_0^{\pm}(\lambda)=2\mp s_0'(\pi,\lambda)\mp c_0(\pi,\lambda),$$

where

$$c_0(x,\lambda)=\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\cos\lambda\mu^{+}(x)+\frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\cos\lambda\mu^{-}(x),$$

$$s_0(x,\lambda)=\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\frac{\sin\lambda\mu^{+}(x)}{\lambda}+\frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\frac{\sin\lambda\mu^{-}(x)}{\lambda},$$

$$\mu^{\pm}(x)=\pm x\sqrt{\rho(x)}+a\left(1\mp\sqrt{\rho(x)}\right).$$

Since

$$s_0'(x,\lambda)=\frac{1}{2}\left(\sqrt{\rho(x)}+1\right)\cos\lambda\mu^{+}(x)-\frac{1}{2}\left(\sqrt{\rho(x)}-1\right)\cos\lambda\mu^{-}(x)$$

we have

$$c_0(\pi,\lambda)=\frac{1}{2}\left(1+\frac{1}{\alpha}\right)\cos\lambda\mu^+(\pi)+\frac{1}{2}\left(1-\frac{1}{\alpha}\right)\cos\lambda\mu^-(\pi),$$

$$s_0'(\pi,\lambda)=\frac{\alpha}{2}\left(1+\frac{1}{\alpha}\right)\cos\lambda\mu^+(\pi)-\frac{\alpha}{2}\left(1-\frac{1}{\alpha}\right)\cos\lambda\mu^-(\pi)$$

which implies

$$\Delta_0^{\pm}(\lambda)=2\mp\frac{(\alpha+1)^2}{2\alpha}\cos\lambda\mu^+(\pi)\pm\frac{(1-\alpha)^2}{2\alpha}\cos\lambda\mu^-(\pi).$$

Now let us prove that the system of eigenfunctions and generalized eigenfunctions of the boundary value problem (1) – (2) is complete in $L_2(0,\pi;\rho)$.

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are solutions of Eq. (1) with conditions

$$\varphi'(\pi,\lambda)=\psi(\pi,\lambda)=0.$$

The following lemma is obtained from the integral representations of the solutions $c(x,\lambda), s(x,\lambda)$ and $\varphi(x,\lambda)$, $\psi(x,\lambda)$ in [12] and the similar fact in [10] (see Lemma 1.3.1).

Lemma 1. *For all $f(x)\in L_1(0,\pi)$ the following equalities are held.*

$$\lim_{|\lambda|\to\infty}e^{-|\mathrm{Im}\lambda\mu^+(\pi)|}\int_0^\pi f(x)c(x,\lambda)dx=\lim_{|\lambda|\to\infty}e^{-|\mathrm{Im}\lambda\mu^+(\pi)|}\int_0^\pi f(x)\varphi(x,\lambda)dx=0,$$

$$\lim_{|\lambda|\to\infty}e^{-|\mathrm{Im}\lambda\mu^+(\pi)|}\int_0^\pi \lambda f(x)s(x,\lambda)dx=\lim_{|\lambda|\to\infty}e^{-|\mathrm{Im}\lambda\mu^+(\pi)|}\int_0^\pi \lambda f(x)\psi(x,\lambda)dx=0.$$

Corollary 1. *For all $f(x)\in L_1(0,\pi)$ the equality*

$$\lim_{|\lambda|\to\infty}e^{-|\mathrm{Im}\lambda\mu^+(\pi)|}w_i^{\pm}(\,f,\lambda)=0,\quad i=1,2,$$

holds.

Proof. From formulas (3) and (4) it is obtained that

$$\begin{aligned}w_1^{\pm}(x,\lambda)&=\mp\left(s(\pi,\lambda)c(x,\lambda)-c(\pi,\lambda)s(x,\lambda)\right)-s(x,\lambda)\\&=\mp\psi(x,\lambda)-s(x,\lambda),\end{aligned}$$

$$\begin{aligned}w_2^{\pm}(x,\lambda)&=\pm\left(c'(\pi,\lambda)s(x,\lambda)\mp s'(\pi,\lambda)c(x,\lambda)\right)+c(x,\lambda)\\&=\pm\varphi(x,\lambda)+c(x,\lambda).\end{aligned}$$

Hence, the assertion is obtained from the Lemma 1. ◀

Lemma 2. *There exists a constant number $C>0$ and a sequence of unboundely expanding contours K_n on which*

$$|\Delta_{\pm}(\lambda)|\geq|\lambda|^{-1}Ce^{|\mathrm{Im}\lambda\mu^+(\pi)|}. \tag{6}$$

Proof. Using representations

$$s(x,\lambda)=s_0(x,\lambda)+\int\limits_0^{\mu^+(x)} N_-(x,t)\frac{\sin\lambda t}{\lambda}dt,$$

$$c(x,\lambda)=c_0(x,\lambda)+\int\limits_0^{\mu^+(x)} N_+(x,t)\cos\lambda t dt$$

(see [12]), where the kernel $N_\pm(x,t)$ and it is first order partial derivatives belong to $L_1(0,\mu^+(x))$ for $\forall x\in[0,\pi]$ we have

$$c(\pi,\lambda)=c_0(\pi,\lambda)+N_+(\pi,t)\frac{\sin\lambda t}{\lambda}\Big|_{t=\mu^-(\pi)-0}^{\mu^-(\pi)+0}$$

$$+N_+(\pi,\mu^+(\pi))\frac{\sin\lambda\mu^+(\pi)}{\lambda}-\int\limits_0^{\mu^+(\pi)} D_tN_+(\pi,t)\frac{\sin\lambda t}{\lambda}dt,$$

$$c'(\pi,\lambda)=c_0'(\pi,\lambda)-\alpha N_+(\pi,t)\cos\lambda t\Big|_{t=\mu^-(\pi)-0}^{t=\mu^-(\pi)+0}$$

$$+\alpha N_+(\pi,\mu^+(\pi))\cos\lambda\mu^+(\pi)+\int\limits_0^{\mu^+(\pi)} D_xN_+(\pi,t)\cos\lambda t dt,$$

$$s(\pi,\lambda)=s_0(\pi,\lambda)+\int\limits_0^{\mu^+(\pi)} N_-(\pi,t)\frac{\sin\lambda t}{\lambda}dt, \tag{7}$$

$$s'(\pi,\lambda)=s_0'(\pi,\lambda)-\alpha N_-(\pi,t)\frac{\sin\lambda t}{\lambda}\Big|_{t=\mu^-(\pi)-0}^{t=\mu^-(\pi)+0}$$

$$+\alpha N_-(\pi,\mu^+(\pi))\frac{\sin\lambda\mu^+(\pi)}{\lambda}+\int\limits_0^{\mu^+(\pi)} D_xN_-(\pi,t)\frac{\sin\lambda t}{\lambda}dt. \tag{8}$$

Consequently,

$$\begin{aligned}\Delta_\pm(\lambda)&=2\mp c(\pi,\lambda)\mp s'(\pi,\lambda)\\&=2\mp c_0(\pi,\lambda)\mp s_0'(\pi,\lambda)+\lambda^{-1}e^{|\mathrm{Im}\lambda\mu^+(\pi)|}\varepsilon(\lambda),\end{aligned}$$

where $\lim_{|\lambda|\to\infty}\varepsilon(\lambda)=0$. Using the corresponding expressions for $c_0(\pi,\lambda)$ and $s_0'(\pi,\lambda)$ we obtain

$$\Delta_\pm(\lambda)=2\mp[1+A]\cos\lambda\mu^+(\pi)\mp[1-A]\cos\lambda\mu^-(\pi)+\lambda^{-1}e^{|\mathrm{Im}\lambda\mu^+(\pi)|}\varepsilon(\lambda), \tag{9}$$

where $A=\frac{1}{2}\left(\alpha+\frac{1}{\alpha}\right)$. The function

$$\Delta_\pm^{(0)}(\lambda)=2\mp[1+A]\cos\lambda\mu^+(\pi)\mp[1-A]\cos\lambda\mu^-(\pi)$$

will be called the main part of the characteristic function $\Delta_{\pm}(\lambda)$ and it defines the behavior of $\Delta_{\pm}(\lambda)$ as $|\lambda| \to \infty$.

Since $\Delta_{\pm}^{(0)}(\lambda)$ is an entire function it has a countable set of zeros $\left\{\lambda_n^0\right\}$, where

$$\lambda_n^0 = \pm\frac{n\pi}{\mu^+(\pi)} + \theta_n^{\pm}$$

and the sequence $\{\theta_n^{\pm}\}$ is bounded. Let Z_r (where $r > 0$ is arbitrary small number) is the domain which is obtained by removing balls of radius r centred at the zeros of the function $\Delta_{\pm}^{(0)}(\lambda)$, i.e. at points λ_n^0 $(n = 0, \pm 1, \pm 2, ...)$. Since the even function $\Delta_{\pm}^{(0)}(\lambda)$ is a quasipolinomial, i.e.

$$\Delta_{\pm}^{(0)}(\lambda) = 2\mp\frac{1}{2}(1+A)e^{i\lambda\mu^+(\pi)}\mp\frac{1}{2}(1+A)e^{-i\lambda\mu^+(\pi)}\mp\frac{1}{2}(1-A)e^{i\lambda\mu^-(\pi)}\mp\frac{1}{2}(1+A)e^{-i\lambda\mu^-(\pi)},$$

then the function

$$\left[\Delta_{\pm}^{(0)}(\lambda)\right]^{-1} e^{-i\lambda\mu^+(\pi)}$$

is holomorphic in the domain $Z_r \cap \{\mathrm{Im}\lambda > 0\}$, tends to $\mp\frac{2}{1+A}$ when $\mathrm{Im}\lambda \to +\infty$ and it is continuous on the boundary of this domain. Then by the maximum modulus principle, the supremum of the modulus is finite. Consequently there is $C_r > 0$ such that

$$\left|\Delta_{\pm}^{(0)}(\lambda)\right| > C_r e^{|\mathrm{Im}\lambda\mu^+(\pi)|}, \quad \lambda \in Z_r.$$

Hence, any sequence K_n of expanding contours contained in the domain Z_r can be taken as contours on which the inequality (6) satisfies. Lemma is proved. ◄

We now turn to the proof of the main results of this section.

Theorem 1. *The system of eigenfunctions and generalized eigenfunctions of the boundary value problem* (1) − (2) *is complete in the space* $L_2(0, \pi; \rho)$.

Proof. We will show that the functions $w_i^{\pm}(f, \lambda)\left[\Delta_{\pm}(\lambda)\right]^{-1}$ $(i = 1, 2)$ are entire functions if and only if $f(x) = 0$ a.e. Suppose $f(x) \in L_2(0, \pi)$ and the functions $w_i^{\pm}(f, \lambda)\left[\Delta_{\pm}(\lambda)\right]^{-1}$ are entire. By Lemma 2, there exists a constant $C > 0$ and a sequence of unboundely expanding contours K_n such that

$$\left|w_i^{\pm}(f, \lambda)\left[\Delta_{\pm}(\lambda)\right]^{-1}\right| \le C|\lambda||w_i^{\pm}(f, \lambda)|e^{-|\mathrm{Im}\lambda\mu^+(\pi)|}$$

satisfies on K_n.

From this inequality and the Corollary 1 it follows that

$$\lim_{n\to\infty} \max_{\lambda\in K_n} \left|w_i^{\pm}(f, \lambda)\left[\Delta_{\pm}(\lambda)\right]^{-1}\right| = 0 \quad (i = 1, 2).$$

Therefore when $|\lambda| \to \infty$, the entire functions $w_i^{\pm}(f, \lambda)\left[\Delta_{\pm}(\lambda)\right]^{-1}$ grow slower than $|\lambda|$, and as a result they are identically equal to constants which we denote by $f_i^{\pm}$. Consequently, $w_i^{\pm}(f, \lambda) = f_i^{\pm}\Delta_{\pm}(\lambda)$ which implies (together with (3), (4))

$$\mp s(\pi, \lambda)c(f, \lambda) - (1 \mp c(\pi, \lambda))s(f, \lambda) = f_1^{\pm}\Delta_{\pm}(\lambda),$$

$$(1 \mp s'(\pi,\lambda)c(f,\lambda) \pm c'(\pi,\lambda)s(f,\lambda) = f_2^{\pm}\Delta_{\pm}(\lambda),$$

where $c(f,\lambda) = \int\limits_0^{\pi} \rho(x)f(x)c(x,\lambda)dx$, $s(f,\lambda) = \int\limits_0^{\pi} \rho(x)f(x)s(x,\lambda)dx$.

Hence

$$s(f,\lambda) = \pm s(\pi,\lambda)f_2^{\pm} \pm s'(\pi,\lambda)f_1^{\pm} - f_1^{\pm}. \tag{10}$$

Consider the identity (10) for real values $\lambda \to \pm\infty$.Using Lemma 1 and formulas (7) , (8) we can express the identity (10) as

$$\lambda^{-1}\delta(\lambda) = -f_1^{\pm} \pm f_2^{\pm}\lambda^{-1}\left(\alpha_+ \sin\lambda\mu^+(\pi) + \alpha_- \sin\lambda\mu^-(\pi) + \varepsilon_1(\lambda)\right)$$
$$\pm f_1^{\pm}\left(\frac{\alpha+1}{2}\cos\lambda\mu^+(\pi) + \frac{1-\alpha}{2}\cos\lambda\mu^-(\pi) + \varepsilon_2(\lambda)\right),$$

where the functions $\delta(\lambda)$, $\varepsilon_1(\lambda)$ and $\varepsilon_2(\lambda)$ tend to zero as $\lambda \to \pm\infty$ which is possible if and only if $f_1^{\pm} = 0$, $f_2^{\pm} = 0$. Consequently, $s(f,\lambda) \equiv 0$, i.e.

$$\int\limits_0^{\pi} \rho(x)f(x)\left\{s_0(x,\lambda) + \int\limits_0^{\mu^+(x)} N_-(x,t)\frac{\sin\lambda t}{\lambda}dt\right\} = 0.$$

We immediately have

$$\int\limits_a^{\mu^+(\pi)} \frac{\sin\lambda t}{\lambda}\left\{Mf(t) + \int\limits_{a+\frac{t-a}{\sqrt{\rho(t)}}} N_-(x,t)f(x)dx\right\} = 0,$$

where the operator M is defined as

$$Mf(t) = \begin{cases} f(t) & , 0 \le t \le \mu^-(\pi), \\ f(t) + \alpha^- f(a + \frac{a-t}{\alpha}) & , \mu^-(\pi) < t \le a, \\ \alpha^+ f(a + \frac{t-a}{\alpha}) & , a < t < \mu^+(\pi). \end{cases}$$

Hence, the Fourier sin-transform of the function

$$Mf(t) + \int\limits_{a+\frac{t-a}{\sqrt{\rho(t)}}}^{\pi} N_-(x,t)f(x)dx \quad \left(0 \le t \le \mu^+(\pi)\right)$$

vanishes identically, and therefore by the uniqueness theorem for Fourier transform

$$Mf(t) + \int\limits_{a+\frac{t-a}{\sqrt{\rho(t)}}}^{\pi} N_-(x,t)f(x)dx = 0 \tag{11}$$

a.e. on the segment $[0, \mu^+(\pi)]$.

It is easy to show that the operator M is a linear and bounded on the space $L_2(0,\mu^+(\pi))$. Moreover it has the inverse operator

$$M^{-1}f(t) = \begin{cases} f(t) & ,0 \le t < \mu^-(\pi), \\ f(t) - \frac{\alpha^-}{\alpha^+} f(2a-t) & ,\mu^-(\pi) < t \le a, \\ \frac{1}{\alpha^+} f(\mu^+(t)) & ,a < t < \pi, \end{cases}$$

which is also bounded on the space $L_2(0,\pi;\rho)$. Then the homogenous equation (11) with the kernel $N_-(x,.) \in L_2(0,\mu^+(\pi))$ has only the trivial solution. We conclude that $f(x) = 0$ for a.e. $x \in [0,\pi]$ as asserted. ◄

3. Asymptotic Formulas for the Solution of the Sturm-Liouville Equation

We consider the Sturm-Liouville equation (1) where

$$\rho(x) = \begin{cases} 1, & 0 \le x \le a, \\ \alpha^2, & x > a, \end{cases}$$

λ is a complex parameter, $q(x) \in W_2^n[0,\pi]$. Here $W_2^n[0,\pi]$ is the Sobolev's space of the complex valued functions that have the square summable absolutely continuous derivatives of $(n-1)$ order and the derivative of n order on the segment $[0,\pi]$.

We need the following lemma(see Lemma 1.4.1 in [10]).

Lemma 3. *The equation*

$$-z'' + q(x)z = \alpha^2\lambda^2 z, \quad a \le x \le \pi, \tag{12}$$

has the solution

$$z(x,\lambda) = e^{i\alpha\lambda(x-a)} \left[\sum_{k=0}^{n} (2i\alpha\lambda)^{-k} u_k(x) + (2i\alpha\lambda)^{-n-1} u_{n+1}(x,\lambda) \right], \tag{13}$$

where

$$u_0(x) = 1, \; u_k(x) = \int_a^x \left(-u''_{k-1}(t) + q(t)u_{k-1}(t)\right) dt, \quad k = 1,2,...n+1 \tag{14}$$

and

$$u_{n+1}(x,\lambda) = u_{n+1}(x) + (2i\alpha\lambda)^{-1} \int_a^x q(t)u_{n+1}(t)dt$$
$$- \int_0^{x-a} \left\{ u'_{n+1}(x-t) + (2i\alpha\lambda)^{-1} K_{n+1}^{(0)}(x,t) \right\} e^{-2i\alpha\lambda t} dt, \tag{15}$$

$$u'_{n+1}(x,\lambda) = 2i\alpha\lambda \int_0^{x-a} \left\{ u'_{n+1}(x-t) + (2i\alpha\lambda)^{-1} K^{(1)}_{n+1}(x,t) \right\} e^{-2i\alpha\lambda t} dt, \tag{16}$$

in which the kernels $K^{(0)}_{n+1}(x,t)$, $K^{(1)}_{n+1}(x,t)$ *are square summable with respect to* t *for each* $x \in [a,\pi]$.

It is very difficult to compute the functions $u_k(x)$ $\left(k=\overline{1,n}\right)$ by the recurrence formulas (14), since these formulas have undesirable integration operations. To remove these difficulties we put

$$\sigma(\lambda,x) = \frac{d}{dx} \ln \left(1 + \frac{u_1(x)}{2i\alpha\lambda} + ... + \frac{u_n(x)}{(2i\alpha\lambda)^n} + \frac{u_{n+1}(\lambda,x)}{(2i\alpha\lambda)^{n+1}} \right) \tag{17}$$

and then we have the following representation for the solution of the equation (12) :

$$z(x,\lambda) = \exp \left\{ i\alpha\lambda (x-a) + \int_a^x \sigma(\lambda,t) dt \right\},$$

where the function $\sigma(\lambda,x)$ also satisfies the equation

$$\sigma'(\lambda,x) + 2i\alpha\lambda\sigma(\lambda,x) + \sigma^2(\lambda,x) - q(x) = 0. \tag{18}$$

Let us set

$$P_n(\lambda,x) = 1 + \sum_{k=1}^{n} (2i\alpha\lambda)^{-k} u_k(x),$$

$$Q_n(\lambda,x) = P_n(\lambda,x) + (2i\alpha\lambda)^{-n-1} u_{n+1}(\lambda,x).$$

Then the formulas (13) and (17) take the forms

$$z(x,\lambda) = e^{i\alpha\lambda(x-a)} Q_n(x,\lambda),$$

$$\sigma(\lambda,x) = \frac{P'_n(\lambda,x)}{P_n(\lambda,x)} + \frac{u'_{n+1}(\lambda,x)P_n(\lambda,x) - u_{n+1}(\lambda,x)P'_n(\lambda,x)}{(2i\alpha\lambda)^{n+1} P_n(\lambda,x) Q_n(\lambda,x)}.$$

As in [10] we can show that

$$\sigma(\lambda,x) = \sum_{k=1}^{n} \frac{\sigma_k(x)}{(2i\alpha\lambda)^k} + \frac{\sigma_n(\lambda,x)}{(2i\alpha\lambda)^n}, \tag{19}$$

where

$$\sigma_1(x) = q(x),\ \sigma_2(x) = -q'(x),\ \sigma_3(x) = q''(x) - q^2(x), ...,$$

$$\sigma_{k+1}(x) = -\sigma'_k(x) - \sum_{j=1}^{k-1} \sigma_{k-j}(x)\sigma_j(x) \quad (k = 2,3,...,n), \tag{20}$$

$$\sigma_n(\lambda,x)=\frac{2i\alpha\lambda\int\limits_0^{x-a}\sigma_{n+1}(x-\xi)e^{-2i\alpha\lambda\xi}d\xi+\int\limits_0^{x-a}\widetilde{K}^{(1)}_{n+1}(x,\xi)e^{-2i\alpha\lambda\xi}d\xi}{2i\alpha\lambda Q_n(\lambda,x)},\quad x>a,\qquad (21)$$

with the kernel $\widetilde{K}^{(1)}_{n+1}(x,\xi)$ which is square summable with respect to the variable ξ.

Now let us return to our problem (1). Using Lemma 3 we can write the following solution of Eq. (1):

$$y(x,\lambda)=e^{i\lambda x+\int\limits_0^x\sigma(\lambda,t)dt},\quad 0\le x\le a,\qquad (22)$$

and

$$y(x,\lambda)=A(\lambda)e^{i\lambda\mu^+(x)+\int\limits_a^x\sigma(\lambda,t)dt}+B(\lambda)e^{i\lambda\mu^-(x)+\int\limits_a^x\sigma(-\lambda,t)dt},\quad x\ge a,\qquad (23)$$

where

$$\mu^{\pm}(x)=\pm\alpha x\mp\alpha a+a,$$

$$A(\lambda)=\frac{(\alpha+1)i\lambda+\sigma(\lambda,a^-)-\sigma(-\lambda,a^+)}{2i\alpha\lambda+\sigma(\lambda,a^+)-\sigma(-\lambda,a^+)}\exp\left(\int\limits_0^a\sigma(\lambda,t)dt\right),$$

$$B(\lambda)=\frac{(\alpha-1)i\lambda+\sigma(\lambda,a^+)-\sigma(\lambda,a^-)}{2i\alpha\lambda+\sigma(\lambda,a^+)-\sigma(-\lambda,a^+)}\exp\left(\int\limits_0^a\sigma(\lambda,t)dt\right).$$

Here $\sigma(\lambda,a^+)$ and $\sigma(\lambda,a^-)$ indicate the right and the left limits at point $x=a$.

We see that by using the formulas (22), (23) it can be obtained two solutions $y(x,\lambda)$ and $y(x,-\lambda)$ of equation (1) for $\lambda\ne 0$. By the formula (22), (23) we have for the Wronskian the following formula:

$$W[y(x,\lambda),y(x,-\lambda)]=D\omega(\lambda,x)e^{\int\limits_0^x[\sigma(\lambda,t)+\sigma(-\lambda,t)]dt},$$

where $D=1$ if $0\le x\le a$, $D=\frac{\omega(\lambda,a^-)}{\omega(\lambda,a^+)}$ if $x>a$ and

$$\omega(\lambda,x)=-2i\lambda\sqrt{\rho(x)}+\sigma(-\lambda,x)-\sigma(\lambda,x),\ 0\le x\le\pi.$$

Since the Wronskian of two solutions of Eq. (1) doesn't depend on x we have

$$W[y(x,\lambda),y(x,-\lambda)]=W[y(x,\lambda),y(x,-\lambda)]|_{x=0}=\omega(\lambda,0),$$

$$W[y(x,\lambda),y(x,-\lambda)]|_{x=0}=W[y(x,\lambda),y(x,-\lambda)]|_{x=a}=\omega(\lambda,a^-)e^{\int\limits_0^a[\sigma(\lambda,t)+\sigma(-\lambda,t)]dt},$$

i.e.

$$\omega(\lambda,a^-)=e^{-\int\limits_0^a[\sigma(\lambda,t)+\sigma(-\lambda,t)]dt}\omega(\lambda,0).\qquad (24)$$

Hence,

$$e^{\int\limits_0^x[\sigma(\lambda,t)+\sigma(-\lambda,t)]dt}=\omega(\lambda,0)\left(D\omega(\lambda,x)\right)^{-1}.\qquad (25)$$

We conclude that the solutions $y(x,\lambda)$ and $y(x,-\lambda)$ are defined for all values of λ which isn't a zero of the polynomial $\lambda^n\omega(\lambda,0)$ and these solutions are linearly independent when λ is distinct from zeros of the polynomial $\lambda^n\omega(\lambda,0)$. Therefore we have proved the following theorem.

Theorem 2. *Equation* (1) *has a solution of the form (22), (23) where the function* $\sigma(\lambda,x)$ *is defined by the formula*

$$\sigma(\lambda,x)=\sum_{k=1}^{n}\frac{\sigma_k(x)}{(2i\sqrt{\rho(x)}\lambda)^k}+\frac{\sigma_n(\lambda,x)}{(2i\sqrt{\rho(x)}\lambda)^n} \tag{26}$$

in which the functions $\sigma_k(x)$ $(k=1,2,...,n)$, *are defined by the formulas (20),* $\sigma_n(\lambda,x)$ *is represented as (21).*

Remark. The function $\sigma_n(\lambda,x)$ is defined by the formula (22) for $x>a$. When $0\le x\le a$ we put $\alpha=1, a=0$ in the formula (22) and other related formulas (see [10]).

4. Asymptotic Formulas for the Eigenvalues

Consider the characteristic function of the boundary value problem (1), (2). Using the representation (22), (23) of the solutions $y(x,\lambda)$, $y(x,-\lambda)$ we can find the following expressions for the solutions $c(x,\lambda)$ and $s(x,\lambda)$:

$$s(x,\lambda)=-\frac{y(x,\lambda)-y(x,-\lambda)}{\omega(\lambda,0)},$$

$$c(x,\lambda)=-\frac{y(x,\lambda)[i\lambda-\sigma(-\lambda,0)]+y(x,-\lambda)[i\lambda+\sigma(\lambda,0)]}{\omega(\lambda,0)}. \tag{27}$$

Recall that the characteristic function $\Delta_\pm(\lambda)$ of the boundary value problem (1), (2) is

$$\Delta_\pm(\lambda)=2\pm c(\pi,\lambda)\pm s'(\pi,\lambda).$$

From the formulas (27) we have

$$\omega(\lambda,0)\Delta_\pm(\lambda)=2\pm y(\pi,\lambda)[i\lambda-\sigma(-\lambda,0)]+y(\pi,-\lambda)[i\lambda+\sigma(\lambda,0)]\pm y'(\pi,\lambda)-y'(\pi,-\lambda).$$

From Eq. (22), (23) it is obtained that the equation $\Delta_\pm(\lambda)=0$ is equivalent to the equation

$$y(\pi,\lambda)[i\lambda-\sigma(-\lambda,0)]+y(\pi,-\lambda)[i\lambda+\sigma(\lambda,0)]y'(\pi,\lambda)-y'(\pi,-\lambda)=\mp 2\omega(\lambda,0)$$

which can be written as

$$\left(G_1(\lambda)e^{i\lambda\mu^+(\pi)}+G_2(\lambda)e^{-i\lambda\mu^-(\pi)}\right)e^{\int_0^\pi\sigma(\lambda,t)dt}-$$

$$-\left(G_1(-\lambda)e^{-i\lambda\mu^+(\pi)}+G_2(-\lambda)e^{i\lambda\mu^-(\pi)}\right)e^{\int_0^\pi\sigma(-\lambda,t)dt}=\mp 2\omega(\lambda,0), \tag{28}$$

where

$$G_1(\lambda) = [(\alpha+1)i\lambda + \sigma(\lambda,\pi) - \sigma(-\lambda,0)]A_0(\lambda),$$

$$G_2(\lambda) = [(1-\alpha)i\lambda + \sigma(\lambda,0) - \sigma(\lambda,\pi)]B_0(-\lambda),$$

$$A_0(\lambda) = \frac{\sigma(-\lambda,a^+) - \sigma(\lambda,a^-) - (\alpha+1)i\lambda}{\omega(\lambda,a^+)},$$

$$B_0(\lambda) = \frac{\sigma(\lambda,a^-) - \sigma(\lambda,a^+) - (\alpha-1)i\lambda}{\omega(\lambda,a^+)} e^{\int\limits_0^a [\sigma(\lambda,t)-\sigma(-\lambda,t)]dt}.$$

Multiplying the equation (28) by $e^{\int\limits_0^\pi \sigma(\lambda,t)dt}$ and taking into our account the formulas (25), (26) we find

$$e^{i\lambda\mu^+(\pi)+\int\limits_0^\pi \sigma(\lambda,t)dt} = \frac{\omega(\lambda,0)}{H(\lambda)}\left[\mp 1 \pm \sqrt{1 + \frac{H(\lambda)H(-\lambda)}{\omega(\lambda,0)\omega(\lambda,\pi)}\frac{\omega(\lambda,a^+)}{\omega(\lambda,a^-)}}\right], \tag{29}$$

where

$$H(\lambda) = G_1(\lambda) + G_2(\lambda)e^{-2i\alpha\lambda}.$$

Clearly, the formula (29) is written as

$$e^{i\lambda\mu^+(\pi)+\frac{1}{2}\int\limits_0^\pi [\sigma(\lambda,t)-\sigma(-\lambda,t)]dt} =$$

$$= \frac{\omega(\lambda,0)}{H(\lambda)}\sqrt{\frac{\omega(\lambda,\pi)\omega(\lambda,a^-)}{\omega(\lambda,0)\omega(\lambda,a^+)}}\left[\mp 1 \pm \sqrt{1 + \frac{H(\lambda)H(-\lambda)}{\omega(\lambda,0)\omega(\lambda,\pi)}\frac{\omega(\lambda,a^+)}{\omega(\lambda,a^-)}}\right]. \tag{29$'$}$$

By the formula (9) of the previous section

$$\Delta_\pm(\lambda) = \Delta_\pm^{(0)}(\lambda) + \lambda^{-1}e^{|\mathrm{Im}\lambda\mu^+(\pi)|}\varepsilon(\lambda), \tag{30}$$

where

$$\Delta_\pm^{(0)}(\lambda) = 2 \pm (1+A)\cos\lambda\mu^+(\pi) \pm (1-A)\cos\lambda\mu^-(\pi).$$

$A = \frac{1}{2}\left(\alpha + \frac{1}{2}\right)$, $\lim\limits_{|\lambda|\to\infty} \varepsilon(\lambda) = 0$ and the zeros of the function $\Delta_\pm^{(0)}(\lambda)$ are

$$\left(\lambda_n^0\right)^\pm = \pm\frac{n\pi}{\mu^+(\pi)} \pm \theta_n^\pm, \;\; \sup|\theta_n^\pm| < +\infty.$$

with $n = 2k$ for '+' case an $n = 2k+1$ for '$-$' case. As it is mentioned in the previous section

$$|\Delta_\pm^{(0)}(\lambda)| \geq C_r e^{|\mathrm{Im}\lambda\mu^+(\pi)|} \tag{31}$$

for some $C_r > 0$ and $\lambda \in Z_r$ where Z_r denotes the domain which is obtained by removing balls of radius r ($r > 0$) centered at the zeros $\left(\lambda_n^0\right)^\pm$. Therefore, using Rouche's theorem and the inequalities (30), (31) we can formulate the following theorem.

Theorem 3. *The asymptotic formula*

$$\sqrt{\lambda_{2k}^{\pm}} = \pm\frac{2k\pi}{\mu^{+}(\pi)} + \theta_{2k}^{\pm} + \varepsilon^{\pm}(2k) \tag{32}$$

is held for the eigenvalues $\sqrt{\lambda_{2k}^{\pm}}$ *of the periodic problem (1), (2_+) and the asymptotic formula*

$$\sqrt{\lambda_{2k+1}^{\pm}} = \pm\frac{2(k+1)\pi}{\mu^{+}(\pi)} + \theta_{2k+1}^{\pm} + \varepsilon^{\pm}(2k+1) \tag{33}$$

is held for the eigenvalues $\sqrt{\lambda_{2k+1}^{\pm}}$ *of the antiperiodic problem (1), (2_-). Here* $\lim\limits_{n\to\infty}\varepsilon^{\pm}(n) = 0$.

Now we will improve the asymptotic formulas (32), (33) by using smoothness of the function $q(x)$. We can write

$$H(\lambda) = -\omega(\lambda,0)\left[1+m(\lambda)\right]A_0(\lambda) + \omega(\lambda,0)m(\lambda)B_0(-\lambda)e^{-2ia\lambda},$$

where

$$m(\lambda) = \frac{\sigma(\lambda,0)-\sigma(\lambda,\pi)+(1-\alpha)i\lambda}{\omega(\lambda,0)}.$$

Then

$$H(\lambda) = -\omega(\lambda,0)A_0(\lambda)\left(1+r(\lambda)\right),$$

where

$$r(\lambda) = m(\lambda)\left(1-\frac{B_0(-\lambda)}{A_0(\lambda)}\right)e^{-2ia\lambda}.$$

Further, since

$$\omega(\lambda,a^-) = -2i\lambda + \sigma(-\lambda,a^-) - \sigma(-\lambda,a^+) - \big(\sigma(\lambda,a^-)-\sigma(\lambda,a^+)\big) + \omega(\lambda,a^+) + 2i\alpha\lambda$$

we have

$$\frac{\omega(\lambda,a^-)}{\omega(\lambda,a^+)} = 1 + \frac{\sigma(\lambda,a^+)-\sigma(\lambda,a^-)}{\omega(\lambda,a^+)} + \frac{\sigma(-\lambda,a^+)-\sigma(-\lambda,a^-)}{\omega(-\lambda,a^+)} + \frac{2i\lambda(\alpha-1)}{\omega(\lambda,a^+)}.$$

If we denote

$$S(\lambda) = \frac{\sigma(\lambda,a^+)-\sigma(\lambda,a^-)}{\omega(\lambda,a^+)} + \frac{i\lambda(\alpha-1)}{\omega(\lambda,a^+)},$$

then

$$\frac{\omega(\lambda,a^-)}{\omega(\lambda,a^+)} = 1 + S(\lambda) + S(-\lambda).$$

Similarly

$$\frac{\omega(\lambda,\pi)}{\omega(\lambda,0)} = 1 + m(\lambda) + m(-\lambda)$$

and we also have

$$A_0(\lambda) = 1 + S(\lambda),\ B_0(\lambda) = -S(\lambda)\exp(\int_0^a [\sigma(\lambda,t) - \sigma(-\lambda,t)]dt).$$

The following lemmas is proved similarly to the Lemma 1.4.3 in [10].

Lemma 4. *Let* $f(x) \in L_2(0,\pi)$ *and* $\lambda_k = \lambda_k^0 + \varepsilon_k$ $(k = 1,2,...)$ *where* $\varepsilon_n = O\left(\frac{1}{n}\right)$, $n \to \infty$ *and* $\lambda_k^0 = \frac{k\pi}{\mu^+(\pi)} + \theta_k$, $\sup|\theta_k| < \infty$. *Then*

$$\int_0^\pi f(x)e^{-2i\lambda_k x}dx = \widetilde{f}(\lambda_k^0) + \frac{g_1(k)}{k},\quad \widetilde{f}(\lambda_k^0) = \int_0^\pi f(x)e^{-2i\lambda_k^0 x}dx,$$

where

$$\sum_{k=1}^{\infty} |g_1(k)|^2 < \infty,\quad \sum_{k=1}^{\infty} \left|\widetilde{f}(\lambda_k^0)\right|^2 < \infty.$$

Lemma 5. *Let* $q(x) \in W_2^n[0,\pi]$ *and the number sequence* $\lambda_k = \lambda_k^0 + \varepsilon_k$ *satisfies the condition* $\varepsilon_k = O\left(\frac{1}{k}\right)$, $k \to \infty$. *Then*

$$\sigma_n(\lambda_k,\pi) = \widetilde{\sigma}_{n+1}(\lambda_k^0) + k^{-1}g_1(k),$$

$$\int_0^\pi \sigma(\lambda_k,x)dx = \sum_{j=1}^{n+2} d_j\,(2i\lambda_k)^{-1} - \frac{\mu^+(\pi)}{(2ik)^{n+1}}\widetilde{\sigma}_{n+1,0}(\lambda_k^0) + k^{-2}g_2(k),$$

where

$$d_j = \int_0^\pi \frac{\sigma_j(x)}{\sqrt{\rho(x)}}dx,\ j = 1,2,...,n+2,$$

$$\widetilde{\sigma}_{n+1,0}(\lambda_k^0) = \int_0^a \sigma_{n+1}(a-t)e^{-2i\lambda_k^0 t}dt + \frac{1}{\alpha}\int_0^{\pi-a} \sigma_{n+1}(\pi-t)e^{-2i\alpha\lambda_k^0 t}dt,$$

$\sum_{k=1}^{\infty} |g_j(k)|^2 < \infty$, $j = 1,2$, *and the coefficients* d_j $(j = 1,2,...n+1)$ *doesn't depend on* n *and* k, *the coefficient* d_{n+2} *doesn't depend on* k *and* $d_{n+2} = 0$ *if* $n = 0$.

Let us denote by $\widetilde{W_2^n}[0,\pi]$ the subspace of the space $W_2^n[0,\pi]$ such that $f^{(k)}(0) = f^{(k)}(\pi)$ $(k = 0,1,2,...,n-1)$ for every $f(x) \in \widetilde{W_2^n}[0,\pi] \subset W_2^n[0,\pi]$. Clearly, $\widetilde{W_2^0}[0,\pi] = L_2(0,\pi)$. Now let $q(x) \in \widetilde{W_2^n}[0,\pi]$ and $\mathrm{Im}q(x) = 0$. Because $\sigma_k(x)$ is a polynomial with respect to $q(x), q'(x), q''(x), ..., q^{(n-1)}(x)$ then $q(x) \in \widetilde{W_2^n}[0,\pi]$ implies that $\sigma_k(0) = \sigma_k(\pi)$ $(k = 1,2,...n)$. Consequently, according to Eq. (33) we have

$$\sigma(\lambda,0) - \sigma(\lambda,\pi) = \sum_{k=1}^{n} \frac{\sigma_k(\pi)}{2i\lambda^k}\left(1 - \frac{1}{\alpha^k}\right) + \frac{\sigma_n(\lambda,\pi)}{(2i\alpha\lambda)^n}.$$

Let $q(x) \in L_2(0,\pi)$ is real. Since $\sigma(\lambda,0)-\sigma(\lambda,\pi)=O\left(\frac{1}{\lambda}\right)$, $\sigma(\lambda,a^+)-\sigma(\lambda,a^-)=O\left(\frac{1}{\lambda}\right)$, $\omega(\lambda,0)=-2i\lambda+O\left(\frac{1}{\lambda}\right)$, $\omega(\lambda,a^+)=-2i\alpha\lambda+O\left(\frac{1}{\lambda}\right)$ we easily compute that

$$m(\lambda)=\left[1+O\left(\frac{1}{\lambda^2}\right)\right]\left[\frac{\alpha-1}{2}+\sum_{k=1}^{n}\left(\frac{1}{\alpha^k}-1\right)\frac{\sigma_k(\pi)}{(2i\lambda)^{k+1}}+\frac{\sigma_n(\lambda,\pi)}{\alpha^n\,(2i\lambda)^{n+1}}\right],$$

$$s(\lambda)=\left[1+O\left(\frac{1}{\lambda^2}\right)\right]\left[\frac{1-\alpha}{2\alpha}-\sum_{k=1}^{n}\left(\frac{1}{\alpha^k}-1\right)\frac{1}{\alpha}\frac{\sigma_k(a)}{(2i\lambda)^{k+1}}+\frac{\sigma_n(\lambda,a^-)}{\alpha\,(2i\lambda)^{n+1}}\right],$$

$$r(\lambda)=m(\lambda)\left(1+\frac{S(-\lambda)}{1+S(\lambda)}\frac{y(a,-\lambda)}{y(a,\lambda)}\right). \tag{34}$$

From Eq. (24) and (27) we have that

$$y(a,-\lambda)y(a,\lambda)=\frac{\omega(\lambda,0)}{\omega(\lambda,a^-)},\quad y(a,-\lambda)-y(a,\lambda)=\omega(\lambda,0)s(a,\lambda).$$

Multiplying the second equality by $y(a,\lambda)$ and taking into account the first one we obtain

$$y(a,\lambda)=-\omega(\lambda,0)s(a,\lambda)\left[1\mp\sqrt{1+\frac{4}{\omega(\lambda,0)\omega(\lambda,a^-)S^2(a,\lambda)}}\right].$$

Since $\omega(-\lambda,0)=-\omega(\lambda,0)$ and $s(a,-\lambda)=-s(a,\lambda)$ we have

$$y(a,-\lambda)=-y(a,\lambda).$$

Consequently,

$$r(\lambda)=m(\lambda)\left(1-\frac{S(-\lambda)}{1+S(\lambda)}\right)=\frac{\alpha(\alpha-1)}{\alpha+1}+O\left(\frac{1}{\lambda^2}\right).$$

From equations (34) we also have

$$1+m(\lambda)+m(-\lambda)=\alpha+O\left(\frac{1}{\lambda^2}\right), \tag{35}$$

$$1+S(\lambda)+S(-\lambda)=\frac{1}{\alpha}+O\left(\frac{1}{\lambda^2}\right), \tag{36}$$

$$1+S(\lambda)-S(-\lambda)=1+O\left(\frac{1}{\lambda^2}\right), \tag{37}$$

$$(1+r(\lambda))\,(1+S(\lambda))=\frac{\alpha^2+1}{2\alpha}+O\left(\frac{1}{\lambda^2}\right). \tag{38}$$

Since

$$(1+S(\lambda))\,(1+r(\lambda))=1+S(\lambda)+m(\lambda)\delta(\lambda),$$

where

$$\delta(\lambda)=1+S(\lambda)-S(-\lambda), \tag{39}$$

the formula (29′) takes the form of

$$e^{i\lambda\mu^{+}(\pi)+\frac{1}{2}\int\limits_0^\pi[\sigma(\lambda,t)-\sigma(-\lambda,t)]dt} = -\frac{\sqrt{[1+m(\lambda)+m(-\lambda)]\,[1+S(\lambda)+S(-\lambda)]}}{1+S(\lambda)+m(\lambda)\delta(\lambda)}\times$$

$$\times\left[\mp 1 \pm i\sqrt{\frac{(S(\lambda)-m(\lambda)\delta(-\lambda))\,(S(-\lambda)-m(-\lambda)\delta(\lambda))}{[1+m(\lambda)+m(-\lambda)]\,[1+S(\lambda)+S(-\lambda)]}}\right]. \tag{40}$$

Since square roots of the eigenvalues are zeros of the equation (40) we have

$$\left(i\theta_k^{\pm}+i\varepsilon^{\pm}(k)\right)\mu^{+}(\pi)+\frac{1}{2}\int\limits_0^\pi[\sigma(\lambda,t)-\sigma(-\lambda,t)]dt|_{\lambda=\frac{k\pi}{\mu^{+}(\pi)}+\theta_k^{\pm}+\varepsilon^{\pm}(k)}$$

$$=\left\{\frac{1}{2}\ln[(1+m(\lambda)+m(-\lambda))\,(1+S(\lambda)+S(-\lambda))]-\ln(1+S(\lambda)+m(\lambda)\delta(\lambda))\right.$$

$$\left.+\ln\left[1-i\frac{(S(\lambda)-m(\lambda)\delta(-\lambda))\,(S(-\lambda)-m(-\lambda)\delta(\lambda))}{[1+S(\lambda)+S(-\lambda)]\,[1+m(\lambda)+m(-\lambda)]}\right]\right\}|_{\lambda=\pm\frac{k\pi}{\mu^{+}(\pi)}+\theta_k^{\pm}+\varepsilon(\pm k)}, \tag{41}$$

as $|\lambda|\to+\infty$. According to equations (34) from Eq. (41) we obtain that

$$\varepsilon^{\pm}(k)+\theta_k^{\pm}=\frac{\beta}{\mu^{+}(\pi)}+O\left(\frac{1}{k}\right),\quad k\to\pm\infty,$$

where

$$\beta=\arctan\left(-\frac{\left|\alpha^2-1\right|}{2\alpha}\right).$$

Using the Lemma 5 and the asymptotic formulas (35) – (39) in the equation (41) by the elementary asymptotic methods (see [10]) we have that

$$\sqrt{\lambda_k^{\pm}}=\frac{k\pi}{\mu^{+}(\pi)}+\theta_k^{\pm}+\varepsilon^{\pm}(k)=\frac{k\pi}{\mu^{+}(\pi)}+\frac{\beta}{\mu^{+}(\pi)}$$

$$+\sum_{1\le y+1\le n+2}a_{2j+1}(2k)^{-2j-1}\pm|e_n\,(2k)|\,(2k)^{-n-1}+\gamma_n^{\pm}k^{-n-2}, \tag{42}$$

where

$$\sum_{k=0}^{\infty}\left|\gamma_n^{\pm}\right|^2<\infty,\quad e_n\,(2k)=\int\limits_0^a q(a-t)e^{-2i\lambda_k^0 t}dt+\frac{1}{\alpha}\int\limits_0^{\pi-a}q(\pi-t)e^{-2i\alpha\lambda_k^0 t}dt,$$

$$a_1=\int\limits_0^\pi\frac{q(t)}{\sqrt{\rho(t)}}dt$$

and the numbers a_{2j+1} do not depend on k. Therefore we have proved the following theorem.

Theorem 4. *If* $q(x) \in \widetilde{W}_2^n[0,\pi]$ *and* $\mathrm{Im}q(x) = 0$ *the eigenvalues* $\lambda_{2k}^{\pm}$ *of the periodic boundary value problem and the eigenvalues* $\lambda_{2k+1}^{\pm}$ *of the antiperiodic boundary value problem have the asymptotic formula* (42).

From this theorem and the formula (42) we have the following corollary.

Corollary 2. *Let* $q(x) \in L_2[0,\pi]$. *Then* $q(x) \in \widetilde{W}_2^n[0,\pi]$ *if and only if*

$$\sum_{k=1}^{\infty} k^{2(n+1)} \left| \sqrt{\lambda_k^+} - \sqrt{\lambda_k^-} \right|^2 < \infty.$$

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